Invariance principles for hybrid systems with connections to detectability and asymptotic stability

Ricardo G. Sanfelice, Rafal Goebel, and Andrew R. Teel

***Abstract*—The paper shows several versions of the (LaSalle's) invariance principle for general hybrid systems. The broad framework allows for nonuniqueness of solutions, Zeno behaviors, and does not insist on continuous dependence of solutions on initial conditions. Instead, only a mild structural property involving graphical convergence of solutions is posed. The general invariance results are then specified to hybrid systems given by set-valued data. Further results involving invariance as well as observability, detectability, and asymptotic stability are given.**

***Index Terms*—Hybrid systems, invariance principles, graphical convergence, detectability, asymptotic stability.**

I. Introduction

A. Hybrid Systems

Hybrid systems theory has been an active research field recently. This is due to the technological advances that require mathematical models allowing for interactions between discrete and continuous dynamics. Hybrid systems, having states that can evolve continuously (flow) and/or discretely (jump), permit modeling and simulation of systems in a wide range of applications including robotics, aircraft control, powertrain automotive systems, etc. Further motivation for studying hybrid systems comes from the recognition of the capabilities of hybrid feedback in robust stabilization of nonlinear control systems; see for example Hespanha and Morse [1], Prieur and Astolfi [2], and Prieur et al. [3].

Several different models and solution concepts for hybrid systems have appeared. See, for example, the work of Tavernini [4], Michel and Hu [5], Lygeros et al. [6], Aubin et al. [7], and van der Schaft and Schumacher [8]. Here, we will work in the framework outlined in [9] (related to concurrent approach in [10]), motivated there by the pursuit of robustness of hybrid control algorithms, and established in [11]. This framework, while similar to [6] and [7], simplifies the data structure somewhat to focus on the dynamics and more importantly, brings to the fore the relationship between properties of the data and the structure of solution sets of a hybrid system. The (mild) properties of the data we will use here were already employed in [12] when showing that asymptotic stability of a hybrid system implies the existence of a smooth Lyapunov function and in [3], where a systematic approach to robust hybrid feedback stabilization of general nonlinear systems was described. The mild regularity properties of the data – which do allow for nonuniqueness of solutions, multiple jumps at a time instant, Zeno behavior, etc. – were further motivated in [13] by accounting for the effects of vanishing noise in a hybrid control system (even when nominal solutions are "well-behaved"). For hybrid control systems that satisfy these regularity properties, results on robustness to a class of singular perturbations, control smoothing, measurement noise, and sample-and-hold implementation of the hybrid controller were recently reported in [14] and [15]. Additionally, in [16], we have developed a general model for simulation of hybrid systems and, relying on the robustness properties shown in [11], we have established sufficient conditions for continuity of asymptotically compact sets of simulated hybrid systems.

B. Invariance Principle Results

LaSalle's invariance principle, presented originally by LaSalle [17], [18] in the setting of differential and difference equations, is one of the most important tools for convergence analysis in dynamical systems. The original principle states that bounded solutions converge to the largest invariant subset of the set where the derivative or the difference, respectively, of a suitable energy function is zero. Byrnes and Martin [19] gave a version stating that bounded solutions converge to the largest invariant subset of the set where an integrable output function is zero. Ryan [20] extended this integral invariance principle to differential inclusions and gave applications to adaptive control. Logemann and Ryan [21] extended the principle for differential inclusions using the notion of meagre functions, alongside a generalization of Barbalat's Lemma. For systems with discontinuous right-hand side, invariance principles based on that of LaSalle were given by Shevitz and Paden [22] and Bacciotti and Ceragioli [23] for Filippov solutions, and by Bacciotti and Ceragioli [24] for Carathéodory solutions. Regarding invariance principles for hybrid systems, in [6] Lygeros et al. extend LaSalle's principle to nonblocking (for each initial condition there exists at least one complete solution), deterministic (the solution is unique), and continuous (see Definition III.3 in [6]) hybrid systems, while Chellaboina et al. [25] work with left-continuous and impulsive systems without multiple jumps at an instant, and with further quasi-continuity properties including uniqueness of solutions. Hespanha, in [26], states an invariance principle for switched linear systems under a specific family of switching signals. The follow-up work, [27], extends some of the results of [26] to a family of nonlinear switched systems under a larger set of

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switching signals. In [28], Bacciotti and Mazzi present invariance principles for nonlinear switched systems with dwell-time signals and state-dependent switching that, in contrast to [26], allow for locally Lipschitz Lyapunov functions.

C. Contributions

In this paper, we identify some basic assumptions that seem necessary to carry out invariance arguments for general hybrid systems, in which nonuniqueness of solutions, multiple jumps at the same time, and Zeno behaviors are possible. These assumptions do not include continuous dependence on initial solutions, whether in the standard uniform metric or in any generalized sense. Instead, we rely on outer semicontinuous, with respect to graphical convergence of solutions, dependence on initial conditions. Whether a given hybrid system possesses this property can be easily verified by checking if the data of the system has some mild regularity. We add that the nonuniqueness of solutions is sometimes necessary in order for outer semicontinuous dependence of solutions on initial conditions to be present. Such nonuniqueness has a physical meaning in hybrid control systems: it comes up naturally when one accounts for small state measurement error (see [13]) and is fundamental in the robustness analysis of hybrid control. The other aspect of the "set-valuedness" of the systems we consider, the set-valued data, serves as an analytical tool to capture nonuniqueness of solutions and is also deeply motivated by the questions of robustness, as outlined in [9]. The usefulness of set-valued data has already been appreciated in the literature of continuous-time systems; see e.g. [29].

As the key to our results is the semicontinuity property of solutions, rather than properties of the data of a hybrid system, we work with abstract systems, defined as sets of hybrid trajectories having the needed property. Only later we specify the results to hybrid systems in the framework of [11] (see also [9], [10]). Such generality allows us to study not only hybrid systems of [11] but also certain subsets of solutions to those, like when the time between jumps is bounded below by a positive constant (dwell-time solutions) or when the number of jumps in a given interval cannot exceed a certain upper bound (average dwell-time solutions). These are usually considered in the switching and hybrid control literature; see [26], [30]. Also, we can obtain specialized results for the classes of Zeno, or of uniformly non-Zeno trajectories.

Our goal is to provide sufficient conditions for convergence of bounded hybrid trajectories. We propose two invariance principles that resemble the original one by LaSalle. The first principle involves a (Lyapunov-like) function that is nonincreasing along all trajectories that remain in a given set. The other relaxes the assumptions, by considering a pair of auxiliary (output) functions satisfying certain conditions only along the hybrid trajectory in question. These conditions seem to be the weakest previously used in invariance principles for continuous-time and discrete-time systems. Thus, in going to the hybrid domain, we do not give up any of the generality. We also invoke observability and detectability for convergence, and we relate this approach to the use of the invariance principles. When coupled with stability, our convergence results give new sufficient conditions for asymptotic stability. Special cases include hybrid versions of Lyapunov's basic theorem and Krasovskii's extension [31]. (For an overview of some other stability results for hybrid systems, see [32] and [33].)

II. Hybrid systems

Throughout this paper, we will study abstract hybrid systems given by a set $\mathcal{S}$ of hybrid trajectories satisfying certain Standing Assumption. Such objects subsume a rich class of hybrid systems defined by generator equations (or inclusions) subject to some weak regularity conditions, and several subsets of solutions to those. Below, $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{N} = \{0, 1, 2, ...\}$, $|\cdot|$ denotes the Euclidean vector norm, and given a nonempty set $\mathcal{A}$, $|\cdot|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$.

A. General framework

Definition 2.1 (hybrid time domain): A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence $0 = t_0 \leq t_1 \leq t_2 \, ... \leq t_J$. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if $\forall (T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. On each hybrid time domain there is a natural ordering of points: $(t, j) \preceq (t', j')$ if $t \leq t'$ and $j \leq j'$.

Definition 2.2 (hybrid trajectory): A *hybrid trajectory* is a pair $(x, \operatorname{dom} x)$ consisting of a hybrid time domain $\operatorname{dom} x$ and a function x defined on $\operatorname{dom} x$ that is continuous in t on $\operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$.

We will often not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid trajectory x comes a hybrid time domain $\operatorname{dom} x$. Alternatively one could think of a hybrid trajectory as a set-valued mapping from $\mathbb{R}_{\geq 0} \times \mathbb{N}$ (or from $\mathbb{R}^2$) whose domain is a hybrid time domain (for a set-valued mapping M, the *domain* $\operatorname{dom} M$ is the set of arguments for which the value is nonempty) and which is single-valued on its domain. We denote the range of x by $\operatorname{rge} x$, i.e. $\operatorname{rge} x = x(\operatorname{dom} x)$.

In what follows, we will rely on a concept of graphical convergence. A sequence of (set-valued) mappings $\{M_i\}_{i=1}^{\infty}$ *converges graphically* to M if the graphs $\operatorname{gph} M_i$ converge to $\operatorname{gph} M$ as sets (for a mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the graph $\operatorname{gph} M$ is $\{(a, b) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ b \in M(a)\}$). For details on set convergence, see Chapter 3 in [34]. When specialized to hybrid trajectories, graphical convergence of a sequence $\{x_i\}_{i=1}^{\infty}$ to a hybrid trajectory x amounts to the following:

(a) for any $(t, j) \in \operatorname{dom} x$ there exists a sequence $(t_i, j_i) \in \operatorname{dom} x_i$ such that $\lim_{i \to \infty} (t_i, j_i, x_i(t_i, j_i)) = (t, j, x(t, j))$,

(b) for any convergent sequence $(t_i, j_i) \in \operatorname{dom} x_i$ such that $\lim_{i \to \infty} x_i(t_i, j_i)$ exists, the limit equals $x(t, j)$ where $(t, j) = \lim_{i \to \infty} (t_i, j_i)$.

When x does not jump multiple times at a single time instant, graphical convergence described above means, intuitively, that the times of j-th jumps of x_i's approach the time of the j-th jump of x, and on time intervals where x does not jump, x_i's do not jump either and converge to x pointwise.

In general, a sequence of hybrid trajectories need not converge graphically, and even when it does, the limit may not be a hybrid trajectory (it can even be set-valued). To carry out invariance principles, we will need to exclude such behavior, and pose some further restrictions, for locally eventually bounded sequences of hybrid trajectories. We call a sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid trajectories *locally eventually bounded* with respect to an open set O if for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset O$ such that for all $i > i_0$, all $(t,j) \in \operatorname{dom} x_i$ with $t + j < m$, $x_i(t,j) \in K$. We can now define our main object of study.

Definition 2.3 (abstract hybrid system): Given an open set $O \subset \mathbb{R}^n$, an *abstract hybrid system* on O is a set $\mathcal{S}$ of hybrid trajectories satisfying the following:

Standing Assumption:

(B1) $\operatorname{rge} x \subset O$ for all $x \in \mathcal{S}$,

(B2) for any $x \in \mathcal{S}$ and any $(\bar{t},\bar{j}) \in \operatorname{dom} x$ we have $\bar{x} \in \mathcal{S}$, where $\operatorname{dom} \bar{x} = \{(t,j) \,|\, (t+\bar{t}, j+\bar{j}) \in \operatorname{dom} x\}$ and $\bar{x}(t,j) = x(t+\bar{t}, j+\bar{j})$ for all $(t,j) \in \operatorname{dom} \bar{x}$,

(B3) for any locally eventually bounded (with respect to O) sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{S}$ that converges graphically, the limit is an element of $\mathcal{S}$.

Remark 2.4: Assumption (B1) identifies O as the state space of the system. (B2) says that tails of trajectories in $\mathcal{S}$ are also in $\mathcal{S}$, and reduces to the standard semi-group property under further existence and uniqueness conditions. (B3) guarantees a kind of semicontinuous dependence of trajectories on initial conditions. More specifically, given a sequence of $x_i \in \mathcal{S}$ with $x_i(0,0)$ convergent to some point x^*, a general property of set convergence (see [34, Theorem 4.18] or Section III in [11]) implies that we can pick a subsequence of x_i's that converge graphically. Under the eventual local boundedness assumption, (B3) guarantees that the graphical limit of that subsequence, say x, is an element of $\mathcal{S}$. As from the very definition of graphical convergence we also get that $x(0,0) = x^*$, this essentially means that a limit of graphically convergent trajectories with initial points convergent to x^* is a trajectory with initial point x^*. (However, this does not mean that every trajectory from x^* is a limit of some trajectories with initial points different from, but convergent to x^*.)

Example 2.5: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function. Consider a differential equation $\dot{x}(t) = f(x(t))$ and, for simplicity, suppose that maximal solutions to it are complete. With each such solution we can identify a hybrid trajectory x with $\operatorname{dom} x = \mathbb{R}_{\geq 0} \times \{0\}$. Let $\mathcal{S}$ be the set of all such hybrid trajectories. (B1) is trivially satisfied, while (B2) follows from the definition of a solution to a differential equation. If f is locally bounded (which is the case if f is continuous), then trajectories $x \in \mathcal{S}$ are uniformly continuous, locally with respect to $\mathbb{R}^n$. Then (B3) is equivalent to assuming that pointwise limits or local uniform limits of (locally eventually bounded) sequences of elements of $\mathcal{S}$ are in $\mathcal{S}$. Classical results say that (B3) is satisfied when f is continuous. When solutions exist and are unique for each initial condition (for example when f is locally Lipschitz continuous) then (B3) reduces to continuous dependence of solutions on initial conditions (in the uniform metric on compact intervals, or pointwise as used by [18]) while (B2) becomes the semigroup property as used by [18]. Hence, (B1)-(B3) is met by the "discontinuous Carathéodory systems" of [24], where f is discontinuous and a solution closure property, corresponding to (B3) but stated in terms of local uniform convergence, is assumed. The importance of properties (B2), (B3) for differential inclusions resulting from Filippov's regularization of a discontinuous f were recognized already in [29, Chapter 3]. $\square$

A hybrid trajectory x is called *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0,0)$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded. We say that x is *continuous complete* if $\operatorname{dom} x = [0,\infty) \times \{0\}$ and *instantaneously Zeno* if $\operatorname{dom} x = \{0\} \times \mathbb{N}$. A trajectory $x \in \mathcal{S}$ is called *maximal* (with respect to $\mathcal{S}$) if there does not exist $x' \in \mathcal{S}$ such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$. A trajectory x is *precompact* if it is complete and $\overline{\operatorname{rge} x} \subset O$ is compact. Finally, we write $\mathcal{S}(x^0)$ as the subset of hybrid trajectories x in $\mathcal{S}$ starting at x^0.

B. Hybrid systems generated by outer semicontinuous data

We now show that the systems in Definition 2.3 subsume those studied in [9], [11], [10]. The latter have the form

$$\mathcal{H}: \qquad \left\{ \begin{array}{ccll} \dot{x} & \in & F(x) & x \in C \\ x^+ & \in & G(x) & x \in D \ , \end{array} \right. \tag{1}$$

where the set-valued mappings F (flow mapping) and G (jump mapping) describe the continuous and the discrete evolutions, respectively, and the sets C (flow set) and D (jump set) say where these evolutions may occur. We will also restrict the solutions to be in a state space O. $\mathcal{S}_{\mathcal{H}}$ denotes the set of all solutions to $\mathcal{H}$. Formally, a *solution to* $\mathcal{H}$ is a hybrid trajectory such that $\operatorname{rge} x \subset O$ and:

(S1) for all $j \in \mathbb{N}$ such that such that I_j has nonempty interior, where $I_j \times \{j\} := \operatorname{dom} x \cap ([0,+\infty) \times \{j\})$, $x(\cdot, j)$ is absolutely continuous in t on I_j and, for almost all $t \in I_j$,

$$x(t,j) \in C, \quad \dot{x}(t,j) \in F(x(t,j));$$

(S2) for all $(t,j) \in \operatorname{dom} x$ such that $(t,j+1) \in \operatorname{dom} x$,

$$x(t,j) \in D, \quad x(t,j+1) \in G(x(t,j)).$$

The following theorem collects some results from [11].

