GRADUATE COURSE ON HYBRID CONTROL SYSTEMS – Homework #2

Suggested reading: First 20 pages of

R. Goebel, R. G. Sanfelice and A. R. Teel. Hybrid Dynamical Systems. IEEE Control Systems Magazine, 2009.

which is available from
`https://hybrid.soe.ucsc.edu/files/preprints/34.pdf`

and Chapter 2 of

R. Goebel, R. G. Sanfelice and A. R. Teel. Hybrid Dynamical Systems: Modeling, Stability, and Robustness, Princeton University Press, 2012

which is available from

`http://press.princeton.edu/chapters/s9759.pdf`

Problem 1 *(20 points)* Consider a simplified model of the Newton's cradle consisting of a pair of pendulums with mass m_L, m_R as shown in Figure 1.

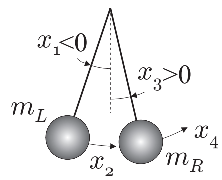

Figure 1: Simplified Newton's cradle.

1.1 Determine the state of the system.

1.2 Define each element of a hybrid system model (C, f, D, g) assuming the following:

- Viscous friction for circular motion.
- Conservation of momentum at impacts.
- Dissipation of energy at impacts.

1.3 Perform the following simulations of the resulting hybrid system for parameters $m_L = 1, m_R = 2$, unitary viscous friction, and a restitution law with parameters such that there is dissipation at impacts:

(a) Plot trajectories as a function of t for pendulums starting at $x_1 = -\pi/4$ and $x_3 = \pi/4$ with zero velocity.

(b) Plot trajectories as a function of t for pendulums starting at $x_1 = 0$ and $x_3 = \pi/4$ with zero velocity.

(c) Plot trajectories as a function of t for pendulums starting at $x_1 = \pi/4$ and $x_3 = \pi/2$ with zero velocity.

Problem 2 *(20 points)* Consider the implementation of a static controller

$$\kappa : \mathbb{R}^n \to \mathbb{R}^m$$

for the continuous-time plant

$$\dot{\xi} = \tilde{f}(\xi, u)$$

in a *digital device*, e.g. computer, microcontroller, digital signal processor, etc. This is depicted in Figure 2, where the controller is interfaced with sample-and-hold devices. The sample-and-hold device that samples the state ξ of the plant is referred to as *sampling device* (or analog-to-digital (A/D) converter), while the sample-and-hold device that stores the output of the controller in between computations is referred to as *hold device* (or digital-to-analog (D/A) converter) which is assumed to be of zero-order type, that is, a zero-order hold (ZOH).

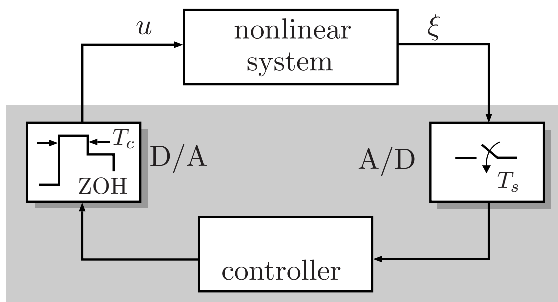

Figure 2: Sample-and-hold control of a nonlinear system.

2.1 Define each element of a hybrid system model (C, f, D, g) and its state assuming the following:

- The computation of the static feedback law takes no time, i.e., is instantaneous.
- The positive constants T_s and T_c are not necessarily equal.

2.2 Explain how the model would change if the computation of the control law takes $\delta > 0$ units of time.

Problem 3 *(20 points)* For the hybrid system $\mathcal{H}$ with data

$$
\begin{aligned}
C &:= \left\{ x \in \mathbb{R}^2 \ : \ |x| < 1 \right\}, & f(x) &:= \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \\
D &:= \left\{ x \in \mathbb{R}^2 \ : \ x_1 = 0, x_2 \in (-\tfrac{1}{2}, 0] \right\}, & g(x) &:= \tfrac{x}{2}
\end{aligned}
$$

with state $x \in \mathbb{R}^2$,

1. Classify all of its solutions following the definitions in Definition 2.5 and Definition 2.7.
2. Indicate the points in $\overline{C} \cup D$ from which nontrivial solutions exist.
3. Indicate the properties of maximal solutions to $\mathcal{H}$.

Hint: apply Proposition 2.10 to answer these questions.

Problem 4 *(20 points)* For each of the hybrid systems in Example 2.8 and Example 2.9 of the main reference, validate the analysis of their solutions done within the examples by simulating each considered solution to each hybrid system.

Problem 5 *(20 points)* Prove the expressions in equations (2.3) and (2.4) in the main reference.