A Nested Matrosov Theorem for Hybrid Systems

Ricardo G. Sanfelice and Andrew R. Teel

***Abstract*— We show that for time-invariant hybrid systems given by a flow map, flow set, jump map, and jump set, uniform global stability of a compact set plus the existence of Lyapunov-like functions and continuous functions satisfying a *nested condition* imply uniform global asymptotic stability of the compact set ("uniform" in the sense that bounds on the solutions and on the convergence time depend only on the distance to the compact set of interest). The required nested condition is a combination of the conditions in nested Matrosov theorems for time-varying continuous-time and discrete-time systems available in the literature. Our result also shows that Matrosov's theorem is a reasonable alternative to LaSalle's invariance principle for time-invariant hybrid systems to conclude attractivity to a compact set. We illustrate the application of our main result by examples.**

I. Introduction

Matrosov's theorem is a powerful tool to establish uniform global asymptotic stability for time-varying differential equations. The result reported by Matrosov in [13] shows that, in addition to other technical conditions, given a continuously differentiable function V that establishes uniform global stability of the origin, the existence of an auxiliary continuous function with derivative that is "definitely nonzero" in the set of points where the derivative of V vanishes is a sufficient condition for uniform global asymptotic stability of the origin. Several extensions of Matrosov's theorem have appeared in the literature; see, e.g., [9] and its references. Matrosov's theorem has been applied to solve several nonlinear control problems, including tracking control [16], output feedback [15], and adaptive control [12], among others.

The extensions of the classical Matrosov theorem that seem to give most flexibility when applied in practice are those allowing for multiple auxiliary functions rather than simply one auxiliary function as in the original result by Matrosov. Such extensions are known as *nested Matrosov theorems* since to assert uniform global asymptotic stability, they require some of the auxiliary functions to be negative at points where other ones vanish. For continuous-time systems see [9], where five auxiliary functions are used in stability analysis for nonholonomic vehicles, and [19], where $3n-2$ auxiliary functions are used for the interconnection of n subsystems. Extensions of Matrosov's theorem with multiple auxiliary functions have also been proposed for discrete-time systems; see [14]. A Matrosov theorem with one auxiliary function but a weaken negativity condition, expressed in terms of persistency of excitation, has been proposed in [11] for a class of single-valued time-varying hybrid systems.

In this paper, we develop a nested Matrosov theorem for time-invariant hybrid systems allowing for set-valued dynamics, nonuniqueness of solutions, multiple jumps at the same instant, and Zeno solutions. Hybrid systems are given by a flow map, a flow set, a jump map, and a jump set. In this context, uniformity of asymptotic stability properties of compact sets indicates that bounds on the solutions and on the convergence time depend only on the distance to the compact set of interest. We show that uniform global stability of a compact set plus the existence of Lyapunov-like functions and continuous functions satisfying a *nested condition* imply uniform global asymptotic stability of the compact set. This result extends the nested Matrosov theorems in [14] and [9] to time-invariant hybrid systems. To the best of our knowledge, all instances of Matrosov's theorem in the literature have focused on time-varying systems. Certainly a Matrosov theorem reaches its full power in the context of time-varying (not necessarily periodic) systems, where general invariance principles are not available. Here, we emphasize that it can be applied to time-invariant systems where it provides a useful alternative to LaSalle's invariance principle for concluding attractivity of a compact set. In particular, no notions of invariance need to be introduced to apply Matrosov's theorem. We provide illustrative examples that emphasize this point. A nested Matrosov theorem for time-varying hybrid systems will be reported elsewhere.

The rest of the paper is organized as follows. Section II introduces the hybrid systems framework as well as stability definitions used in this paper. Section III presents a motivational example and states our main result. In Section IV, we illustrate its applicability by examples.