Theorem 2.6: If the data (F,G,C,D,O) of $\mathcal{H}$ satisfies

(A0) $O \subset \mathbb{R}^n$ is an open set;

(A1) C and D are closed sets relative to O;

(A2) $F : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex $\forall x \in C$;

(A3) $G : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and $G(x)$ is nonempty and $G(x) \subset O$ for all $x \in D$;

then $\mathcal{S}_\mathcal{H}$ satisfies (B1)-(B3).

The set-valued mapping $F : O \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for every convergent sequence of x_i's with $\lim x_i \in O$, and every convergent sequence of $\zeta_i \in F(x_i)$, $\lim \zeta_i \in F(\lim x_i)$. Similarly for G. F is *locally bounded* if for every compact $K \subset O$ there exists a compact $K' \subset \mathbb{R}^n$ such that $F(K) \subset K'$. For locally bounded mappings that have closed (and hence compact) values, outer semicontinuity agrees with what is often called upper semicontinuity; see [34] or [11].

Even if one originally considers a hybrid system with single-valued but discontinuous flow and jump maps (as often is the case in hybrid feedback control of nonlinear systems), accounting for arbitrarily small measurement noise leads to systems with set-valued data satisfying (A1)-(A3); see [13].

Lemma 2.7: Suppose that $\mathcal{H}$ satisfies (A0)-(A3) and that $D \cap G(D) = \emptyset$. Then for any precompact $x \in \mathcal{S}_\mathcal{H}$ there exists $\gamma > 0$ such that $t_{j+1} - t_j \geq \gamma$ for all $j \geq 1$, $(t_j, j), (t_{j+1}, j) \in \operatorname{dom} x$ (i.e. the elapsed time between jumps is uniformly bounded below by a positive constant).

Proof: By local boundedness of F and precompactness of x, $F(\operatorname{rge} x)$ is bounded and for some $\delta > 0$, $|\dot{x}(t,j)| < \delta$ for all $(t,j) \in \operatorname{dom} x$. Let $E := \cup_{j=0}^{J-1}(t_{j+1}, j)$ be the set of all points in $\operatorname{dom} x$ at which a jump occurs (J can be finite or infinite). Then $\overline{x(E)} \subset O$ is compact by precompactness of x, $x(E) \subset D$, and by relative closedness of D in O, $\overline{x(E)} \subset D$. By outer semicontinuity of G, $G(\overline{x(E)}) \subset G(D)$ is closed, and as $\overline{x(E)} \cap G(\overline{x(E)}) = \emptyset$, the distance between $\overline{x(E)}$ and $G(\overline{x(E)})$ is positive, say $\epsilon > 0$. Then, for $j = 0, 1, 2, \dots, J-1$, the time interval between t_j and t_{j+1} is at least ϵ/δ (as the distance between $x(t_j, j)$ and $x(t_{j+1}, j)$ is at least ϵ). ■

Various subsets of $\mathcal{S}_\mathcal{H}$ also satisfy the Standing Assumption.

Corollary 2.8: Suppose that $\mathcal{H}$ satisfies Assumption (A0)-(A3). Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to [-\infty, +\infty]$ be a lower semicontinuous function. Then the subset of $\mathcal{S}_\mathcal{H}$ consisting of all solutions x to $\mathcal{H}$ such that

$(\diamond)$ $\phi(s,i,t,j) \leq 0$ *for all* $(s,i), (t,j) \in \operatorname{dom} x$,

satisfies (B3) of the Standing Assumption. If furthermore ϕ is such that for some function Φ we have $\phi(s,i,t,j) = \Phi(t-s, j-i)$ for all $(s,i,t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}_{\geq 0} \times \mathbb{N}$, then the subset of solutions satisfies (B2) of the Standing Assumption.

Proof: If $\{x_k\}_{k=1}^\infty$ is a locally eventually bounded and a graphically convergent sequence is $\mathcal{S}_\mathcal{H}$, then by [9, Lemma 4.3], the limit, which we call x, is a solution to $\mathcal{H}$. Moreover, the sets $\operatorname{dom} x_k$ converge (in the sense of set convergence) to $\operatorname{dom} x$; see the proof of [11, Lemma 4.3]. In particular, given any $(s,i), (t,j) \in \operatorname{dom} x$, there exist $(s_k, i_k), (t_k, j_k) \in \operatorname{dom} x_k$ for all large enough k's, so that $(s_k, i_k) \to (s,i)$ and $(t_k, j_k) \to (t,j)$. If each of x_k's satisfies $(\diamond)$, then by lower semicontinuity of ϕ, so does x. This shows the first claim. Now, let $x \in \mathcal{S}_\mathcal{H}$ satisfy $(\diamond)$ and $\phi(s,i,t,j) = \Phi(t-s, j-i)$ for all (s,i,t,j). For any $(T,J) \in \operatorname{dom} x$, let $\overline{x}(t,j) := x(t+T, j+J)$. Then, for any $(s,i), (t,j) \in \operatorname{dom} \overline{x}$, $\phi(s,i,t,j) = \Phi(t-s, j-i) = \Phi((t+T)-(s+T), (j+J)-(i+J)) = \phi(s+T, i+J, t+T, j+J) \leq 0$ since $(s+T, i+J), (t+T, j+J) \in \operatorname{dom} x$. This shows the second claim. ■

Example 2.9 (autonomous differential/difference inclusions): Given a closed set $K \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, let $\phi(s,i,t,j)$ be

$$\phi(s,i,t,j) = \begin{cases} 0 & (t,j) \in K \\ +\infty & (t,j) \notin K. \end{cases}$$

Such ϕ is lower semicontinuous, and $(\diamond)$ means just that $\operatorname{dom} x \subset K$. In particular, for $K = \mathbb{R}_{\geq 0} \times \{0\}$ (respectively, $K = \{0\} \times \mathbb{N}$), the set of all solutions to $\mathcal{H}$ satisfying $(\diamond)$ can be identified with the set of absolutely continuous functions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfying $\dot{x}(t) \in F(x(t))$ and $x(t) \in C$ for all $t \in \mathbb{R}_{\geq 0}$ (respectively, with the set of sequences $x : \mathbb{N} \to \mathbb{R}^n$ satisfying $x(j+1) \in G(x(j))$ and $x(j) \in D$) for all $j \in \mathbb{N}$. For the special cases of K just mentioned, (B2) is satisfied (tails of solutions to autonomous differential/difference inclusions are also solutions). □

Example 2.10 (dwell-time solutions): Consider

$$\phi(s,i,t,j) = \begin{cases} a(j-i) - b(t-s) - c & i < j \\ -\infty & i \geq j \,. \end{cases}$$

Note that for such ϕ, $\phi(s,i,t,j) = \Phi(t-s, j-i)$ with $\Phi(\tau, \iota) = a\iota - b\tau - c$ if $\iota > 0$, $\Phi(\tau, \iota) = -\infty$ if $\iota \leq 0$. When $a = c = 1$ and $b = 1/\tau_D > 0$, then $(\diamond)$ reduces to $(j-i-1)\delta \leq t-s$ when $i < j$, which requires that the jumps be separated by at least τ_D amount of "dwell-time". This class of solutions is known as dwell-time solutions. Bounds of the type $j - i \leq b(t-s) + c$ for $i < j$ describe solutions with bounded average dwell time. See [26] and [30]. □

Example 2.11 (switched systems): Fix an integer $m > 0$ and for each $q \in Q := \{1, 2, \dots, m\}$ let $f_q : O \to \mathbb{R}^n$ be a continuous function where the set $O \subset \mathbb{R}^n$ is open. Consider a hybrid system $\mathcal{H}$ in the form (1), with a variable (x,q) and data $(\dot{x}, \dot{q}) = (f_q(x), 0)$, $(x^+, q^+) \in (x, Q)$, $C = D = O \times Q$. Then $\mathcal{H}$ meets the conditions in Theorem 2.6. The set $\mathcal{S}_\mathcal{H}$ includes representations, on hybrid time domains, of all solutions to the switched system $\dot{x}(t) = f_q(x(t))$ for which the increasing (and finite or infinite) sequence of switching times t_i, $i = 1, 2, \dots$ has no accumulation points or has one accumulation point equal to $\sup_i t_i$. (Note that each solution to a switched system can be represented on a hybrid time domain, but some solutions to $\mathcal{H}$ – those with multiple jumps at an instant – do not correspond to a solution of a switched system.) For background on switched systems, see for example [26]. Corollary 2.8 and Example 2.10 show that hybrid time domain representations of certain classes of solutions to the switched system $\dot{x} = f_q(x)$ do satisfy the Standing Assumption. In particular, such classes include solutions with dwell-time τ_D for each $\tau_D > 0$, and also solutions with bounded average dwell-time or reverse average dwell-time (cf. [30]). □

Example 2.12 (Lyapunov-like inequalities): Different kinds of families of solutions to $\mathcal{H}$, also meeting the Standing Assumption, can be generated by various Lyapunov-like inequalities. For example, for any continuous function $V : O \to \mathbb{R}$, and any fixed $\overline{j} \in \mathbb{N}$, the set of all $x \in \mathcal{S}_\mathcal{H}$ such that, if $(t, \overline{j}-1), (t, \overline{j}) \in \operatorname{dom} x$ then $V(x(t, \overline{j})) \leq V(x(t, \overline{j}-1))$ meets (B3) of the Standing Assumption. (In other words, this is the set of all x such that, if x has a j-th jump, then V does not increase during that jump.) Consequently, the set of all $x \in \mathcal{S}_\mathcal{H}$ such that $V(x(t,j)) \leq V(x(t,j-1))$ for all

$(t, j) \in \operatorname{dom} x$ such that $(t, j-1) \in \operatorname{dom} x$ meets (B3); it can be easily verified that this set also satisfies (B2). $\square$

For any subset K close relative to O, the subset of all solutions x to $\mathcal{H}$ such that $\operatorname{rge} x \subset K$ meets the Standing Assumption. More generally, if $\mathcal{S}$ satisfies the Standing Assumption and $K \subset O$ is closed relative to O, then $\mathcal{S}_K := \{x \in \mathcal{S} \mid \operatorname{rge} x \subset K\}$ satisfies the Standing Assumption.

In contrast, the set of all $x \in \mathcal{S}_{\mathcal{H}}$ for which $t_{j+1} - t_j > 0$ may not meet the Standing Assumption. Indeed, a sequence of such solutions can converge graphically to an instantaneous Zeno solution. Another negative example is the set of all x which have exactly $J > 0$ jumps (or at least J jumps). One can construct a system and a convergent sequence of its solutions, $\{x_i\}_{i=1}^{\infty}$, so that all J jumps for x_i occur at time i. The graphical limit will have no jumps.

III. Weak invariance and Ω-limit sets

We define invariance for the set of hybrid trajectories $\mathcal{S}$.

Definition 3.1 (weak invariance): For the set of hybrid trajectories $\mathcal{S}$, the set $\mathcal{M} \subset O$ is said to be

(a) *weakly forward invariant* (with respect to $\mathcal{S}$) if for each $x^0 \in \mathcal{M}$, there exists at least one complete hybrid trajectory $x \in \mathcal{S}(x^0)$ with $x(t, j) \in \mathcal{M}$ for all $(t, j) \in \operatorname{dom} x$;

(b) *weakly backward invariant* (with respect to $\mathcal{S}$) if for each $q \in \mathcal{M}$, $N > 0$, there exist $x^0 \in \mathcal{M}$ and at least one hybrid trajectory $x \in \mathcal{S}(x^0)$ such that for some $(t^*, j^*) \in \operatorname{dom} x$, $t^* + j^* \geq N$, we have $x(t^*, j^*) = q$ and $x(t, j) \in \mathcal{M}$ for all $(t, j) \preceq (t^*, j^*)$, $(t, j) \in \operatorname{dom} x$;

(c) *weakly invariant* (with respect to $\mathcal{S}$) if it is both weakly forward invariant and weakly backward invariant.

Our weak forward invariance essentially agrees with the concept of viability used in [7], and if one insists on uniqueness of trajectories, with invariance as used in [6]. In [7], a set K is viable for a impulsive differential inclusion if for each initial condition in K there exists a complete solution that stays in K. Invariance of a set for impulsive difference inclusions is also defined and it is based on a viable set but requires all complete solutions starting in K to stay in K. Similarly, Lygeros et al. in [6] define invariance of a set but do not restrict the solutions to be complete. Requiring completeness in forward invariance and arbitrarily large $N > 0$ in backward invariance leads to the "smallest" possible invariant sets. To verify the forward invariance for sets of trajectories closed under concatenation (see Assumption (B5) in Section VII), it is sufficient to test every point x^0 of $\mathcal{M}$ for the existence of a complete hybrid trajectory x starting at x^0 such that $x(t, j) \in \mathcal{M}$ for all $t + j \leq 1$, $(t, j) \in \operatorname{dom} x$.

Given a hybrid trajectory $x \in \mathcal{S}$, a sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ of points in $\operatorname{dom} x$ is *unbounded* if the sequence of $t_i + j_i$'s is unbounded, and *increasing* if for $i = 1, 2, \ldots$, $(t_i, j_i) \prec (t_{i+1}, j_{i+1})$ in the natural ordering on $\operatorname{dom} x$.

Definition 3.2 (ω-limit set): For a complete hybrid trajectory $x \in \mathcal{S}$, its *ω-limit* set, denoted $\Omega(x)$, is the set of all ω-limit points, that is points $x^* \in O$ for which there exists an increasing and unbounded sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ in $\operatorname{dom} x$ so that $\lim_{i \to \infty} x(t_i, j_i) = x^*$.

The next lemma extends the results on ω-limit sets in [35, Chapter VII], [18, Chapter 1 §5, Chapter 2 §5], and [29, Chapter 3 §12.4] to hybrid trajectories. It can be also seen as a generalization of [6, Lemma IV.1].

Lemma 3.3: If $x \in \mathcal{S}$ is a precompact hybrid trajectory of $\mathcal{S}$ then its ω-limit set $\Omega(x)$ is nonempty, compact, and weakly invariant. Moreover, the hybrid trajectory x approaches $\Omega(x)$, which is the smallest closed set approached by x. That is, for all $\epsilon > 0$ there exists $(\bar{t}, \bar{j}) \in \operatorname{dom} x$ such that for all (t, j) satisfying $(t, j) \succeq (\bar{t}, \bar{j})$, $(t, j) \in \operatorname{dom} x$, $x(t, j) \in \Omega(x) + \epsilon\mathbb{B}$.