Notation: $\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} := [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, \ldots\}$. $\mathbb{Z}$ denotes the integers. $\mathbb{Z}_{\geq k}$ denotes integers greater than or equal to the integer k. Given a set S, $\overline{S}$ denotes its closure. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$. Given a set $S \subset \mathbb{R}^n$ and constants $\delta, \Delta, 0 \leq \delta \leq \Delta$, $\Omega_S(\delta, \Delta) := \{x \in \mathbb{R}^n \mid \delta \leq |x|_S \leq \Delta\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

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II. Hybrid Systems

Hybrid systems are dynamical systems with both continuous and discrete dynamics. Several frameworks to model hybrid systems have been proposed in the literature, including [18], [3], [10], [2], [6], to just list a few. In this paper, we follow the hybrid systems framework introduced in [6], where a hybrid system $\mathcal{H}$ with state space $\mathbb{R}^n$ is given by four objects defining its *data*:

- *Flow map* given by a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the flows (or continuous evolution) of $\mathcal{H}$.
- *Flow set* $C \subset \mathbb{R}^n$ specifying the points where flows are possible.
- *Jump map* given by a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the jumps (or discrete evolution) of $\mathcal{H}$.
- *Jump set* $D \subset \mathbb{R}^n$ specifying the points where jumps are possible.

A hybrid system $\mathcal{H} := (F, C, G, D)$ can be written in the compact form:

$$\mathcal{H}: \qquad x \in \mathbb{R}^n \qquad \left\{ \begin{array}{ccccc} \dot{x} & \in & F(x) & & x \in C \\ x^+ & \in & G(x) & & x \in D \ , \end{array} \right. \tag{1}$$

where the state x can contain both continuous and discrete states. That is, the state x can be given by $x := [\xi^\top \ \ q]^\top$ where $\xi \in \mathbb{R}^{n-1}$ is the continuous state and $q \in \{1, 2, \ldots, N\} \subset \mathbb{R}$ is the discrete (or logic) state.

Solutions can evolve continuously (or flow) and/or discretely (or jump) depending on the continuous and discrete dynamics and the sets where those dynamics apply. We treat the number of jumps as an independent variable j and we parametrize the state by (t, j). Solutions to $\mathcal{H}$ will be given by *hybrid arcs* on *hybrid time domains*.

Definition 2.1: (hybrid time domain) A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. ■

Definition 2.2: (hybrid arc) A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is locally absolutely continuous. ■

Definition 2.3: (solution to $\mathcal{H}$) A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in \overline{C} \cup D$ and:

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j)) \ .$$

(S2) For all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)) \ .$$

■

A solution x is said to be *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *maximal* if there does not exists a solution x' such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded.

Stability, uniform attractivity, and uniform asymptotic stability of compact sets for hybrid systems $\mathcal{H} = (F, C, G, D)$ are defined as follows.

Definition 2.4: (UGAS) Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. The set $\mathcal{A}$ is said to be

- *uniformly globally stable* (UGS) for $\mathcal{H}$ if there exists a class-$\mathcal{K}_\infty$ function α such that any solution x to $\mathcal{H}$ satisfies $|x(t, j)|_\mathcal{A} \leq \alpha(|x(0, 0)|_\mathcal{A})$ for all $(t, j) \in \operatorname{dom} x$;
- *uniformly globally attractive* (UGA) for $\mathcal{H}$ if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution x to $\mathcal{H}$ with $|x(0, 0)|_\mathcal{A} \leq r$, $(t, j) \in \operatorname{dom} x$ and $t + j \geq T$ imply $|x(t, j)|_\mathcal{A} \leq \varepsilon$;
- *uniformly globally asymptotically stable* (UGAS) for $\mathcal{H}$ if it is both UGS and UGA.

The stability and attractivity notions in Definition 2.4 do not insist that solutions to $\mathcal{H}$ exist from every point in $\mathbb{R}^n$. In fact, by the very definition of solutions in Definition 2.3, solutions to $\mathcal{H}$ can only exist from points in $\overline{C} \cup D$, which does not necessarily cover $\mathbb{R}^n$. Moreover, maximal solutions to $\mathcal{H}$ are not necessarily complete. For more details about existence of solutions to hybrid systems, see [7].

The results for hybrid systems $\mathcal{H}$ in [7] give mild conditions on the data (F, C, G, D) to guarantee certain regularity properties for the set of solutions to $\mathcal{H}$. These conditions are critical for things like inherent robustness of asymptotic stability [7], establishing that asymptotic stability implies uniform asymptotic stability [7], invariance principles [17], and converse Lyapunov theorems [4]. However, these conditions are not required to establish sufficient conditions for nominal asymptotic stability of compact sets, like those proposed in this paper. Therefore, we will not insist on them.

III. Nested Matrosov Theorem

A. Motivational example

Consider the so-called bouncing ball system shown in Figure 1. Let x_1 be the vertical position of the ball and x_2

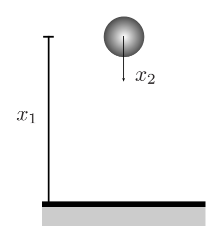


Fig. 1. Bouncing ball system.

be its vertical velocity. A model of the bouncing ball system is as follows. In between bounces, the equations of motion are given by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\gamma\ ,\end{aligned}$$

where $\gamma > 0$ is the gravity constant. In between bounces, we have that $x_1 > 0$. The bouncing condition of the ball can be modeled by the condition

$$x_1 = 0 \text{ and } x_2 < 0\ ,$$

and after the bounce (or jump), the ball's state is mapped by

$$\begin{aligned}x_1^+ &= 0,\\ x_2^+ &= -\varrho x_2,\end{aligned}$$

where $\varrho \in [0, 1)$ is the restitution coefficient. This defines a hybrid system, which we denote by $\mathcal{H}_{BB}$. Let $x := [x_1\ x_2]^\top$. Then, $\mathcal{H}_{BB}$ is given by

$$\mathcal{H}_{BB}\ :\ x \in \mathbb{R}^2 \left\{\begin{array}{rcll}\dot{x} &=& f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} & x \in C\\ x^+ &=& g(x) := \begin{bmatrix} 0 \\ -\varrho x_2 \end{bmatrix} & x \in D\ ,\end{array}\right. \tag{2}$$

where

$$\begin{aligned}C &:= \left\{x \in \mathbb{R}^2\ \mid x_1 > 0\right\},\\ D &:= \left\{x \in \mathbb{R}^2\ \mid x_1 = 0, x_2 < 0\right\}\ .\end{aligned}$$

To assert that $\mathcal{A} = (0, 0)$ is uniformly globally asymptotically stable for $\mathcal{H}_{BB}$, one could take the energy of the system given the continuously differentiable function

$$V_1(x) \ := \ \frac{1}{2}x_2^2 + \gamma x_1\ , \tag{3}$$

evaluate it along solutions to $\mathcal{H}_{BB}$, and try to conclude from those that the origin is UGS and UGA. It follows that along flows

$$\langle \nabla V_1(x), f(x)\rangle = 0 \qquad \forall x \in C \tag{4}$$

and that at jumps

$$V_1(g(x)) - V_1(x) \le -\frac{1}{2}(1-\varrho^2)x_2^2 \qquad \forall x \in D\ . \tag{5}$$

From (4) and (5), using, for example, the sufficient conditions for stability of compact sets for hybrid systems in [17], it follows that $\mathcal{A}$ is UGS. However, classical Lyapunov arguments cannot be used to establish (uniform) attractivity since the function V does not decrease along flows when the state is away from the origin. Instead, one could appeal to invariance principles, for which certain technical conditions must be verified and also some rudimentary knowledge of solutions is needed to compute invariant sets. For invariance principles for hybrid systems, see [10], [5], and [17]. Instead, we take the continuously differentiable function

$$V_2(x) \ := \ g x_2 \tag{6}$$

and note that

$$\langle \nabla V_2(x), f(x)\rangle = -\gamma^2 \qquad \forall x \in C\ , \tag{7}$$

in particular, for each point x in C such that $\langle \nabla V_1(x), f(x)\rangle = 0$. We show that (4), (5), and (7) imply that the pair V_1, V_2 establishes uniform asymptotic stability of $\mathcal{A}$ for $\mathcal{H}_{BB}$ via a *nested Matrosov theorem*. This result parallels the original one proposed by Matrosov in [13]. While asymptotic stability of the origin for the bouncing ball has been established by other means in the literature (see, for example, [1], [17], and [4]), the appeal of Matrosov's theorem is that it is expressed in terms of less stringent Lyapunov-like conditions and requires no knowledge about the solutions of the hybrid system.

B. Main result

Since our main theorem assumes UGS, we start by establishing a UGS result for closed sets of hybrid systems.