Proof: For any increasing and unbounded sequence (t_i, j_i), the sequence $x(t_i, j_i)$ is bounded and has a convergent subsequence. Thus $\Omega(x) \neq \emptyset$. Boundedness of x implies that of $\Omega(x)$. Pick $x_k^* \in \Omega(x)$ with $x_k^* \to x^*$. By the definition of $\Omega(x)$, for each k there exists an increasing and unbounded sequence (t_k^i, j_k^i) such that $x_k(t_k^i, j_k^i) \to x_k^*$ as $i \to \infty$. Let $\bar{i_k}$ be such that $|x_k(t_k^i, j_k^i) - x_k^*| \leq k^{-1}$ for all k, all $i \geq \bar{i_k}$. Pick i_k's so that for each k, $i_k \geq \bar{i_k}$ and $(t_k^{i_k}, j_k^{i_k}) \prec (t_k^{i_{k+1}}, j_k^{i_{k+1}})$. As $x_k^* \to x^*$, we must have $x_k(t_k^{n_{i_k}}, j_k^{n_{i_k}}) \to x^*$ as $k \to \infty$. Thus $x^* \in \Omega(x)$, and $\Omega(x)$ is closed.

We now show the weak invariance. Pick $N > 0$ and $x^* \in \Omega(x)$. Let (t_i, j_i) be an increasing and unbounded sequence such that $x(t_i, j_i) \to x^*$ as $i \to \infty$. For all large i's, pick $(\underline{t}_i, \underline{j}_i) \in \operatorname{dom} x$ such that $t_i + j_i - (N+1) \leq \underline{t}_i + \underline{j}_i \leq t_i + j_i - N$ and let $\bar{x}_i(t, j) = x(t + \underline{t}_i, j + \underline{j}_i)$ for all $(t, j) \in \operatorname{dom} \bar{x}_i$. Then $\bar{x}_i \in \mathcal{S}$ by (B2) of the Standing Assumption. Since x is bounded and $\operatorname{rge} \bar{x}_i \subset \operatorname{rge} x$, $\{\bar{x}_i\}_{i=0}^{\infty}$ is locally eventually bounded with respect to O. By (B3), there exists a subsequence $\{\bar{x}_{i_k}\}_{k=0}^{\infty}$ of $\{\bar{x}_i\}_{i=0}^{\infty}$, graphically converging to some $\tilde{x} \in \mathcal{S}$. As each $\bar{x}$ is complete, so is $\tilde{x}$; see [11, Lemmas 3.5 and 4.5]. The subsequence can be picked so that $(\bar{t}_{i_k}, \bar{j}_{i_k})$ converge to some (t^*, j^*) with $t^* + j^* \geq N$, where $\bar{t}_{i_k} = t_{i_k} - \underline{t}_{i_k}$ and $\bar{j}_{i_k} = j_{i_k} - \underline{j}_{i_k}$. By the definition of graphical convergence, $\tilde{x}(t^*, j^*) = \lim_{k \to \infty} \bar{x}_{i_k}(\bar{t}_{i_k}, \bar{j}_{i_k})$ and so $\tilde{x}(t^*, j^*) = x^*$. Now define a hybrid arc $\hat{x}$ by $\hat{x}(t, j) = \tilde{x}(t + t^*, j + j^*)$. Then $\hat{x}$ is complete, and by (B2), $\hat{x} \in \mathcal{S}$. Thus, $\hat{x}$ verifies weak forward invariance (at x^*) and $\tilde{x}$, since N is arbitrary, verifies weak backward invariance, as long as we show that $\tilde{x}(\tilde{t}, \tilde{j}) \in \Omega(x)$ for all $(\tilde{t}, \tilde{j}) \in \operatorname{dom} \tilde{x}$. By the graphical convergence of $\bar{x}_{i_k}$ to $\tilde{x}$, there exist $(\tilde{t}_{i_k}, \tilde{j}_{i_k}) \in \operatorname{dom} \bar{x}_{i_k}$, $(\tilde{t}_{i_k}, \tilde{j}_{i_k}) \to (\tilde{t}, \tilde{j})$ such that $\bar{x}_{i_k}(\tilde{t}_{i_k}, \tilde{j}_{i_k}) \to \tilde{x}(\tilde{t}, \tilde{j})$. By construction, $\bar{x}_{i_k}(\tilde{t}_{i_k}, \tilde{j}_{i_k}) = x(\tilde{t}_{i_k} + t_{i_k}, \tilde{j}_{i_k} + j_{i_k})$ where (t_{i_k}, j_{i_k}) is increasing and unbounded. Thus, the sequence $(\tilde{t}_{i_k} + t_{i_k}, \tilde{j}_{i_k} + j_{i_k})$ in $\operatorname{dom} x$ is increasing and unbounded, and so $\tilde{x}(\tilde{t}, \tilde{j})$ is an ω-limit point of x.

Finally, we show convergence of x to its ω-limit set. Suppose that for some $\epsilon > 0$ there exists an increasing and unbounded sequence $(t_i, j_i) \in \operatorname{dom} x$ such that $x(t_i, j_i) \notin \Omega(x) + \epsilon\mathbb{B}$ for $i = 1, 2, \ldots$. By precompactness of x, there exists a convergent subsequence of $x(t_i, j_i)$'s. Its limit is, by definition, in $\Omega(x)$. This is a contradiction. ∎

IV. An invariance principle involving a nonincreasing function

The invariance principles we formulate in this section rely on properties of certain functions not only on the range of the trajectory in question, but also on the neighborhood of its range. Invariance principles relying only on the properties of certain functions on the range of the trajectory will be the subject of Section V. In what follows, given a hybrid trajectory x with domain $\operatorname{dom} x$, $t(j)$ will denote the least time t such that $(t,j) \in \operatorname{dom} x$, while $j(t)$ will denote the least index j such that $(t,j) \in \operatorname{dom} x$.

A. Sets of hybrid trajectories

We say that a function $V : O \to \mathbb{R}$ is *nonincreasing along a hybrid trajectory* x if $V(x(t,j)) \geq V(x(t',j'))$ for all $(t,j),(t',j') \in \operatorname{dom} x$ such that $(t,j) \preceq (t',j')$. The notation $f^{-1}(r)$ will stand for the r-level set of f on $\operatorname{dom} f$, the domain of definition of f, i.e. $f^{-1}(r) := \{z \in \operatorname{dom} f \mid f(z) = r\}$.

Lemma 4.1: Suppose a function $V : O \to \mathbb{R}$ is nonincreasing along a hybrid trajectory x. If V is lower semicontinuous, then for some $r \in \mathbb{R}$, $V(\Omega(x)) \subset (-\infty, r]$. If V is continuous, then for some $r \in \mathbb{R}$, $V(\Omega(x)) = r$.

Proof: If $\Omega(x) = \emptyset$, there is nothing to prove. Otherwise, pick any $\bar{z} \in \Omega(x)$. By the definition of $\Omega(x)$, there exists $\{(t_i,j_i)\}_{i=1}^{\infty}$, an increasing and unbounded sequence in $\operatorname{dom} x$, satisfying $x(t_i,j_i) \to \bar{z}$. Let $\bar{r} = \liminf_{i\to\infty} V(x(t_i,j_i))$. Pick any $z \in \Omega(x)$, and an increasing and unbounded sequence $\{(t_k,j_k)\}_{k=1}^{\infty}$ in $\operatorname{dom} x$ with $x(t_k,j_k) \to z$. There exists a subsequence $\{(t_{k_i},j_{k_i})\}_{i=1}^{\infty}$ of $\{(t_k,j_k)\}_{k=1}^{\infty}$ such that for $i = 1,2,\ldots$, $(t_i,j_i) \preceq (t_{k_i},j_{k_i})$, and as V is nonincreasing along x, $V(x(t_i,j_i)) \geq V(x(t_{k_i},j_{k_i}))$. If V is lower semicontinuous, taking limits as $i \to \infty$ yields $\bar{r} \geq \liminf_{i\to\infty} V(x(t_{k_i},j_{k_i})) \geq V(z)$. If V is continuous, then let $\bar{r} = \lim V(x(t_i,j_i)) = V(\bar{z})$, and considering a subsequence $\{(t_{i_k},j_{i_k})\}_{k=1}^{\infty}$ of $\{(t_i,j_i)\}_{i=1}^{\infty}$ so that $(t_k,j_k) \preceq (t_{i_k},j_{i_k})$ and $V(x(t_k,j_k)) \geq V(x(t_{i_k},j_{i_k}))$ yields, in the limit, that $V(z) \geq \bar{r}$. Thus, if V is continuous, $V(z) = \bar{r}$. ■

Lemma 4.2: Let $V : O \to \mathbb{R}$, $u_c, u_d : O \to [-\infty,+\infty]$ be any functions, the set $\mathcal{U} \subset O$ be such that $u_c(z) \leq 0$, $u_d(z) \leq 0$ for all $z \in \mathcal{U}$ and such that for any trajectory $\xi \in \mathcal{S}$ with $\operatorname{rge}\xi \subset \mathcal{U}$,

$$V(\xi(t',j')) - V(\xi(t,j)) \leq \int_t^{t'} u_c(\xi(s,j(s)))\,ds + \sum_{i=j+1}^{j'} u_d(\xi(t(i),i-1)) \quad (2)$$

holds for any $(t,j),(t',j') \in \operatorname{dom}\xi$ such that $(t,j) \preceq (t',j')$. Let $\mathcal{M} \subset \mathcal{U}$ be a set such that $V(\mathcal{M}) = r$ for some $r \in \mathbb{R}$. If $\mathcal{M}$ is weakly forward invariant, then $\mathcal{M} \subset \overline{u_c^{-1}(0) \cup u_d^{-1}}(0)$. If $\mathcal{M}$ is weakly backward invariant, then $\mathcal{M} \subset \overline{u_c^{-1}(0)} \cup R^{(0,1)}_{u_d^{-1}(0)}$, where $R^{(0,1)}_{u_d^{-1}(0)} := \{z \in O \,|\, z = x(0,1), x \in \mathcal{S}(u_d^{-1}(0)), (0,1) \in \operatorname{dom} x\}$, that is, $R^{(0,1)}_{u_d^{-1}(0)}$ is the reachable set from $u_d^{-1}(0)$ in hybrid time $(0,1)$. If $\mathcal{M}$ is weakly invariant, then

$$\mathcal{M} \subset \overline{u_c^{-1}(0)} \cup \left(u_d^{-1}(0) \cap R^{(0,1)}_{u_d^{-1}(0)}\right).$$

Proof: For any trajectory $x \in \mathcal{S}$ such that $x(t,j) \in \mathcal{M}$ for $(t,j) \in \operatorname{dom} x$ with $(\underline{t},\underline{j}) \preceq (t,j) \preceq (\overline{t},\overline{j})$ for some $(\underline{t},\underline{j})$ and $(\overline{t},\overline{j})$ in $\operatorname{dom} x$, the fact that V is constant along trajectories in $\mathcal{M}$ gives

$$\int_{\underline{t}}^{\overline{t}} u_c(x(s,j(s)))\,ds + \sum_{i=\underline{j}+1}^{\overline{j}} u_d(x(t(i),i-1)) = 0.$$

Pick any $z \in \mathcal{M}$. If $\mathcal{M}$ is weakly forward invariant, then there exists a nontrivial $x \in \mathcal{S}(z)$ with $\operatorname{rge} x \subset \mathcal{M}$. If $(0,1) \in \operatorname{dom} x$, applying the above equation to $(\underline{t},\underline{j}) = (0,0)$, $(\overline{t},\overline{j}) = (0,1)$ yields $u_d(x(0,0)) = 0$, which shows that $z \in u_d^{-1}(0)$. If $(T,0) \in \operatorname{dom} x$ for some $T > 0$, then applying the equation to $(0,0)$, $(T,0)$ yields $\int_0^T u_c(x(s,0))\,ds = 0$. As u_c is nonpositive, it must be the case that $u_c(x(s,0)) = 0$ for almost all $s \in [0,T]$. Hence, $z \in \overline{u_c^{-1}(0)}$. If $\mathcal{M}$ is weakly backward invariant, then there exists $x \in \mathcal{S}(z^*)$, $z^* \in \mathcal{M}$, such that $x(t^*,j^*) = z$, $t^*+j^* > 1$, and $x(t,j) \in \mathcal{M}$ for all $(t,j) \preceq (t^*,j^*)$. If $(t^*,j^*-1) \in \operatorname{dom} x$, then the inequality above with $(\underline{t},\underline{j}) = (t^*,j^*-1)$, $(\overline{t},\overline{j}) = (t^*,j^*)$ shows that $u_d(x(t^*,j^*-1)) = 0$ and so $z = x(t^*,j^*) \in R^{(0,1)}_{u_d^{-1}(0)}$. If $(t^*-T,j^*) \in \operatorname{dom} x$ for some $T > 0$, then an argument similar to the one for forward invariance can be given. ■

The previous two lemmas allow us to establish the first invariance principle for hybrid trajectories.

Theorem 4.3: (V invariance principle) Suppose that there exist a continuous function $V : O \to \mathbb{R}$, a set $\mathcal{U} \subset O$, and functions $u_c, u_d : O \to [-\infty,+\infty]$ such that for any hybrid trajectory $\xi \in \mathcal{S}$ with $\operatorname{rge}\xi \subset \mathcal{U}$,

$$u_c(\xi(t,j)) \leq 0, \quad u_d(\xi(t,j)) \leq 0$$

for all $(t,j) \in \operatorname{dom}\xi$ and (2) holds for any $(t,j),(t',j') \in \operatorname{dom}\xi$ such that $(t,j) \preceq (t',j')$.

Let $x \in \mathcal{S}$ be a precompact hybrid trajectory such that

$$\overline{\{x(t,j) \mid (t,j) \in \operatorname{dom} x, (T,J) \preceq (t,j)\}} \subset \mathcal{U},$$

for some $(T,J) \in \operatorname{dom} x$, which holds when $\overline{\operatorname{rge} x} \subset \mathcal{U}$. Then, for some $r \in V(\mathcal{U})$, x approaches the largest weakly invariant subset of

$$V^{-1}(r) \;\cap\; \mathcal{U} \;\cap \left[\overline{u_c^{-1}(0)} \cup \left(u_d^{-1}(0) \cap R^{(0,1)}_{u_d^{-1}(0)}\right)\right]. \quad (3)$$

Proof: For any precompact trajectory x, from Lemma 3.3 we know that x approaches its ω-limit, which is weakly invariant. This ω-limit is the same as the ω-limit of the truncation of x to (t,j)'s with $(T,J) \preceq (t,j) \in \operatorname{dom} x$. By (2), the function V is nonincreasing along the truncation. Thus V is constant on $\Omega(x)$ by Lemma 4.1. Now note that $\Omega(x)$ is a subset of $\mathcal{U}$ intersected with $\{\operatorname{rge}\xi \mid \xi \in \mathcal{S}, \operatorname{rge}\xi \subset \mathcal{U}\}$. In turn, this intersection meets the conditions placed on the set $\mathcal{U}$ in Lemma 4.2. Thus, invoking Lemma 4.2, with $\mathcal{M}$ also replaced by $\Omega(x)$, finishes the proof. ■

Corollary 4.4: Under the assumptions of Theorem 4.3,

(a) *if x is Zeno, then, for some $r \in V(\mathcal{U})$, it approaches the largest weakly invariant subset of*

$$V^{-1}(r) \;\cap\; \mathcal{U} \;\cap\; u_d^{-1}(0) \cap R^{(0,1)}_{u_d^{-1}(0)}; \quad (4)$$

(b) *if x is s.t., for some $\gamma > 0$, $J \in \mathbb{N}$, and all $j \geq J$, $t_{j+1} - t_j \geq \gamma$ (i.e. the elapsed time between jumps is eventually bounded below by γ), then, for some $r \in V(\mathcal{U})$, x approaches the largest weakly invariant subset of*

$$V^{-1}(r) \ \cap \ \mathcal{U} \ \cap \ \overline{u_c^{-1}(0)}. \qquad (5)$$

Proof: If x is Zeno, then the weak invariance of $\Omega(x)$ can be verified by instantaneous Zeno trajectories. More specifically, given $z \in \Omega(x)$ with $x(t_i, j_i) \to z$ for some increasing and unbounded sequence of (t_i, j_i)'s, the sequence of trajectories $x_i(t,j) := x(t + t(j_i), j + j_i - 1)$ has a graphically convergent subsequence, the limit ξ of which has the domain equal to $\{0\} \times \mathbb{N}$ (see also the proof of Lemma 3.3) and is such that $\xi(0,1) = z$. Using this limit in the proof of Lemma 4.2 shows that $z \in u_d^{-1}(0) \cap R^{(0,1)}_{u_d^{-1}(0)}$.