*Theorem 3.1: (*UGS conditions*) The closed set $\mathcal{A} \subset \mathbb{R}^n$ is UGS for the hybrid system $\mathcal{H} = (F, C, G, D)$ if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, continuously differentiable on an open set containing $\overline{C}$, and class-$\mathcal{K}_\infty$ functions α_1, α_2 such that $\alpha_1(|x|_\mathcal{A}) \le V(x) \le \alpha_2(|x|_\mathcal{A})$ for all $x \in \overline{C} \cup D \cup G(D)$ and*

$$\begin{array}{rcll}\langle \nabla V(x), f\rangle &\le& 0 & \forall x \in C\ ,\ f \in F(x)\\ V(g) - V(x) &\le& 0 & \forall x \in D\ ,\ g \in G(x)\ .\end{array}$$

We are now ready to state our main result.

Theorem 3.2: (hybrid nested Matrosov) Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact, UGS set for the hybrid system $\mathcal{H} = (F, C, G, D)$. Then, $\mathcal{A}$ is UGAS if there exist $m \in \mathbb{Z}_{\ge 1}$ and, for each $0 < \delta < \Delta$,

- *a number $\mu > 0$,*
- *continuous functions $u_{c,i} : \overline{C} \cap \Omega_\mathcal{A}(\delta, \Delta) \to \mathbb{R}$, $u_{d,i} : \overline{D} \cap \Omega_\mathcal{A}(\delta, \Delta) \to \mathbb{R}$, $i \in \{1, \ldots, m\}$,*
- *functions $V_i : \mathbb{R}^n \backslash \mathcal{A} \to \mathbb{R}$, $i \in \{1, \ldots, m\}$, C^1 on an open set containing $\overline{C} \cap \Omega_\mathcal{A}(\delta, \Delta)$,*

such that, for each $i \in \{1, \ldots, m\}$,

$$|V_i(x)| \le \mu \qquad \forall x \in (\overline{C} \cup D \cup G(D)) \cap \Omega_\mathcal{A}(\delta, \Delta) \tag{8}$$

$$\langle \nabla V_i(x), f\rangle \le u_{c,i}(x) \qquad \begin{array}{l}\forall x \in C \cap \Omega_\mathcal{A}(\delta, \Delta)\ ,\\ \forall f \in F(x)\end{array} \tag{9}$$

$$V_i(g) - V_i(x) \le u_{d,i}(x) \qquad \begin{array}{l}\forall x \in D \cap \Omega_\mathcal{A}(\delta, \Delta)\ ,\\ \forall g \in G(x) \cap \Omega_\mathcal{A}(\delta, \Delta)\end{array} \tag{10}$$

and, with the definitions $u_{c,0}, u_{d,0} : \mathbb{R}^n \to \{0\}$ and $u_{c,m+1}, u_{d,m+1} : \mathbb{R}^n \to \{1\}$, we have, for each $j \in \{0, \ldots, m\}$,

1) *if $x \in \overline{C} \cap \Omega_\mathcal{A}(\delta, \Delta)$ and $u_{c,i}(x) = 0$ for all $i \in \{0, \ldots, j\}$ then $u_{c,j+1}(x) \le 0$,*
2) *if $x \in \overline{D} \cap \Omega_\mathcal{A}(\delta, \Delta)$ and $u_{d,i}(x) = 0$ for all $i \in \{0, \ldots, j\}$ then $u_{d,j+1}(x) \le 0$.*

The theorem imposes a nested negative semi-definite condition on the functions $u_{c,i}$ and $u_{d,i}$, which bound the change in V_i along flows and jumps, respectively. Through the definition of $u_{c,0}$ and $u_{d,0}$, the nested condition requires that $u_{c,1}$ and $u_{d,1}$ are never positive. The function $u_{c,2}$ (respectively, $u_{d,2}$) can be positive only where $u_{c,1}$ (respectively,

$u_{d,1}$) is negative. In other words, when $u_{c,1}$ is zero, $u_{c,2}$ should be nonpositive. Similarly for $u_{d,2}$. Continuing, $u_{c,3}$ should be nonpositive when $u_{c,1}$ and $u_{c,2}$ are zero, but $u_{c,3}$ can be positive elsewhere. And so on. Finally, through the definitions of $u_{c,m+1}$ and $u_{d,m+1}$, there are no points in $\Omega_{\mathcal{A}}(\delta, \Delta)$ where all of the $u_{c,i}$ (respectively, $u_{d,i}$) are zero.