Regarding (b), note that we can truncate x (and we won't relabel it) so that, for some $\gamma > 0$, $t_{j+1} - t_j \geq \gamma$ for all $j \geq 0$ such that $(t_j, j), (t_{j+1}, j) \in \operatorname{dom} x$. Pick $\bar{z} \in \Omega(x)$ and an increasing and unbounded sequence (t_i, j_i) with the property that $x(t_i, j_i) \to \bar{z}$. Suppose that the sequence given by $x_i(t,j) := x(t + t_i, j + j_i)$ graphically converges, say to a trajectory $\bar{x} \in \mathcal{S}$, and $\bar{x}(0,0) = \bar{z}$ (consult the proof of Lemma 3.3). If $[0, \gamma/3] \times \{0\} \subset \operatorname{dom} \bar{x}$, then using $\bar{x}$ in the proof of Lemma 4.2 shows that $\bar{z} \in \overline{u_c^{-1}(0)}$. In the opposite case, a graphically convergent subsequence can be extracted from the sequence given by $x_i'(t,j) := x(t + t_i - \gamma/3, j + j_i)$ so that its limit $\bar{x}'$ is such that $[0, \gamma/3] \times \{0\} \subset \operatorname{dom} \bar{x}'$. Furthermore $\bar{x}'(\gamma/3, 0) = \bar{z}$ and $\operatorname{rge} \bar{x}' \subset \Omega(x)$ (so $\bar{x}'$ verifies the weak backward invariance of $\Omega(x)$ at $\bar{z}$), and using $\bar{x}'$ in the proof of Lemma 4.2 shows that $\bar{z} \in \overline{u_c^{-1}(0)}$. ■

Corollary 4.4 relies on the character of the trajectories verifying the weak invariance of $\Omega(x)$, rather that on whether x jumps infinitely many times or whether x is not Zeno. The example below illustrates this, among other things.

Example 4.5: Consider the hybrid system on $O = \mathbb{R}^2$ given by $f(x) := [-x_2 \ \ x_1]^T$, $C := \mathbb{R} \times [0, \infty)$, $g(x) := [-x_2 \ \ x_1]^T$, and $D := \mathbb{R} \times (-\infty, 0]$. Any solution to this system (recall (S1) and (S2) in Section II-B) satisfies (2) with $V(x) = |x|$. Let $u_c(x) = 0$ if $x_2 \geq 0$, $u_c(x) = -\infty$ if $x_2 < 0$, and $u_d(x) = 0$ if $x_2 \leq 0$, $u_d(x) = -\infty$ if $x_2 > 0$. Functions u_c, u_d are the natural bounds on the decrease of V, see (8) and (9). For these functions, $\overline{u_c^{-1}(0)}$ is the (closed) upper half plane, $u_d^{-1}(0)$ is the (closed) lower half plane, and $R^{(0,1)}_{u_d^{-1}(0)}$ is the (closed) right half plane. For the periodic solution given by $x(t,j) = (\cos t, \sin t)$ for $t \in [0, \pi]$, $x(\pi, 1) = (0, -1)$, and $x(t,j) = x(t - \pi, j - 2)$ for $t \geq \pi$, $j \geq 2$, the ω-limit set is just $\operatorname{rge} x$: the (closed) upper half of the unit circle and $(0, -1)$. Note that the domain of this solution is unbounded in both t and j directions. For this solution, $V(x(t,j)) = 1$ for all $(t,j) \in \operatorname{dom} x$. Suppose that Corollary 4.4 were applicable. Taking $r = 1$ and $\mathcal{U} = \mathbb{R}^2$, the set (4) would be the unit circle in the closed fourth quadrant and the set (5) would be the unit circle in the (closed) upper half plane. In particular, x does not approach either of these two sets even though $\operatorname{dom} x$ is unbounded in both t and j directions, and therefore, it will not approach an invariant set included in those sets. Of course, x approaches the largest weakly invariant set contained in the union of the sets (4) and (5) (as dictated by Theorem 4.3). This set turns out to be $\operatorname{rge} x$. We note that if $R^{(0,1)}_{u_d^{-1}(0)}$ is not used in Theorem 4.3 then we must search for the largest weakly invariant subset of $V^{-1}(1) \cap \mathcal{U} \cap \left(\overline{u_c^{-1}(0)} \cup u_d^{-1}(0) \right)$. This turns out to be the unit circle, which is larger than $\operatorname{rge} x$. □

Note that the strong conclusion in the example above relies both on the strong (forward and backward) invariance notion and the set $R^{(0,1)}_{u_d^{-1}(0)}$ in (4). In contrast, the invariance principle in [6] would only conclude that the trajectory in the example converges to the unit circle.

B. Hybrid Systems

For the hybrid systems as in Section II-B, the functions $u_c(x)$ and $u_d(x)$ of Section IV-A will be constructed from a Lyapunov-like function V and will be denoted by $u_C(x)$ and $u_D(x)$, respectively. One will be determined by the "derivative" of V at x in directions belonging to $F(x)$, the other by the difference between V at x and at points belonging to $G(x)$. These functions will be used to bound the increment of V as in equation (2). We begin by formulating the infinitesimal inequality version of this. Let $V : O \to \mathbb{R}$ be continuous on O and locally Lipschitz on a neighborhood of C. Let x be any solution to the hybrid system $\mathcal{H}$, and let $(\underline{t}, \underline{j}), (\bar{t}, \bar{j}) \in \operatorname{dom} x$ be such that $(\underline{t}, \underline{j}) \preceq (\bar{t}, \bar{j})$. The increment $V(x(\bar{t}, \bar{j})) - V(x(\underline{t}, \underline{j}))$ is given by

$$V(x(\bar{t}, \bar{j})) - V(x(\underline{t}, \underline{j})) = \int_{\underline{t}}^{\bar{t}} \frac{d}{dt} V(x(t, j(t)))\, dt + \sum_{j = \underline{j}+1}^{\bar{j}} \left[V(x(t(j), j)) - V(x(t(j), j-1)) \right], \qquad (6)$$

which takes into account the "continuous increment" due to the integration of the time derivative of $V(x(t,j))$ and the "discrete increment" due to the difference in V before and after the jump. The integral above expresses the desired quantity since $t \mapsto V(x(t, j(t)))$ is locally Lipschitz and absolutely continuous on every interval on which $t \mapsto j(t)$ is constant.

The generalized gradient (in the sense of Clarke) of V at $x \in C$, denoted by $\partial V(x)$ is a closed, convex, and nonempty set equal to the convex hull of all limits of sequences $\nabla V(x_i)$ where x_i is any sequence converging to x while avoiding an arbitrary set of measure zero containing all the points at which V is not differentiable (as V is locally Lipschitz, ∇V exists almost everywhere). The (Clarke) generalized directional derivative of V at x in the direction of v can be expressed as

$$V^{\circ}(x, v) = \max_{\zeta \in \partial V(x)} \langle \zeta, v \rangle. \qquad (7)$$

One of its basic properties is that for any solution $z(\cdot)$ to $\dot{z}(t) \in F(z(t))$,

$$\frac{d}{dt} V(z(t)) \leq V^{\circ}(z(t), \dot{z}(t))$$

for almost all t. (Note that as V is locally Lipschitz, the derivative on the left above can be understood in the standard sense.) For more details see [36].

Consequently, the function $u_C : O \to [-\infty, +\infty)$ given by

$$u_C(x) := \begin{cases} \max_{v \in F(x)} \max_{\zeta \in \partial V(x)} \langle \zeta, v\rangle & x \in C \\ -\infty & \text{otherwise} \end{cases} \quad (8)$$

can be used to bound the increase of V along solutions to the hybrid system. That is, for any solution to the hybrid system, and any t where $\frac{d}{dt}V(x(t, j(t)))$ exists, we have $\frac{d}{dt}V(x(t, j(t))) \le u_C(x(t, j(t)))$.

To bound the "discrete contribution" to the change in V from (6), we will use the following quantity:

$$u_D(x) := \begin{cases} \max_{\zeta \in G(x)} \{V(\zeta) - V(x)\} & x \in D \\ -\infty & \text{otherwise.} \end{cases} \quad (9)$$

Even without any regularity on V, one gets the bound $V(x(t_{j+1}, j+1)) - V(x(t_{j+1}, j)) \le u_D(x(t_{j+1}, j))$ for any solution to the hybrid system.

Lemma 4.6: (upper semicontinuity of u_C and u_D) If V is continuous on O and locally Lipschitz on a neighborhood of C, then u_C and u_D are upper semicontinuous on O.

For the continuous evolution, better bounds on $\frac{d}{dt}V(x(t, j(t)))$ can be obtained if one does not insist on upper semicontinuity of the bounding function. We describe two such improvements. Both are based on the observation that not all vectors $v \in F(x)$ may be selected as the velocity of some solution to $\dot{x}(t) \in F(x(t))$ for some t. The second alternative we present also relies on the concept of nonpathological functions.

For any solution $z(t)$ to the differential inclusion $\dot{z}(t) \in F(z(t))$, whenever $\dot{z}(t)$ exists, we have $\dot{z}(t) \in T_C(z(t))$. Here, $T_C(x)$ is the *tangent cone* to C at $x \in C$. It is the set of all $v \in \mathbb{R}^n$ for which there exists a sequence of real numbers $\alpha_i \searrow 0$ and a sequence $v_i \to v$ such that for every $i = 1, 2, ...,$ $x + \alpha_i v_i \in C$. For further details see [37] or Chapter 6 in [34]. Hence, for any solution to the hybrid system, $\frac{d}{dt}V(x(t, j(t))) \le v_C(x(t, j(t)))$ for almost all t, where $v_C : O \to [-\infty, +\infty)$ is defined by

$$v_C(x) := \begin{cases} \max_{v \in F_T(x)} \max_{\zeta \in \partial V(x)} \langle \zeta, v\rangle & x \in C, F_T(x) \neq \emptyset \\ -\infty & \text{otherwise,} \end{cases} \quad (10)$$

where $F_T(x) := F(x) \cap T_C(x)$. Obviously, $v_C(x) \le u_C(x)$. We note though that $v_C(x) = u_C(x)$ for all $x \in \operatorname{int} C$ since for such x, $T_C(x) = \mathbb{R}^n$. Still, different values of v_C and u_C on the boundary of C may lead to different invariant sets.

The next construction relies on a concept proposed by Valadier [38]. A function $f : O \to \mathbb{R}$ is called *nonpathological* if it is locally Lipschitz and for every absolutely continuous $z : [a, b] \to O$ the set $\partial(f \circ z)$ is a subset of an affine subspace orthogonal to $\dot{z}(t)$ for almost every $t \in [a, b]$. (For recent results involving nonpathological functions see Bacciotti and Ceragioli [24].) Locally Lipschitz functions that are (Clarke) regular, semiconcave, or semiconvex are nonpathological. In particular, finite-valued convex functions are nonpathological.

When V is nonpathological on an open set containing C, for any absolutely continuous $z : [a, b] \to \mathbb{R}^n$, the set of points $\{\langle \partial V(z(t)), \dot{z}(t)\rangle\}$ reduces to the singleton $\frac{d}{dt}V(z(t))$ for almost all $t \in [a, b]$; see [38], Proposition 3. Consequently, the following function can replace u_C in the bounds on the increase of V:

$$w_C(x) := \begin{cases} \max_{v \in F_\perp(x)} \langle \partial V(x), v\rangle & x \in C, F_\perp(x) \neq \emptyset \\ -\infty & \text{otherwise,} \end{cases} \quad (11)$$

where $F_\perp(x) := \{\nu \in F_T(x) \mid \exists c \text{ s.t. } \langle \partial V(x), \nu\rangle = c\}$. Clearly, $w_C(x) \le v_C(x)$ for all $x \in C$. The condition that there exists c such that $\langle \partial V(x), v\rangle = c$ means just that $\partial V(x)$ is in an affine subspace orthogonal to v. Note that to make (11) resemble (8) more, we can replace (without really changing anything) the expression $\langle \partial V(x), v\rangle$ above by $\max_{\zeta \in \partial V(x)} \langle \zeta, v\rangle$. Note that equivalently, $\min$ can be used instead of $\max$. In any case, for any solution to the hybrid system, we have $\frac{d}{dt}V(x(t, j(t))) \le w_C(x(t, j(t))) \le v_C(x(t, j(t))) \le u_C(x(t, j(t)))$ almost everywhere.

As mentioned before, both v_C and w_C can fail to be upper semicontinuous. The reason for this is that the set-valued mapping F_T, and consequently $F_\perp$, does not need to be outer semicontinuous ($T_C(x)$ is not an outer semicontinuous map). If one defines $w'_C(x)$ similarly to $w_C(x)$, but with the maximum over v in $F'(x) := \{\nu \in F(x) \mid \exists c \text{ s.t. } \langle \partial V(x), \nu\rangle = c\}$, the function still need not be upper semicontinuous. Indeed, consider the function $V : \mathbb{R} \to \mathbb{R}_{\ge 0}$ given by $V(x) = |x|$ and the set valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(x) = [-1, 1]$ for all $x \in \mathbb{R}$. Then $F'(0) = 0$ while $F'(x) = F(x)$ for $x \neq 0$ (in general, $F'(x) = F(x)$ for all x at which $\partial V(x) = \nabla V(x)$). Thus, F' is not outer semicontinuous. Evaluating w'_C yields $w'_C(0) = 0$ while $w'_C(x) = 1$ for all $x \neq 0$. This function is not upper semicontinuous.

We now state the invariance principle for hybrid systems $\mathcal{H}$ satisfying (A0)-(A3) when a Lyapunov-like function V is provided that is locally Lipschitz and possibly nonpathological.

Theorem 4.7: (hybrid V invariance principle) Given a hybrid system $\mathcal{H}$, let $V : O \to \mathbb{R}$ be continuous on O and locally Lipschitz on a neighborhood of C. Suppose that $\mathcal{U} \subset O$ is nonempty, and that $x \in \mathcal{S}_{\mathcal{H}}$ is precompact with $\overline{\operatorname{rge} x} \subset \mathcal{U}$. If

$$u_C(z) \le 0, \quad u_D(z) \le 0$$

for all $z \in \mathcal{U}$, then for some constant $r \in V(\mathcal{U})$, x approaches the largest weakly invariant set in

$$V^{-1}(r) \;\cap\; \mathcal{U} \;\cap\; \left[\overline{u_C^{-1}(0)} \cup \left(u_D^{-1}(0) \cap G(u_D^{-1}(0))\right)\right]. \quad (12)$$

If $u_D(z) \le 0$ for all $z \in U$ and either $v_C(z) \le 0$ for all $z \in \mathcal{U}$, or V is nonpathological on a neighborhood of C and $w_C(z) \le 0$ for all $z \in \mathcal{U}$, then the conclusion holds with $u_C^{-1}(0)$ replaced by $\overline{v_C^{-1}(0)}$, respectively by $\overline{w_C^{-1}(0)}$, in equation (12).

Proof: The bound (2) holds with u_c, u_d replaced by u_C, u_D, for any trajectory x with $\operatorname{rge} x \subset \mathcal{U}$. Consequently, by Theorem 4.3, any precompact trajectory x with $\overline{\operatorname{rge} x} \subset \mathcal{U}$ approaches the largest weakly invariant set in (3) for some $r \in V(\mathcal{U})$ (with $G = g$), and here, the reachable set $R^{(0,1)}_{u_D^{-1}(0)}$ is just $G(u_D^{-1}(0))$. Since u_C is upper semicontinuous and nonpositive on $\mathcal{U}$, the set $u_C^{-1}(0)$ is closed, and the closure can be omitted. The same reasoning applies when assumptions involve v_C or w_C, however since these functions need not be upper semicontinuous, the closures are necessary. ∎

Consequences of Theorem 4.7, similar to those of Theorem 4.3 stated in Corollary 4.4, can be given. As in Theorem 4.3, Theorem 4.7 can be written for the case that, for some $(T, J) \in \operatorname{dom} x$, $\overline{\{x(t, j) \mid (t, j) \in \operatorname{dom} x, (T, J) \preceq (t, j)\}} \subset \mathcal{U}$.

C. Relation to previous results

As noted in Example 2.5, continuous-time systems parameterized by $t \in \mathbb{R}_{\geq 0}$ can be viewed as hybrid trajectories with domains in $\mathbb{R}_{\geq 0} \times \{0\}$ and that the set of all hybrid trajectories corresponding to solutions of $\dot{x}(t) = f(x(t))$ with f continuous meets the Standing Assumption. Thus, Theorem 4.7 implies the original invariance principle of LaSalle, [17, Theorem 1], by setting $F = f$, $C = O$, and $D = \emptyset$. Taking $C = O$ and $D = \emptyset$ but letting F be a set-valued map satisfying (A2) of Theorem 2.6 reduces Theorem 4.7 to the invariance principle in [20, Theorem 2.11]. Theorem 4.3 implies the principle as stated by LaSalle in [18, Chapter 2, Theorem 6.4] — the general notion of a derivative used in [18, Chapter 2, Theorem 6.4] can take the place of u_c in inequality (2); see [18, Chapter 2, Lemma 6.2] and the comment following it. Theorem 4.3 also implies [24, Proposition 3], by using the nonpathological derivative of the Lyapunov function as u_c in (2) and relying on the solutions closure property, [24, Definition 5], to satisfy our Standing Assumption.

Setting $C = \emptyset$, $G = g$ where g is a function, and $D = O$ reduces Theorem 4.7 to a discrete-time systems invariance principle [18, Theorem 6.3, Chapter 1]. Indeed, the term $G(u_D^{-1}(0))$ in (12) is irrelevant for discrete-time systems, but it is important in truly hybrid systems; see Example 4.5.

Theorems 4.3, 4.7 and their corollaries can also be used to deduce convergence of trajectories of switched systems. Let $\dot{x}(t) = f_{q(t)}(x(t))$, $q(t) \in Q := \{1, 2, \ldots, m\}$ be a switched system and $\mathcal{H}$ be a corresponding hybrid system, as in Example 2.11. Let $\mathcal{S}(\tau_D)$ be the set of all solutions to this hybrid system with dwell-time τ_D (recall Example 2.10).

Proposition 4.8: For each $q \in Q$ let $f_q : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and $V_q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function such that $\nabla V_q(x) \cdot f_q(x) \leq 0$ for all $x \in \mathbb{R}^n$. Let $\mathcal{S}^* \subset \mathcal{S}(\tau_D)$ for some $\tau_D > 0$ be such that Standing Assumption holds holds for $\mathcal{S}^*$ and $V_{q(t,j+1)}(x(t, j+1)) \leq V_{q(t,j)}(x(t, j))$ for all solutions $(x, q) \in \mathcal{S}^*$. Then each precompact solution $(x, q) \in \mathcal{S}^*$ approaches the largest subset K of $\bigcup_{q=1}^{m} \{\nabla V_q(x) \cdot f_q(x) = 0\}$ that is invariant in the following sense: for each $x^0 \in K$ there exists $\varepsilon > 0$ and: i) $q \in Q$ and a solution x to $\dot{x}(t) = f_q(x(t))$ such that $x(0) = x^0$ and $x(t) \in K$ for all t in $[0, \varepsilon)$; ii) $q \in Q$ and a solution x to $\dot{x}(t) = f_q(x(t))$ such that $x(0) = x^0$ and $x(t) \in K$ for all t in $(-\varepsilon, 0]$.

Proof: The bound (2) holds for each $(x, q) \in \mathcal{S}^*$ with $u_c(x, q) = \nabla V_q(x) \cdot f_q(x)$ and $u_d(x, q) = 0$ for all $(x, q) \in \mathbb{R}^n \times Q$. Corollary 4.4 implies that (x, q) approaches L, the largest weakly invariant (with respect to $\mathcal{S}^*$, and thus with respect to the larger set $\mathcal{S}(\tau_D)$) subset of $\bigcup_{q \in Q} \{\nabla V_q(x) \cdot f_q(x) = 0\} \times \{q\}$. Thus x approaches the projection L' of L onto $\mathbb{R}^n$. It remains to show that this projection is invariant in the sense stated in the proposition. Pick any $x^0 \in L'$ and a corresponding $(x^0, q^0) \in L$. By weak forward invariance of L, there exists a complete $(x, q) \in \mathcal{S}(\tau_D)$ with $(x(0, 0), q(0, 0)) = (x^0, q^0)$ and $(x(t, j), q(t, j)) \in L$ for all $(t, j) \in \operatorname{dom}(x, q)$. As $(x, q) \in \mathcal{S}(\tau_D)$, we either have $(t, 0) \in \operatorname{dom}(x, q)$ and $q(t, 0) = q^0$ for some $\varepsilon > 0$ and all $t \in [0, \varepsilon]$, in which case $\dot{x}(t, 0) \in f_{q^0}(x(t, 0))$ and $x(t, 0) \in L'$ for $t \in [0, \varepsilon]$, or $(0, 1) \in \operatorname{dom}(x, q)$ in which case $\dot{x}(t, 1) \in f_{q(0,1)}(x(t, 1))$ and $x(t, 1) \in L'$ for $t \in [0, \tau_D]$. Either $x(\cdot, 0)$ or $x(\cdot, 1)$, with the corresponding values of q, provide the needed (forward) solutions. Arguments involving backward invariance are similar. ■

When $V_1, V_2, \ldots, V_m$ are identical, then the condition $V_{q(t,j+1)}(x(t, j+1)) \leq V_{q(t,j)}(x(t, j))$ is trivially satisfied for any solution to the switched system. Thus, the result above implies that any solution with a positive dwell-time (i.e., an element of $\mathcal{S}(\tau_D)$ for some $\tau_D > 0$) approaches the set K; see Section V-C for further generalizations. This is essentially the invariance principle for switched systems as stated in [28, Theorem 1]; our result is stronger as the concept of invariance in Proposition 4.8 involves both forward and backward parts, and not forward or backward, as in [28]. See [26] for related results involving multiple Lyapunov functions.

Regarding hybrid systems, a result closely related to our work, in particular to Theorem 4.7, is [6, Theorem IV.1]. [6, Theorem IV.1] assumes continuous dependence of solutions on initial conditions, properties quite hard to verify by looking at the data (see [39] and [40] for some results in that direction). Theorem 4.7 of this paper relies on semicontinuous dependence, both weaker and easier to verify (recall Theorem 2.6). Another difference is the sharper notion of invariance (which includes backward invariance) used in Theorem 4.7 and the presence of the term $G(u_D^{-1}(0))$ in (12) which leads to a tighter characterization of the set to which trajectories converge; recall Example 4.5.

V. A MEAGRE-LIMSUP INVARIANCE PRINCIPLE

Below, we use the concept of a weakly meagre function. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *weakly meagre* if $\lim_{n\to\infty}(\inf_{t \in I_n} |f(t)|) = 0$ for every family $\{I_n \mid n \in \mathbb{N}\}$ of nonempty and pairwise disjoint closed intervals I_n in $\mathbb{R}_{\geq 0}$ with $\inf_{n \in \mathbb{N}} \mu(I_n) > 0$. Here, μ stands for the Lebesgue measure. Weak meagreness was used previously by Logemann et al. in [21] to formulate extensions of the Barbalat's lemma and resulting invariance principles. Following [21], f is weakly meagre if for some $\tau > 0$,

$$\lim_{M \to +\infty} \int_{M}^{M+\tau} |f(t)|\, dt = 0. \tag{13}$$

In particular, any L^1 function is weakly meagre.

A. Case of a general hybrid trajectory

Lemma 5.1: (meagre-limsup conditions) Let x be a complete hybrid trajectory such that

(*) *For each $z \in \Omega(x)$ and $\epsilon > 0$ there exist $\delta > 0$ and $T > 0$ such that, if $x(t, j) \in z + \delta\mathbb{B}$ for some $(t, j) \in \operatorname{dom} x$ then $x(t', j) \in z + \epsilon\mathbb{B}$ for all $t' \in [t - T, t + T]$ such that $(t', j) \in \operatorname{dom} x$.*

Furthermore, suppose that for some set $\mathcal{U} \subset O$ with $\operatorname{rge} x \subset \mathcal{U}$ there exist functions $\ell_c, \ell_d : \mathcal{U} \to [0, +\infty]$ that, for the hybrid trajectory x, satisfy the meagre-limsup conditions given by

(a) *if the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is unbounded then $t \mapsto \ell_c(x(t, j(t)))$ is weakly meagre,*
(b) *if the projection of $\operatorname{dom} x$ onto $\mathbb{N}$ is unbounded then for all large enough j there exists $t_j^* \in [t(j), t(j+1)]$ such that $\limsup_{j\to\infty} \ell_d(x(t_j^*, j)) = 0$.*

Then $\Omega(x) \subset E_{x,\ell_c} \cup E_{x,\ell_d}$, where E_{x,ℓ_c} and E_{x,ℓ_d} are respectively defined by

$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \liminf_{i\to\infty} \ell_c(z_i) = 0\}$,
$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \liminf_{i\to\infty} \ell_d(z_i) = 0\}$.

Proof: Suppose otherwise, that for some $x^* \in \Omega(x)$ and $\epsilon, \gamma > 0$, $\ell(z) := \min\{\ell_c(z), \ell_d(z)\} \geq \gamma$ for all $z \in x^* + \epsilon\mathbb{B}$, $z \in \operatorname{rge} x$. By definition of ω-limit point, there is an increasing and unbounded sequence $(t_i, j_i) \in \operatorname{dom} x$ with $x(t_i, j_i) \to x^*$ as $i \to \infty$. We can assume that for all i, $t_i + j_i + 1 \leq t_{i+1} + j_{i+1}$. Let $\delta, T > 0$ be related to x^*, ϵ as in condition (*) and, without loss of generality, suppose that $T < 1$. Ignoring initial terms if necessary, we have $x(t_i, j_i) \in x^* + \delta\mathbb{B}$ for all $i \in \mathbb{N}$. Consequently, $x(t, j_i) \in x^* + \delta\mathbb{B}$ for all $t \in [t_i - T, t_i + T]$, $(t, j_i) \in \operatorname{dom} x$. For each i, either of the two conditions holds:

(1') either $t(j_i) \leq t_i - T$ or $t(j_i + 1) \geq t_i + T$ (x flows for time T either before t_i or after t_i)
(2') $t(j_i) > t_i - T$ and $t(j_i + 1) < t_i + T$ (x jumps within time T before and after t_i)

Either (1') or (2') has to occur for infinitely many i's. Suppose it is (1') and that $t(j_i) \leq t_i - T$ for such i's (the other case is treated similarly). Then, $\operatorname{dom} x$ must be unbounded in the t-direction. The fact that $\ell(x(t, j(t))) > \gamma$ for any $t \in [t_i - T, t_i]$ for infinitely many i's contradicts weak meagreness of $t \mapsto \ell_c(x(t, j(t)))$ (note that intervals $[t_i - T, t_i]$ are disjoint). If (2') holds for infinitely many i's, then $\operatorname{dom} x$ is unbounded in the j-direction, and for infinitely many i's and all $t \in [t(j_i), t(j_i + 1)]$ we have $\ell_d(x(t, j_i)) > \gamma$. This contradicts (b). ■

The condition (*) in Lemma 5.1 can be viewed as a sort of continuity of $x(t, j)$ in t, uniform "near each point of $\Omega(x)$". The condition automatically holds if x is a solution to a hybrid system that satisfies (S1) and (S2) in Example II-B and subject to (A0)-(A3). In fact, since F is locally bounded, $x(t, j)$ is Lipschitz in t, locally "near each point of $\Omega(x)$". Also, (*) holds if x is precompact, and $\mathcal{S}$ is any family satisfying our standing assumption. Indeed, suppose that in such a case, for some $z \in \Omega(x)$ and $\epsilon > 0$ there exist increasing and unbounded sequences $(t_i, j_i), (t_i', j_i) \in \operatorname{dom} x$ with $x(t_i, j_i) \to z$, $t_i' - t_i \to 0$, and $x(t_i', j_i) \notin z + \epsilon\mathbb{B}$. By passing to a subsequence, we can suppose that $t_i' \geq t_i$ (the opposite case is treated similarly). Since x is precompact, the sequence of trajectories $x_i(t, j) := x(t + t_i, j + j_i)$ is locally eventually bounded. Let $\xi \in \mathcal{S}$ be the graphical limit of x_i's. Then $\xi(0, 0)$ contains both z and some point w with $|z - w| = \epsilon$. This is impossible.

In Lemma 5.1, $E_{x,\ell_c} \subset \{z \in \overline{\operatorname{rge} x} \mid \underline{\ell_c}(z) = 0\}$, where $\underline{\ell_c}$ is the lower semicontinuous closure of ℓ_c. (Given a set $\mathcal{U}$ and a function $\ell : \mathcal{U} \to [-\infty, +\infty]$, its *lower semicontinuous closure* $\underline{\ell} : \overline{\mathcal{U}} \to [-\infty, +\infty]$, is the greatest lower semicontinuous function defined on $\overline{\mathcal{U}}$, bounded above by ℓ on $\mathcal{U}$. Equivalently, for any $x \in \overline{\mathcal{U}}$, $\underline{\ell}(x) = \liminf_{x_i \to x} \ell(x_i)$. In this terminology, $E_{x,\ell}$ is the zero-level set of the lower semicontinuous closure of the function ℓ truncated to $\operatorname{rge} x$.) In particular, if both ℓ_c and ℓ_d are lower semicontinuous, and $\overline{\operatorname{rge} x} \subset \mathcal{U}$, then the conclusion of Lemma 5.1 implies that $\Omega(x)$ is a subset of

$$\{z \in \overline{\operatorname{rge} x} \mid \ell_c(z) = 0\} \cup \{z \in \overline{\operatorname{rge} x} \mid \ell_d(z) = 0\}.$$

However, if the assumption that ℓ_c, ℓ_d are nonnegative was weakened to say that they are nonnegative only on $\operatorname{rge} x$, the last conclusion above may fail.