The existence of μ satisfying (8) is guaranteed when V_i is continuous on $\Omega_{\mathcal{A}}(\delta, \Delta)$. However, continuity is not required in general. The theorem is stated for functions V_i that are continuously differentiable at each point in $\overline{C} \cap \Omega_{\mathcal{A}}(\delta, \Delta)$, but a similar result holds for functions locally Lipschitz on this set. Such a result requires working with a generalized notion of derivative.

Note that when the first function in Matrosov's theorem is positive definite and radially unbounded as in Theorem 3.1, it can be used to establish UGS for the given compact set.

Under mild regularity assumptions, hybrid systems with UGAS compact sets admit smooth, strict Lyapunov functions [4]. However, such functions can be difficult to construct. As highlighted above, the Matrosov theorem relaxes the requirements on the functions that need to be constructed.

The proof of Theorem 3.2 follows the proofs of both the continuous-time and discrete-time nested Matrosov theorems in [14], [9]. In fact, the main argument of the proof is to recursively exploit the negativity guaranteed in the nested conditions in 1) and 2) of the $(i+1)$-th Matrosov functions $u_{c,i+1}, u_{d,i+1}$ at points where the i previous Matrosov functions vanish for each $i < m$. Then, UGA follows from the construction of a function V, which is obtained from a linear combination of the V_i functions, with the property that it has a strictly negative decrease along flows and jumps on $\Omega_{\mathcal{A}}(\delta, \Delta)$. The details of the proof are omitted due to space constraints. Finally, note that Theorem 3.2 when specialized to the continuous-time case, i.e., taking $G, D = \emptyset$, recovers the result in [9] for the time-invariant case, while when specialized to the discrete-time case, i.e., taking $F, C = \emptyset$, recovers the result in [14] for the time-invariant case.

IV. EXAMPLES

We now apply our main result to the following hybrid systems.

Example 4.1: (Bouncing ball revisited) Consider the bouncing ball system in Section III-A given by $\mathcal{H}_{BB}$. Let $\mathcal{A} = (0,0)$ and $V_1 : \mathbb{R}^2 \setminus \mathcal{A} \to \mathbb{R}$ be the continuously differentiable function in (3). It follows that conditions (9) and (10) in Theorem 3.2 hold for $i = 1$ with

$$\begin{array}{rcll} u_{c,1}(x) & := & 0 & \text{for each } x \in \overline{C} \setminus \mathcal{A} \ , \\ u_{d,1}(x) & := & -\frac{1}{2}(1-\varrho^2)x_2^2 & \text{for each } x \in \overline{D} \setminus \mathcal{A} \ . \end{array}$$

Moreover, with $u_{c,0}(x) := 0$ for all $\overline{C} \setminus \mathcal{A}$ and $u_{d,0}(x) := 0$ for all $\overline{D} \setminus \mathcal{A}$, $u_{c,1}(x)$ and $u_{d,1}(x)$ establish items 1 and 2 in Theorem 3.2 for $j = 0$.

Let $V_2 : \mathbb{R}^2 \setminus \mathcal{A} \to \mathbb{R}$ be the continuously differentiable function given in (6). Conditions (9) and (10) hold for $i = 2$ with

$$\begin{array}{rcll} u_{c,2}(x) & := & -g^2 & \text{for each } x \in \overline{C} \setminus \mathcal{A} \ , \\ u_{d,2}(x) & := & -g\gamma x_2 & \text{for each } x \in \overline{D} \setminus \mathcal{A} \ . \end{array}$$

For every $x \in \overline{C} \setminus \mathcal{A}$ such that $u_{c,i}(x) = 0$, $i = 0, 1$, then $u_{c,2}(x) < 0$; and for every $x \in \overline{D} \setminus \mathcal{A}$ such that $u_{d,i}(x) = 0$, $i = 0, 1$, then $u_{d,2}(x) < 0$. Then, items 1 and 2 hold for $j = 1, 2$. Moreover, UGAS of $\mathcal{A}$ follows from Theorem 3.2 with $m = 2$.