Let x be a precompact hybrid trajectory for which there exist functions $u_c, u_d : O \to [-\infty, 0]$ and $V : O \to \mathbb{R}$ such that (2) holds for the hybrid trajectory x for all $(t, j), (t', j') \in \operatorname{dom} x$ such that $(t, j) \preceq (t', j')$. Then $\ell_c = -u_c$, $\ell_d = -u_d$ satisfy conditions (a) and (b) of Theorem 5.1. In fact, there exists a constant $M > 0$ for which

$$\int_0^T \ell_c(x(t, j(t)))\, dt < M, \quad \sum_{j=0}^{J} \ell_d(x(t(j+1), j)) < M,$$

for any $(T, J) \in \operatorname{dom} x$ (this shows that $\ell_c(t, j(t))$ is integrable on $[0, \infty)$ and thus weakly meagre, while to satisfy (b), one can take $t_j^* = t(j+1)$).

Based on the previous discussion, the next result shows that, when a function V with the right properties exists, the conditions (a) and (b) of Lemma 5.1 are guaranteed.

Corollary 5.2: Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Suppose that there exists a continuous function $V : O \to \mathbb{R}$, and functions $u_c, u_d : O \to [-\infty, +\infty]$ such that for some $(T, J) \in \operatorname{dom} x$,

$$u_c(x(t, j)) \leq 0, \quad u_d(x(t, j)) \leq 0$$

for all $(t, j) \in \operatorname{dom} x$ with $(T, J) \preceq (t, j)$, and (2) holds for the hybrid trajectory x for all $(t, j), (t', j') \in \operatorname{dom} x$ such that $(T, J) \preceq (t, j) \preceq (t', j')$. Then $\Omega(x) \subset E^{x,u_c} \cup E^{x,u_d}$, where E^{x,u_c} and E^{x,u_d} are respectively defined by

$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \limsup_{i\to\infty} u_c(z_i) = 0\}$,
$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \limsup_{i\to\infty} u_d(z_i) = 0\}$.

More precise results can be obtained if the domain of the hybrid trajectory is bounded in one of the directions.

Corollary 5.3: Let x be a complete hybrid trajectory for which (*) holds.

(a) *If the projection of $\operatorname{dom} x$ onto $\mathbb{N}$ is bounded and there exists a function $\ell_c : \operatorname{rge} x \to [0, +\infty]$ such that $t \mapsto \ell_c(x(t, j(t)))$ is weakly meagre, then $\Omega(x) \subset E_{x,\ell_c}$.*
(b) *If the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded and there exists a function $\ell_d : \operatorname{rge} x \to [0, +\infty]$ such that, for all large enough j, there exists $t_j^* \in [t(j), t(j+1)]$ such that $\limsup_{j\to\infty} \ell_d(x(t_j^*, j)) = 0$, then $\Omega(x) \subset E_{x,\ell_d}$.*

Proof: For (a) use $\ell_d(z) = r > 0$ for all z in the Theorem above, for (b) use $\ell_c(z) = r > 0$. ■

If, for a hybrid trajectory, the time between jumps is uniformly positive then only (a) of the meagre-limsup conditions needs to be checked to draw the conclusion of Lemma 5.1.

Corollary 5.4: Let x be a complete hybrid trajectory such that (*) holds and $t_{j+1} - t_j \geq \gamma > 0$ for all $j = 1, 2, \ldots$. If there exists a function $\ell_c : \operatorname{rge} x \to [0, +\infty]$ such that *condition (a) of the meagre-limsup conditions holds, then* $\Omega(x) \subset E_{x,\ell_c}$.

Proof: In the proof of Lemma 5.1, T can be chosen arbitrarily small. Picking $T < \frac{\gamma}{2}$ shows that (2') in the proof of Lemma 5.1 cannot hold; hence (1') has to hold for infinitely many times. The proof follows that of Lemma 5.1. ■

If multiple instantaneous jumps can occur "only on the zero level set of ℓ_d" (for a hybrid system $\mathcal{H}$, this is equivalent to $\ell_d(G(D) \cap D) = 0$) and $x \in \mathcal{S}$ is precompact, then only (a) of the meagre-limsup conditions needs to be checked to draw the conclusion of Lemma 5.1. This is because under such assumption on the jumps, on each compact set away from the zero level set of ℓ_d, the elapsed time between jumps is uniformly bounded below by a positive constant.

Corollary 5.5: Given the function $\ell_d : O \mapsto \mathbb{R}_{\geq 0}$, assume that for all $\tilde{x} \in \mathcal{S}$, if $(t, j-1), (t, j), (t, j+1) \in \operatorname{dom} \tilde{x}$, then $\ell_d(\tilde{x}(t, j)) = 0$. Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Suppose that there exists a function $\ell_c : \operatorname{rge} x \to [0, +\infty]$ such that condition (a) of the meagre-limsup conditions holds. Then the conclusion of Lemma 5.1 is true.

Proof: The first paragraph of the proof of Lemma 5.1 can be repeated. Then, we claim that there exists $\tau \in (0, T]$ such that for all large enough i's, the following holds:

(1') either $t(j_i) \leq t_i - \tau$ or $t(j_i + 1) \geq t_i + \tau$ (x flows for time τ either before t_i or after t_i)

Otherwise, for some sequence of $\tau_k \searrow 0$ and a subsequence t_{i_k} there is a jump at $\bar{t}_-(k) \in [t_{i_k} - \tau_k, t_{i_k}]$ and at $\bar{t}_+(k) \in [t_{i_k}, t_{i_k} + \tau_k]$, so that $(\bar{t}_-(k), j_{i_k} - 1)$, (t_{i_k}, j_{i_k}), and $(\bar{t}_+(k), j_{i_k} + 1)$ are all in $\operatorname{dom} x$. Now, consider a sequence of trajectories given by $x_k(t, j) = x(t + \bar{t}_-(k), j + j_{i_k} - 1)$, and pick a graphically convergent subsequence. For the limit $\bar{x}$ we must have that $(0,0)$, $(0,1)$, and $(0,2)$ are in $\operatorname{dom} \bar{x}$, while $\bar{x}(0,1) = x^*$. This contradicts the assumption. Now, as (1') has to occur infinitely many times, the proof can be completed as for Lemma 5.1. ■

Based on the results stated so far in this section, various invariance principles can be stated.

Corollary 5.6: (meagre-limsup invariance principle) Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Suppose that for $\mathcal{U} \subset O$, $\operatorname{rge} x \subset \mathcal{U}$, there exist functions $\ell_c, \ell_d : \mathcal{U} \to [0, +\infty]$ for which the meagre-limsup conditions hold. Then x converges to the largest weakly invariant subset of

$$\{z \in \overline{\mathcal{U}} \mid \underline{\ell_c}(z) = 0\} \cup \{z \in \overline{\mathcal{U}} \mid \underline{\ell_d}(z) = 0\}.$$

If $\overline{\operatorname{rge} x} \subset \mathcal{U}$ and ℓ_c, ℓ_d are lower semicontinuous, then all the closure operations above can be removed.

The difference between Theorem 4.3 and Corollary 5.6 is that, in the latter, properties of ℓ_c, ℓ_d are only relevant on the range of the hybrid trajectory x in question. In the former, we require properties of u_c, u_d, and V to hold for other trajectories than the one in question (in particular, for the trajectories verifying forward invariance of $\Omega(x)$). One may ask whether the conclusions of Corollary 5.6 can be strengthened if assumptions were made on all trajectories; i.e., whether the following is true:

Suppose that there exist functions $\ell_c, \ell_d : O \to [0, +\infty]$ such that, for all $x \in \mathcal{S}$, conditions (a) and (b) of Lemma 5.1 hold. Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Then x converges to the largest weakly invariant subset of $\overline{\ell_c^{-1}(0) \cup \ell_d^{-1}(0)}$.

This turns out to be impossible. Such a conclusion is not a byproduct of the trajectories considered here being hybrid; rather, it is caused by ℓ not being lower semicontinuous. We illustrate this with an example in continuous time.

Example 5.7: Consider the nonlinear system given by $\dot{x}_1 = \phi(x)\left(x_2 - x_1(|x| - 1)\right)$, $\dot{x}_2 = \phi(x)\left(-x_1 - x_2(|x| - 1)\right)$, where $x := [x_1\ x_2]^T \in \mathbb{R}^2$, $\phi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is any smooth function such that $\phi(0,1) = 0$, $\phi(x) > 0$ when $x \neq (0,1)$. Except for the trajectory $x(t) = (0,0)$ for all $t \geq 0$, the trajectories with initial points not on the unit circle rotate and get closer to the unit circle (while "slowing down" in the neighborhood of $(0,1)$). In particular, their omega limit set is the unit circle. The trajectories originating on the unit circle converge to $(0,1)$ (and so their omega limit set is $(0,1)$). Let $\ell : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be given by $\ell(x) = (|x| - 1)^2$ when $|x| \neq 1$ and $x \neq 0$, $\ell(x) = 0$ when $|x| = 1$ and $x_2 \geq 0$ or $x = 0$, $\ell(x) = 1$ when $|x| = 1$ and $x_2 < 0$. One can verify that for all trajectories of the system, $\ell(x(t))$ is weakly meagre. However, it is not true that the omega limit of any nonzero trajectory originating not on the unit circle (such omega limit is the unit circle) is in the closure of the zero level set of ℓ (which is the union of the upper unit semicircle and the origin). □

B. Case of a solution to a hybrid system

For hybrid systems, the natural counterparts of ℓ_c, ℓ_d, that is the functions u_C and u_D, as defined by (8), (9), are upper semicontinuous. This does not lead to significant improvements over the results in the previous subsection.

Corollary 5.8: Given a hybrid system $\mathcal{H}$, let $V : O \to \mathbb{R}$ be continuous on O and locally Lipschitz on a neighborhood of C. Suppose that $\mathcal{U} \subset O$ is nonempty and x is a precompact solution to $\mathcal{H}$ with $\operatorname{rge} x \subset \mathcal{U}$. If

$$u_C(z) \leq 0, \quad u_D(z) \leq 0$$

for all $z \in \mathcal{U}$, then for some constant $r \in V(\overline{\mathcal{U}})$, x approaches the largest weakly invariant set in

$$V^{-1}(r) \;\cap\; \overline{\mathcal{U}} \;\cap\; \left(E^{x,u_C} \cup E^{x,u_D}\right). \tag{14}$$

If $u_D(x) \leq 0$ for all $z \in \mathcal{U}$ and either $v_C(z) \leq 0$ for all $z \in \mathcal{U}$ or V is nonpathological on a neighborhood of $C \cap \mathcal{U}$ and $w_C(z) \leq 0$ for all $z \in \mathcal{U}$, then E^{x,u_C} in (14) can be replaced by E^{x,v_C} (respectively, E^{x,w_C}), defined analogously, with v_C (respectively, w_C) replacing u_C.

If, in Corollary 5.8, we have $\overline{\operatorname{rge} x} \subset \mathcal{U}$, then E^{x,u_C} can be replaced by $\{z \in \overline{\operatorname{rge} x} \mid u_C(z) = 0\}$, based only on upper semicontinuity arguments; similarly for E^{x,u_D}. The resulting conclusion for locally Lipschitz V (about the invariant set approached by x) is the same as that of Corollary 4.7. Furthermore, if $\overline{\operatorname{rge} x} \subset \mathcal{U}$ then $E^{x,v_C} \subset \{z \in \mathcal{U} \mid \overline{v_C}(z) = 0\}$, and, if V is nonpathological, $E^{x,w_C} \subset \{z \in \mathcal{U} \mid \overline{w_C}(z) = 0\}$ (here, $\overline{v_C}$ ($\overline{w_C}$) is the upper semicontinuous closure of v_C (respectively, w_C)). The resulting conclusion is weaker than Corollary 4.7, where $\overline{v_C^{-1}(0)}$ (and $\overline{w_C^{-1}(0)}$) appears. This shows that relying on properties of V along all trajectories in $\mathcal{U}$, rather than just along x, leads to stronger results when continuity or semicontinuity (of v_C and w_C) can not be used. We add that if u_C is continuous on C, then E^{x,u_C} reduces to $\overline{\mathrm{rge}\, x} \cap u_C^{-1}(0)$; similarly for E^{x,u_D}, E^{x,v_D}, and E^{x,w_C}.

A similar result to Corollary 4.4 can be written when a single trajectory is considered.

Corollary 5.9: Let the assumptions of Corollary 5.8 hold.

(a) *If x is Zeno, then, for some constant $r \in V(\mathcal{U})$, it approaches the largest weakly invariant set in*

$$V^{-1}(r) \ \cap \ \overline{\mathcal{U}} \ \cap \ E^{x,u_D}.$$

(b) *If x is such that, for some $\gamma > 0$, $J \in \mathbb{N}$, and all $j \geq J$, $t_{j+1} - t_j \geq \gamma$ (i.e. the elapsed time between jumps is eventually bounded below by a positive γ), then, for some $r \in V(\mathcal{U})$, x approaches the largest weakly invariant subset of*

$$V^{-1}(r) \ \cap \ \overline{\mathcal{U}} \ \cap \ E^{x,u_C}.$$

Proof: Part (a) is just a restatement of (b) in Corollary 5.3. Part (b) follows from Corollary 5.4. ■

Of course, as in Theorem 4.3, Corollary 5.8 can be written for the case that x stays in $\mathcal{U}$ after some $(T, J) \in \mathrm{dom}\, x$.

C. Relation to previous results

A reduction of the results of this section to continuous-time systems, much like what we noted in Section IV-C, is also possible. Lemma 5.1 implies [20, Theorem 2.10] (which in turn implies the result of [19]) because condition (*) of Lemma 5.1 is satisfied for solutions of differential inclusions discussed in [20] and the set E_{x,ℓ_c} is exactly $\{z \in \overline{\mathrm{rge}\, x} \mid \ell_c(z) = 0\}$ when ℓ_c is lower semicontinuous, as assumed in [20, Theorem 2.10]. Furthermore, results of this section can also be applied to switched systems. For example, via Corollary 5.4 and a simple trick of building a solution with dwell-time $\tau_D > 0$ from what [26] calls a p-dwell solution with parameters $\tau_D > 0, T > 0$, we can recover results like [26, Theorem 4 and 8].

VI. Locating weakly invariant sets using observability, or stability and detectability

Now we extend results on stability and convergence, and the implications of observability and detectability, from differential equations to sets of hybrid trajectories $\mathcal{S}$.

A. Observability

Definition 6.1 (observability): Given sets $\mathcal{A}, K \subset O$, the distance to $\mathcal{A}$ is *observable relative to* K for the set of trajectories $\mathcal{S}$ if for every nontrivial trajectory $x \in \mathcal{S}$ such that $\mathrm{rge}\, x \subset K$ we have $|x(t,j)|_{\mathcal{A}} = 0$ for all $(t,j) \in \mathrm{dom}\, x$.

Classically, (zero-state) observability means that if the output of a system is zero, the state is identically zero. If, for a certain (output) function $h : O \to \mathbb{R}^k$, $K = h^{-1}(0)$, we say that the distance to $\mathcal{A}$ is observable through (the output) h.

Basic properties based on observability are stated below, under the assumption that $\mathcal{A}$ and K are compact subsets of O and the distance to $\mathcal{A}$ is observable relative to K for the sets of hybrid trajectories $\mathcal{S}$:

(a) the largest weakly invariant set in K is a subset of $\mathcal{A}$;
(b) if $\omega : O \to \mathbb{R}_{\geq 0}$ is a continuous and positive definite function with respect to K, $x \in \mathcal{S}$ is precompact, and the meagre-limsup conditions hold for x with ℓ_c, ℓ_d replaced by ω, then x converges to $\mathcal{A}$.