Example 4.2: (Non-Zeno bouncing ball) The previous bouncing ball model exhibits Zeno solutions. See, for example, [8]. Moreover, if the flow and jump sets are extended to their closures, the model exhibits purely discrete (jumping) solutions. In the current example, an alternative model for the bouncing ball is developed that does not exhibit Zeno solutions. It is related to a model arising from regularization procedures in [8]. The model is developed to preserve UGAS of the origin, relying on Matrosov's theorem. As before, let γ represent the gravitational constant and let $\varrho \in [0, 1)$ represent the restitution coefficient. Consider a model parametrized by the positive constants ε_i, $i = 1, 2, 3$, with data

$$f(x) := \begin{bmatrix} x_2 \\ -M(x_1) - N(x_1)x_2 \end{bmatrix} \ , \ g(x) := \begin{bmatrix} 0 \\ -\varrho x_2 \end{bmatrix} \ ,$$
$$C := \left\{ x \in \mathbb{R}^2 \ \mid x_1 \geq 0 \text{ or } (x_2 \geq -\varepsilon_2 \text{ and } x_1 \geq -\varepsilon_1) \right\} ,$$
$$D := \left\{ x \in \mathbb{R}^2 \ \mid x_1 = 0 \ , \ x_2 \leq -\varepsilon_2 \right\} \ ,$$

where

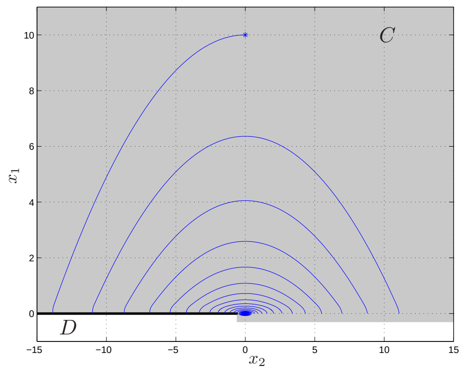
- $M : \mathbb{R} \to \mathbb{R}$ is continuous with $M(x_1) = \gamma$ for $x_1 \geq \varepsilon_3$ and $x_1 M(x_1) > 0$ for all $x_1 \neq 0$;
- $N : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is continuous with $N(x_1) = 0$ for $x_1 \geq \varepsilon_3$, and $N(x_1) > 0$ for $x_1 < 0$.

The jump map takes points in the jump set, which is already closed, to points outside of the jump set. Thus, there are no Zeno solutions (see [7, Corollary 4.9]). Indeed, after a finite number of jumps, the ball's trajectory asymptotically converges to the origin by flowing only. Figure 2 depicts trajectories evolving on the flow and jump sets using the functions M and N shown in Figure 3, which satisfy the conditions above. When $x_1 \geq \varepsilon_3$, the flow map here matches the flow map of the previous bouncing ball model. When $x_1 \leq \varepsilon_3$, the modifications to the model are aimed at generating forces corresponding to compression of the ball and energy dissipation. Ideally, the functions M and N and the values ε_1 and ε_2 would be such that the solution of $\dot{x} = f(x)$ starting at $(x_1, x_2) = (0, -\varepsilon_2)$ would satisfy:

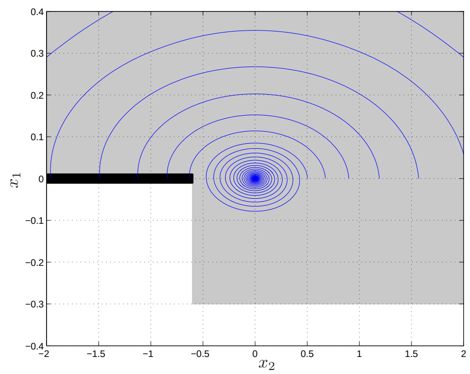
- $x_1(t) \geq -\varepsilon_1$ for all $t \geq 0$, in order to guarantee complete solutions for the hybrid system from the set where $x_1 \geq 0$,
- $x_2(t) = \varrho\varepsilon_2$ for the first $t > 0$ such that $x_1(t) = 0$, in order to replicate the dissipation caused by jumping from $(x_1, x_2) = (0, \varepsilon_2)$.

However, neither of these conditions is required for the derivation that follows.

Let $\mathcal{A}$ denote the origin and let $V_1 : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be the C^1 function given by $V_1(x) := \frac{1}{2}x_2^2 + \int_0^{x_1} M(s)ds$. We find that the conditions of Theorem 3.1 hold, so that the origin is UGS, and conditions (9) and (10) in Theorem 3.2 hold for



(a) Jumps occur in D, given by the thick black line. Jumps decrease velocity. Flows occur in C, denoted by the solid gray region.



(b) After a finite number of jumps, the ball's trajectory remains in the flow set and asymptotically converges to $\mathcal{A}$ by only flowing.

Fig. 2. Phase plot of a trajectory to the non-Zeno bouncing ball starting from $x(0,0) = [10\ ,\ 0]^\top$. Parameters: $\gamma = 9.8\text{m/s}^2$, $\varrho = 0.8$, $\varepsilon_1 = \varepsilon_3 = 0.3$, $\varepsilon_2 = 0.6$. Functions M and N as given in Figure 3.

$i = 1$ with

$$\begin{array}{rcll} u_{c,1}(x) & := & -N(x_1)x_2^2 & \forall x \in \overline{C}\ , \\ u_{d,1}(x) & := & -\frac{1}{2}\left(1-\varrho^2\right)\varepsilon^2 & \forall x \in \overline{D}\ . \end{array}$$

Since these functions are never positive, items 1 and 2 in Theorem 3.2 hold for $j = 0$. In fact, since there are no points where $u_{d,1}$ is zero, item 2 in Theorem 3.2 will hold for all j no matter what $u_{d,i}$ is for $i > 1$. Also note that $u_{c,1}(x) < 0$ for all $x \in \left\{x \in \mathbb{R}^2 \ \middle|\ x_2 > 0\ ,\ x_1 < 0\right\}$.

To define V_2, we let $\mathcal{N}$ denote the open cone in $\left\{x \in \mathbb{R}^2 \ \middle|\ x_2 > 0\ ,\ x_1 < 0\right\}$ given by

$$\mathcal{N} := \left\{x \in \mathbb{R}^2 \ \middle|\ x = r\begin{bmatrix} \lambda - 2 \\ \lambda + 1 \end{bmatrix},\ r > 0, \lambda \in (0,1)\right\}\ ,$$

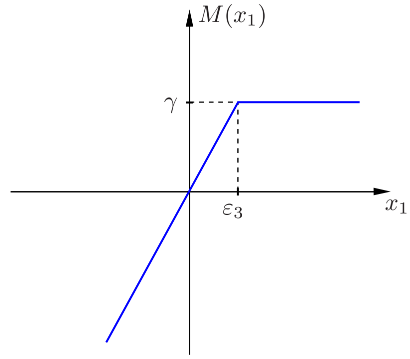
and let $\sigma : \mathbb{R} \to [-2\pi, 2\pi]$ be a continuously differentiable, 2π-periodic function such that $\frac{d\sigma(s)}{ds} = 1$ when $s = \angle(x/|x|)$ and $x \in \mathbb{R}^2 \backslash (\mathcal{N} \cup \mathcal{A})$, where $\angle : S^1 \to [0, 2\pi)$ is such that $\angle z$ denotes the angle, positive in the counter-clockwise direction, between z and the positive horizontal axis and S^1 denotes the unit circle. See, for example, the function plotted in Figure 4(b). Then, for all $x \in \mathbb{R}^2 \setminus \mathcal{A}$, define

$$V_2(x) := \sigma\left(\angle\left(\frac{x}{|x|}\right)\right),\ \ u_{c,2}(x) := \langle \nabla V_2(x), f(x)\rangle\ .$$

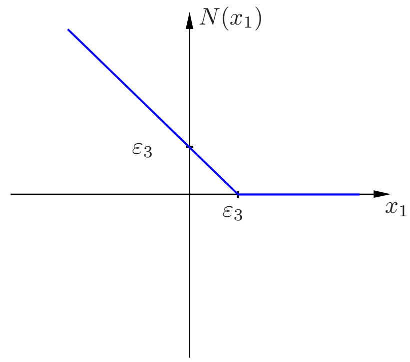
If $u_{c,1}(x) = 0$ then $x \in \mathbb{R}^2 \setminus (\mathcal{N} \cup \mathcal{A})$, $\nabla V_2(x) = \left[-x_2 \quad x_1\right]^T / |x|^2$, and $N(x_1)x_2 = 0$. Therefore,

$$u_{c,1}(x) = 0 \qquad \Longrightarrow \qquad u_{c,2}(x) = \frac{-x_2^2 - x_1 M(x_1)}{|x|^2}\ .$$