B. Relative stability and detectability

In differential equations, detectability is the property that when the output is held to zero, complete solutions x satisfy $\lim_{t\to\infty} |x(t)|_{\mathcal{A}} = 0$. Below, we generalize this notion.

Definition 6.2 (detectability): Given sets $\mathcal{A}, K \subset O$, the distance to $\mathcal{A}$ is *detectable relative to* K for the set of trajectories $\mathcal{S}$ if for every complete trajectory $x \in \mathcal{S}$ such that $\mathrm{rge}\, x \subset K$ we have $\liminf_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

As discussed in [41], this detectability condition can be understood as x having an ω-limit point at $\mathcal{A}$. As for observability, if $K = h^{-1}(0)$ for some function $h : O \to \mathbb{R}^k$, then we say that the distance to $\mathcal{A}$ is detectable through h.

Definition 6.3 (relative stability): Given sets $\mathcal{A}, K \subset O$, $\mathcal{A}$ is *stable relative to* K for the set of trajectories $\mathcal{S}$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that any trajectory $x \in \mathcal{S}(x^0)$ with $\mathrm{rge}\, x \subset K$ and $x^0 \in \mathcal{A} + \delta\mathbb{B}$ is such that $\mathrm{rge}\, x \subset \mathcal{A} + \epsilon\mathbb{B}$.

Stability of $\mathcal{A}$ is the same as stability relative to O. When detectability (as in Definition 6.2) is combined with relative stability, the usual detectability is recovered.

Lemma 6.4: (detectability and relative stability) Let $\mathcal{A}, K \subset O$ be compact. If the distance to $\mathcal{A}$ is detectable relative to K and $\mathcal{A}$ is stable relative to K, then each complete trajectory $x \in \mathcal{S}$ with $\mathrm{rge}\, x \subset K$ converges to $\mathcal{A}$.

Example 6.5: For $x \in \mathbb{R}^n$, $A_1, A_2 \in \mathbb{R}^{n\times n}$, and closed $C, D \subset \mathbb{R}^n$, consider the hybrid system $\mathcal{H}$ given by

$$\dot{x} = A_1 x \text{ when } x \in C, \quad x^+ = A_2 x \text{ when } x \in D.$$

For simplicity, assume that $C \cup D = \mathbb{R}^n$. The motivation for this type of systems comes from many applications, like sample-data control, reset systems, etc. Suppose that:

(∘) Let $\tilde{C} \in \mathbb{R}^{m\times n}$ be such that there exists matrices L_1, L_2, and $P = P^T > 0$ that satisfy

$$x^T \left(\left(A_1 + L_1\tilde{C}\right)^T P + P^T \left(A_1 + L_1\tilde{C}\right) \right) x < 0,$$

$$x^T \left(\left(A_2 + L_2\tilde{C}\right)^T P \left(A_2 + L_2\tilde{C}\right) - P \right) x < 0,$$

where the first inequality is for all $x \in C \setminus \{0\}$ and the second one for all $x \in D \setminus \{0\}$.

This assumption holds in particular when the pairs $(\tilde{C}, A_1)$ and $(\tilde{C}, A_2)$ are detectable (in the linear sense) and the detectability of both pairs can be verified with a common Lyapunov function (which is quadratic and given by P).

Let $\mathcal{S}_{\mathcal{H}}$ be the set of solutions to $\mathcal{H}$, K any subset of $\left\{ z \in \mathbb{R}^n \;\middle|\; \tilde{C}z = 0 \right\}$, and $\mathcal{A} = \{0\} \subset \mathbb{R}^n$. By definition of

K, trajectories that remain in K are also trajectories of the output injected hybrid system $\mathcal{H}_O$ defined as

$$\dot{x} = (A_1 + L_1\tilde{C})x \quad x \in C, \; x^+ = (A_2 + L_2\tilde{C})x \quad x \in D.$$

Stability of $\mathcal{A}$ (for the system above, and hence for $\mathcal{H}$ relative to K) can be easily verified with the use of the common quadratic Lyapunov function $V(x) = x^T P x$. Moreover, by Corollary 4.7 with $\mathcal{U} = \mathbb{R}^n$ and $V(x)$, every trajectory that stays in K converges to $\mathcal{A}$. Hence, the distance to $\mathcal{A}$ is detectable relative to K for the set of hybrid trajectories $\mathcal{S}_{\mathcal{H}}$. We point out though that ($\circ$) is not a necessary condition for detectability of $\mathcal{H}$ relative to K, it is only sufficient.

Note that for LTI systems the concepts of relative stability and detectability introduced above reduces to the standard one in the literature. For instance, for the continuous-time LTI system $\dot{x} = Ax$ with output $y = \tilde{C}x$, detectability of the pair $(\tilde{C}, A)$ is equivalent to the distance to $\mathcal{A} := \{0\}$ being detectable relative to subsets of $\left\{ z \in \mathbb{R}^n \;\middle|\; \tilde{C}z = 0 \right\}$. □

Theorem 6.6: (detectability and invariance principle) Let $\mathcal{A}, K \subset O$ be compact, and suppose that $\mathcal{A}$ is stable relative to K for the set of trajectories $\mathcal{S}$. Then the following statements are equivalent:

1) *The distance to $\mathcal{A}$ is detectable relative to K.*
2) *The largest weakly invariant set in K is a subset of $\mathcal{A}$.*

Proof: ($1 \Rightarrow 2$) Let $\mathcal{M}$ be the largest weakly invariant set in K. Suppose that there exists $z \in \mathcal{M} \setminus \mathcal{A}$. Let $\epsilon = |z|_{\mathcal{A}}$. By stability of $\mathcal{A}$ relative to K, there exists $\delta > 0$ such that every hybrid trajectory $\xi \in \mathcal{S}$ with $\operatorname{rge} \xi \subset K$ and $\xi(0,0) \in \mathcal{A} + \delta\mathbb{B}$ satisfies $\operatorname{rge} \xi \subset \mathcal{A} + \frac{\epsilon}{2}\mathbb{B}$. By weak backward invariance of $\mathcal{M}$, there exists a trajectory $x_1 \in \mathcal{S}$ such that for some $(t_1, j_1) \in \operatorname{dom} x_1$, $t_1 + j_1 \geq 1$, $x_1(t_1, j_1) = z$ and $x_1(t,j) \in \mathcal{M}$ for all $(t,j) \preceq (t_1, j_1)$, $(t,j) \in \operatorname{dom} x_1$ (in particular $x_1(0,0) \in \mathcal{M}$). Note that by stability, since $x_1(t_1, j_1) \notin \mathcal{A} + \frac{\epsilon}{2}\mathbb{B}$, we have $x_1(t,j) \in \mathcal{M} \setminus (\mathcal{A} + \delta\mathbb{B})$ for all $(t,j) \in \operatorname{dom} x_1$, $(t,j) \preceq (t_1, j_1)$. In this way, we can construct a sequence $x_i \in \mathcal{S}$ such that for every $i > 0$, there exists $(t_i, j_i) \in \operatorname{dom} x_i$, $t_i + j_i \geq i$ with $x_i(t_i, j_i) = z$ and $x_i(t,j) \in \mathcal{M} \setminus (\mathcal{A} + \delta\mathbb{B})$ for all $(t,j) \preceq (t_i, j_i)$, $(t,j) \in \operatorname{dom} x_i$. As K is compact, the sequence $\{x_i\}_{i=1}^{\infty}$ is locally eventually bounded. By the Standing Assumption, it has a subsequence (that we won't relabel) converging to some $x \in \mathcal{S}$, with $x_i(0,0) \to x(0,0) \in \mathcal{M}$. Since $\operatorname{dom} x_i$ are "increasing", x is complete; see [11, Lemma 3.5]. Finally, $\operatorname{rge} x \subset \mathcal{M} \setminus (\mathcal{A} + \delta\mathbb{B})$, and also $\operatorname{rge} x \subset K$. The second inclusion, by detectability of $\mathcal{A}$ relative to K, relative stability of $\mathcal{A}$, and Lemma 6.4, implies that x converges to $\mathcal{A}$. This is a contradiction with the first inclusion.

($2 \Rightarrow 1$) Any trajectory $x \in \mathcal{S}$ with $\operatorname{rge} x \subset K$ is precompact, by compactness of K, and as such, it converges to its ω-limit. Since the ω-limit is invariant and a subset of K, it must be a subset of $\mathcal{A}$. Hence, x converges to $\mathcal{A}$. ■

Corollary 6.7: Let $\mathcal{A}$, K be compact subsets of O, with $\mathcal{A}$ stable relative to K and with the distance to $\mathcal{A}$ detectable on K, and let $\omega : O \to \mathbb{R}_{\geq 0}$ be a continuous and positive definite function with respect to K. If $x \in \mathcal{S}$ is precompact and the meagre-limsup conditions hold for x with ℓ_c, ℓ_d replaced by ω, then x converges to $\mathcal{A}$.

C. Uniform Convergence

Stability and detectability of the distance to a compact set $\mathcal{A}$ relative to a compact set K leads to uniform convergence.

Theorem 6.8: (uniform convergence) Let $\mathcal{A}, K \subset O$ be compact. Suppose that $\mathcal{A}$ is stable relative to K and the distance to $\mathcal{A}$ is detectable relative to K. Then, for each $\epsilon > 0$, there exists $M > 0$ such that for each complete trajectory $x \in \mathcal{S}$ with $\operatorname{rge} x \subset K$ we have $|x(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t,j) \in \operatorname{dom} x$, $t + j \geq M$.

Proof: Suppose otherwise. Then, for some $\epsilon > 0$, there exist a sequence of complete trajectories $x_i \in \mathcal{S}$ such that $\operatorname{rge} x_i \subset K$ and a sequence $(t_i, j_i) \in \operatorname{dom} x_i$ with $t_i + j_i \geq i$ such that $|x_i(t_i, j_i)|_{\mathcal{A}} > \epsilon$. By relative stability of $\mathcal{A}$, there exists $\delta > 0$ such that for each $i = 1, 2, \ldots$, $|x_i(t,j)|_{\mathcal{A}} > \delta$ for all $t + j \leq i$, $(t,j) \in \operatorname{dom} x_i$. Since K is compact, the sequence $\{x_i\}_{i=1}^{\infty}$ is locally eventually bounded, and, by the Standing Assumption, it has a graphically convergent subsequence. Its limit, let us call it x, is complete (since each x_i is complete; see [11, Lemma 3.5]) and such that $\operatorname{rge} x \subset K$. Furthermore, for all $(t,j) \in \operatorname{dom} x$, $|x(t,j)|_{\mathcal{A}} \geq \delta$. This contradicts the detectability assumption. ■

VII. Asymptotic Stability

A. Definitions and a $\mathcal{KLL}$-characterization

For results on uniform convergence without a priori restriction of the trajectories to a compact set, we need an additional condition. Besides the Standing Assumption, from now on we also suppose the following:

(B4) any sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid trajectories in $\mathcal{S}$ for which initial points $x_i(0,0)$ converge to a point x^0 where every maximal solution $x \in \mathcal{S}(x^0)$ is complete, is locally eventually bounded.

For solutions to hybrid systems, this property requires local boundedness of G. With the other growth properties of G and the fact that G maps to O, its local boundedness is equivalent to *local boundedness with respect to O*: for any compact $K \subset O$ there exists a compact $K' \subset O$ such that $G(K) \subset K'$.

Theorem 7.1: ([11], Theorem 4.6) If the hybrid system $\mathcal{H}$ with state space O satisfies (A0)-(A3) and $G : O \rightrightarrows O$ is locally bounded, then $\mathcal{S}_{\mathcal{H}}$ satisfies (B4).

Definition 7.2 (attractivity): A set $\mathcal{A} \subset O$ is *attractive* for the set of trajectories $\mathcal{S}$ if there exists $\rho > 0$ such that for any $x^0 \in \mathcal{A} + \rho\mathbb{B}$, each maximal trajectory $x \in \mathcal{S}(x^0)$ is complete and satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

We denote by $\mathcal{B}_{\mathcal{A}}$ the basin of attraction of a compact set $\mathcal{A}$, i.e. the set of all points x^0 for which $\mathcal{S}(x^0)$ is nonempty, each $x \in \mathcal{S}(x^0)$ is complete and such that $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. The set $\mathcal{A}$ is said to be *asymptotically stable* if it is both stable and attractive. For the basin of attraction of an asymptotically stable set to be forward invariant, another assumption needs to be placed on $\mathcal{S}$:

(B5) For any $x_1 \in \mathcal{S}$, any $(T,J) \in \operatorname{dom} x_1$, and any $x_2 \in \mathcal{S}(x_1(T,J))$, the hybrid trajectory x defined on

$$\operatorname{dom} x := \{(t,j) \in \operatorname{dom} x_1 \mid (t,j) \preceq (T,J)\} \cup \{(t+T, j+J) \mid (t,j) \in \operatorname{dom} x_2\}$$

is given by $x(t,j) = x_1(t,j)$ for $(t,j) \in \operatorname{dom} x_1$, $(t,j) \preceq (T,J)$, and $x(t,j) = x_2(t-T, j-J)$ for (t,j) such that $(t-T, j-J) \in \operatorname{dom} x_2$ is an element of $\mathcal{S}$.

The assumption means that a concatenation of two solutions is still a solution. (Recall that assumption (B2) required that tails of solutions be solutions.) It automatically holds for the hybrid system $\mathcal{H}$ as in Section II-B, and here, it guarantees that if $x \in \mathcal{S}(\mathcal{B}_{\mathcal{A}})$ then for any $(t,j) \in \operatorname{dom} x$ we have $x(t,j) \in \mathcal{B}_{\mathcal{A}}$.

Given an open set $X \subset O$ and a compact set $\mathcal{A} \subset X$, a proper indicator $\omega : X \to \mathbb{R}_{\geq 0}$ for $\mathcal{A}$ on X is a continuous function that is positive definite with respect to $\mathcal{A}$ and proper with respect to X. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KLL}$ if it is continuous, $\beta(\cdot, t, j)$ is zero at zero and nondecreasing, $\beta(s, \cdot, j)$ and $\beta(s, t, \cdot)$ are nonincreasing and converge to zero as t, respectively, j go to ∞. We say that the set of hybrid trajectories $\mathcal{S}$ is forward complete on K if for all $x^0 \in K$, every $x \in \mathcal{S}(x^0)$ is complete.

Definition 7.3: ($\mathcal{KLL}$ stability) Let $\omega : K \to \mathbb{R}_{\geq 0}$ be continuous. The set of hybrid trajectories $\mathcal{S}$ is said to be *$\mathcal{KLL}$-stable with respect to* ω if it is forward complete on K and there exists $\beta \in \mathcal{KLL}$ such that, for each $x^0 \in K$, every $x \in \mathcal{S}(x^0)$ satisfies

$$\omega(x(t,j)) \leq \beta(\omega(x^0), t, j) \quad \text{for each } (t,j) \in \operatorname{dom} x.$$

Theorem 7.4: (asymptotic stability implies $\mathcal{KLL}$ stab.) Suppose that, for the set of trajectories $\mathcal{S}$, the compact set $\mathcal{A}$ is locally asymptotically stable, and its basin of attraction $\mathcal{B}_{\mathcal{A}}$ can be expressed as $X \cap Y$ where $X \subset O$ is open and $Y \subset O$ is closed relative to O. Then, for each proper indicator ω for $\mathcal{A}$ on X, $\mathcal{S}$ is $\mathcal{KLL}$-stable with respect to ω.