Since $M(x_1)x_1 > 0$ for all $x_1 \neq 0$, it follows that item 1 of Theorem 3.2 is satisfied for $j = 1, 2$. Hence, the origin is UGAS. $\triangle$



(a) Gravity force function M.



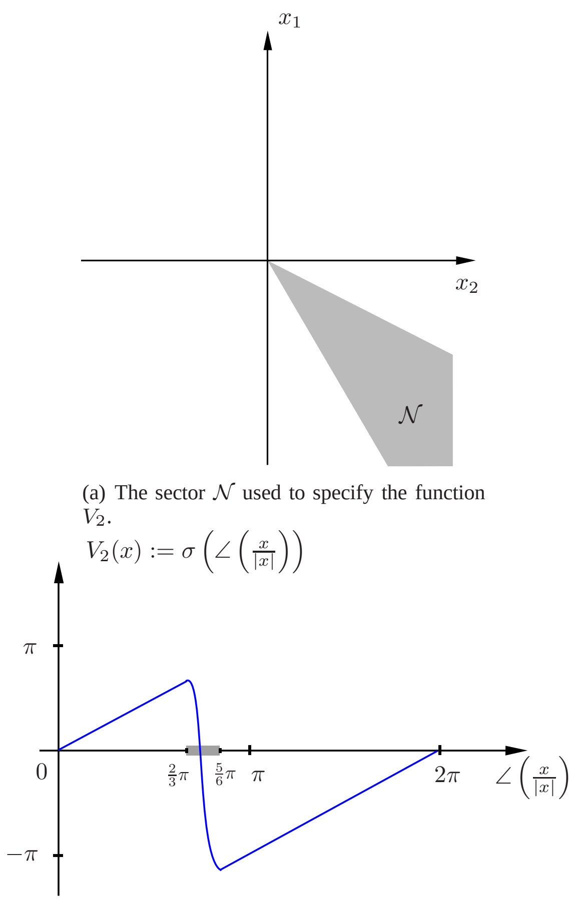
(b) Friction force function N.

Fig. 3. Functions used in Non-Zeno bouncing ball model. They agree with the Zeno bouncing ball model in Example 4.1 when $x_1 > \varepsilon_3 > 0$. Otherwise, these functions capture compression of the ball at impacts with the floor.

Remark 4.3: In Example 4.1, using Theorem 3.2, we establish that the origin of the bouncing ball is globally asymptotically stable. This fact can be also established using invariance principles in [17], or using a strict Lyapunov function as shown in [4]. The model in Example 4.2 is an alternative, which is realistic to some extent, to the bouncing ball model in Example 4.1. It has the property that every solution to it is non-Zeno. Note that away from the origin, both models behave similarly.

V. Conclusions

For hybrid systems allowing for set-valued dynamics, nonuniqueness of solutions, multiple jumps at the same instant, and Zeno solutions we introduced a nested Matrosov theorem as a tool to establish uniform global asymptotic



(a) The sector $\mathcal{N}$ used to specify the function V_2.

(b) V_2 as a function of $\angle\left(\frac{x}{|x|}\right)$. The set $\left[\frac{2}{3}\pi, \frac{5}{6}\pi\right]$ denotes the angles in the cone $\mathcal{N}$.

Fig. 4. Function V_2 for the application of Theorem 3.2 to the non-Zeno bouncing ball system.

stability of compact sets. The required nested condition is a combination of the conditions in nested Matrosov theorems for time-varying continuous-time and discrete-time systems available in the literature. The conditions constitute a relaxation of classical Lyapunov conditions. Moreover, in contrast to invariance principles, no knowledge about solutions of the hybrid system are required. Indeed, like Lyapunov theorems, only bounds on derivatives and differences must be established. Our result was demonstrated on two examples: the classical bouncing ball system and a non-Zeno version of it.

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