Proof: The proof follows that of Theorem 6.5 in [11]. The expression $X \cap Y$ for $\mathcal{B}_{\mathcal{A}}$ is used to conclude that the set $\{z \in \mathcal{B}_{\mathcal{A}} \,|\, \omega(z) \leq r\}$ is compact for any $r \geq 0$, while with (B5), the reachable set from any subset of $\mathcal{B}_{\mathcal{A}}$ is a subset of $\mathcal{B}_{\mathcal{A}}$. Other arguments, including those leading to Propositions 6.2 and 6.3 in [11], carry over essentially without change. ■

A particular case when a basin of attraction can be described as in the theorem above is when the range of $\mathcal{S}$, i.e. $\mathcal{R} := \{\operatorname{rge} x \mid x \in \mathcal{S}\}$, is closed relative to O and when every maximal solution in $\mathcal{S}$ is complete. Indeed, then it can be shown that $\mathcal{B}_{\mathcal{A}}$ is open relative to $\mathcal{R}$ (related result for hybrid systems $\mathcal{H}$ is in Proposition 6.4 of [11]). Then one can take $Y = \mathcal{R}$ and an appropriate open X exists by the definition of a relatively open set. We add that by (B2) of the Standing Assumption, $\mathcal{R} = \{x(0,0) \mid x \in \mathcal{S}\}$, and by (B3), $\mathcal{R}$ is closed whenever it is bounded with respect to O. For hybrid systems $\mathcal{H}$ as described in Section II-B, $\mathcal{R}$ is always closed under (A0)-(A3). We conclude this section by illustrating the need for some closedness assumptions on $\mathcal{B}_{\mathcal{A}}$.

Example 7.5: Let $\mathcal{S}$ be the set of all solutions to the hybrid system given on $O = (-\infty, 5)$ by $D = [0,1] \cup (2,3] \cup [4,5)$ and $G(x) = x/2$ if $x \in [0,1]$, $G(x) = 7 - x$ if $x \in (2,3]$, $G(x) = x - 4$ if $x \in [4,5)$. (Such hybrid system can be identified with a difference equation, and thus in what follows we do not mention t). Note that such a system does not meet (A0)-(A3) since D is not closed relative to O. However, the set $\mathcal{S}$ does meet the assumptions (B0)-(B5). Indeed, the only potential "trouble" is the sequence of solutions x_i with $x_i(0) \to 2$. For example, take $x_i(0) = 2 + 1/i$. Then $x_i(1) = 5 - 1/i$, and $x_i(2+k) = (1-1/i)2^{-k}$ for $k = 0, 1, \ldots$. The graphical limit of such a sequence is given by $x(0) = 2$, $x(1) = 5$, $x(2+k) = 2^{-k}$ for $k = 0, 1, \ldots$. Clearly, $x \notin \mathcal{S}$. However, the sequence of x_i's is not locally eventually bounded (as $x_i(2) \to 5$ as $i \to \infty$), so (B3) is not violated. The fact that x_i's are not locally eventually bounded does not violate (B4), as there are no solutions starting from 2 at all.

Now, notice that for $\mathcal{S}$, $\mathcal{A} = \{0\}$ is asymptotically stable, and $\mathcal{B}_{\mathcal{A}} = D$ (and thus $\mathcal{B}_{\mathcal{A}} = O \cap D$). However, there is no $\mathcal{KL}$ bound if ω is a proper indicator of $\mathcal{A}$ with respect to O. Indeed, any $\mathcal{KL}$ function β such that $\omega(x_i(1)) \leq \beta(\omega(x_i(0)), 1)$ for $i = 1, 2, \ldots$ would need to satisfy $\beta(s, 1) = \infty$ for all $s \geq \omega(2)$, which is impossible. In common words, trajectories originating far from the boundary of O get arbitrarily close to the boundary (before approaching $\mathcal{A}$).

In order to make D (and thus $\mathcal{R}$, the range of $\mathcal{S}$) closed, one could include a solution $x(0) = 2$, $x(1) = 1.5$ (maximal, but not complete) in $\mathcal{S}$. Assumptions (B0)-(B5) are still satisfied. The set $\mathcal{R}$ now equals $[0,1] \cup \{1.5\} \cup [2,3] \cup [4,5)$, but $\mathcal{B}_{\mathcal{A}}$ remains unchanged. Any open set X such that $\mathcal{B}_{\mathcal{A}} = X \cap \mathcal{R}$ must not contain 2. The solutions x_i considered above are now such that $x_i(0)$ approach the boundary of X as $i \to \infty$, and so $\omega(x_i(0)) \to \infty$ for any proper indicator of $\mathcal{A}$ with respect to X. For such ω, a $\mathcal{KL}$ bound can be written down. □

B. Lyapunov and Krasovskii theorems for hybrid systems

In what follows, we work with hybrid systems $\mathcal{H}$ with data (F, G, C, D, O) as described in Section II-B. We replace the Standing Assumption by the following assumptions: (A0)-(A3), G locally bounded (see Theorem 7.1), and

- (VC) for each $x^0 \in C$ and for some neighborhood U of x^0, for every $x' \in U \cap C$, $T_C(x') \cap F(x') \neq \emptyset$,
- (VD) for each $x^0 \in D$, $G(x^0) \subset C \cup D$.

The conditions (VC) and (VD) guarantee existence of solutions; see [11, Proposition 2.4]. A particular consequence of them is that any maximal solution to $\mathcal{H}$ is either complete or eventually leaves any compact subset of O.

Theorem 7.6: (hybrid Krasovskii) Given a hybrid system $\mathcal{H}$, suppose that

- ($\star$) *$\mathcal{A} \subset O$ is compact, $\mathcal{U} \subset O$ is a neighborhood of $\mathcal{A}$, $V : O \to \mathbb{R}_{\geq 0}$ is continuous on O, locally Lipschitz on a neighborhood of C, and positive definite on $C \cup D$ with respect to $\mathcal{A}$, and u_C and u_D satisfy $u_C(z) \leq 0$, $u_D(z) \leq 0$ for all $z \in \mathcal{U}$.*

Then $\mathcal{A}$ is stable. Suppose additionally that

- ($\star\star$) *there exists $r^* > 0$ such that for all $r \in (0, r^*)$ the largest weakly invariant subset in (12) is empty.*

Then $\mathcal{A}$ is locally asymptotically stable.

Proof: Assume ($\star$) and let $\epsilon > 0$ be small enough so that $\mathcal{A} + 2\epsilon\mathbb{B} \subset \mathcal{U}$. We claim that there exists c_ϵ such that

$$\begin{aligned} &V(z) \leq c_\epsilon, z \in (\mathcal{A} + 2\epsilon\mathbb{B}) \cap (C \cup D) \\ &\Rightarrow z \in (\mathcal{A} + \epsilon\mathbb{B}) \cap (C \cup D), G(z) \subset (\mathcal{A} + \epsilon\mathbb{B}) \cap (C \cup D). \end{aligned} \tag{15}$$

Certainly, as V is positive definite on $C \cup D$ with respect to $\mathcal{A}$, there exists $r'_\epsilon > 0$ so that for $z \in (\mathcal{A}+2\epsilon\mathbb{B}) \cap (C \cup D)$, $V(z) \leq r'_\epsilon$ implies $z \in (\mathcal{A}+\epsilon\mathbb{B}) \cap (C \cup D)$. Now note that as $u_D(z) \leq 0$ for all $z \in \mathcal{A}$ and V is positive definite on $C \cup D$ with respect to $\mathcal{A}$, $G(\mathcal{A} \cap (C \cup D)) \subset \mathcal{A} \cap (C \cup D)$. By outer semicontinuity and local boundedness, the mapping G is "upper semicontinuous", in particular there exists $\gamma > 0$ so that $G(\mathcal{A}+\gamma\mathbb{B}) \subset \mathcal{A}+\epsilon\mathbb{B}$. Using positive definiteness of V again, one can find $r''_\epsilon > 0$ so that $z \in (\mathcal{A}+2\epsilon\mathbb{B}) \cap (C \cup D)$ and $V(z) \leq r''_\epsilon$ imply $z \in (\mathcal{A}+\gamma\mathbb{B}) \cap (C \cup D)$. To make the implication (15) true, one now takes $r_\epsilon = \min\{r'_\epsilon, r''_\epsilon\}$.

Based on (15), we claim that the set

$$\mathcal{N} = \{z \in (\mathcal{A}+\epsilon\mathbb{B}) \cap (C \cup D) \mid V(z) \leq r_\epsilon\} \tag{16}$$

is (strongly) forward invariant for $\mathcal{H}$, that is for any $x \in \mathcal{S}(z)$ with $z \in \mathcal{N}$, $\operatorname{rge} x \subset \mathcal{N}$. Indeed, pick any $z \in \mathcal{N}$ and let $x \in \mathcal{S}(z)$. If $(0,1) \in \operatorname{dom} x$, then $x(0,1) \in G(z) \subset (\mathcal{A}+\epsilon\mathbb{B}) \cap (C \cup D)$. If $[0,T] \times \{0\} \subset \operatorname{dom} x$ and for some $t' \in (0,T]$, $x(t',0) \notin \mathcal{N}$, then by continuity of $t \mapsto x(t,0)$, for some $t'' \in (0,t']$, $x(t'',0) \notin \mathcal{N}$ but $x(t'',0) \in (\mathcal{A}+\epsilon\mathbb{B}) \cap (C \cup D)$ and $V(x(t'',0)) \leq r_\epsilon$ (the latter is true as V is nonincreasing along x). By equation (15), $x(t'',0) \in \mathcal{N}$. This is a contradiction. Thus $x([0,T],0) \subset \mathcal{N}$. The facts just shown are enough to conclude that $\mathcal{N}$ is forward invariant.

Finally, by continuity of V, given any small enough $\epsilon > 0$ and $r_\epsilon > 0$ so that (15) holds, we can find $\delta \in (0,\epsilon)$ so that $z \in (\mathcal{A}+\delta\mathbb{B}) \cap (C \cup D)$ implies $V(z) \leq r_\epsilon$. Relying on forward invariance of $\mathcal{N}$, each maximal $x \in \mathcal{S}(z)$ with $z \in \mathcal{A}+\delta\mathbb{B}$ is so that $\operatorname{rge} x \subset \mathcal{A}+\epsilon\mathbb{B}$. Thus, $\mathcal{A}$ is stable.

Now assume $(\star)$ and $(\star\star)$. To show attractivity, note that given $\epsilon > 0$ with $\mathcal{A}+2\epsilon\mathbb{B} \subset \mathcal{U}$, we can find $r_\epsilon \in (0,r)$, r as in condition $(\star\star)$ so that $\mathcal{N}$ in (16) is forward invariant (i.e. one can pick r_ϵ in the proof of stability of $\mathcal{A}$ arbitrarily small). In particular, if δ is associated with ϵ as in the paragraph above, any $x \in \mathcal{S}(z)$ with $z \in \mathcal{A}+\delta\mathbb{B}$ is precompact. As such, by Theorem 4.7, it converges to the largest weakly invariant subset of the set given by (12). It must be the case that $r' \leq r_\epsilon$ as V is nonincreasing along x, and then $r' < r^*$. As $\Omega(x)$ is nonempty, x converges to the largest weakly invariant subset of (12) with $r = 0$ which, by positive definiteness of V, is a subset of $\mathcal{A}$. Hence, $\mathcal{A}$ is attractive. ∎

We note that in Theorem 7.6, the function u_C could be replaced by v_C, or w_C if V is also nonpathological, as long as $u_C^{-1}(0)$ in equation (12) is replaced by $v_C^{-1}(0)$ or $w_C^{-1}(0)$. (In fact, the result could be stated in terms of any functions u_c, u_d for which (2) holds for all solutions to the hybrid system $\mathcal{H}$; the justification would involve Theorem 4.3.) Similarly, v_C or w_C could be used in the results below; for the sake of clarity we choose not to state them in the greatest generality.

Corollary 7.7: (hybrid Lyapunov) For a hybrid system $\mathcal{H}$, suppose that $(\star)$ of Theorem 7.6 holds, and that furthermore $u_C(z) < 0$, $u_D(z) < 0$ for all $z \in \mathcal{U} \setminus \mathcal{A}$. Then $\mathcal{A}$ is attractive, and hence locally asymptotically stable.

The following result states that when u_C (respectively, u_D) is negative in points near a compact set and instantaneous Zeno solutions (respectively, complete continuous solutions) converge to the compact set, then it is asymptotically stable.

Theorem 7.8: For the hybrid system $\mathcal{H}$, suppose that $(\star)$ of Theorem 7.6 holds. Suppose that either

(a) $u_C(z) < 0$ *for each* $z \in \mathcal{U} \setminus \mathcal{A}$,

(b) *any instantaneous Zeno solution x to $\mathcal{H}$ with $\operatorname{rge} x \subset \mathcal{U}$ converges to $\mathcal{A}$;*

or

(a') $u_D(z) < 0$ *for each* $z \in \mathcal{U} \setminus \mathcal{A}$,

(b') *any complete continuous solution x to $\mathcal{H}$ with $\operatorname{rge} x \subset \mathcal{U}$ converges to $\mathcal{A}$.*

Then $\mathcal{A}$ is locally asymptotically stable.

Proof: Stability of $\mathcal{A}$ is guaranteed by Theorem 7.6. To show attractivity, pick $\delta > 0$ as in the last paragraph of the proof of Theorem 7.6. Pick any $z \in \mathcal{A}+\delta\mathbb{B}$ and any $x \in \mathcal{S}(z)$. Then $\Omega(x) \subset \mathcal{N}$, where $\mathcal{N}$ is given by (16), and in particular, $\Omega(x) \subset \mathcal{U}$. Given any $z' \in \Omega(x)$, let $\xi \in \mathcal{S}(z')$ be any solution to $\mathcal{H}$ verifying the forward invariance of $\Omega(x)$, i.e. $\operatorname{rge}\xi \subset \Omega(x)$. By Lemma 4.1, V is constant along ξ. Suppose that $V(\xi(t,j)) = d > 0$ for all $(t,j) \in \operatorname{dom}\xi$, so in particular $\Omega(x) \cap \mathcal{A} = \emptyset$. If assumptions (a) and (b) hold, then by (a) and Lemma 4.1, ξ is instantaneously Zeno since $\Omega(x) \subset \mathcal{N}$. Hence, by (b), it converges to $\mathcal{A}$. But this contradicts V being constant along ξ. If assumptions (a') and (b') hold, then by (a') and Lemma 4.1, ξ has no jumps, i.e. it is a complete continuous solution. Hence, by (b'), it converges to $\mathcal{A}$. This again contradicts V being constant along ξ. Thus, $V(\xi(t,j)) = 0$ for all $(t,j) \in \operatorname{dom}\xi$ and consequently, $\Omega(x) \subset \mathcal{A}$. This implies that x converges to $\mathcal{A}$. ∎

VIII. Conclusions

In the appropriate framework, the most general invariance principles from continuous-time or discrete-time dynamical systems can be extended, with no loss of generality, to the setting of hybrid systems. As an application, these extensions enrich the set of tools available for establishing asymptotic stability of compact sets in hybrid control systems. They permit stability proofs using Lyapunov functions that do not strictly decrease along both flows and jumps, and also trajectory-based proofs, perhaps based on small-gain theorems expressed in terms of detectable outputs. These tools can be used to readily assist in the analysis of many physical examples, including the bouncing ball system, Newton's cradle, and swing-up of an inverted pendulum on a cart.

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