Hybrid Dynamical Systems

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ROBUST STABILITY AND CONTROL FOR SYSTEMS THAT COMBINE CONTINUOUS-TIME AND DISCRETE-TIME DYNAMICS

Many dynamical systems combine behaviors that are typical of continuous-time dynamical systems with behaviors that are typical of discrete-time dynamical systems. For example, in a switched electrical circuit, voltages and currents that change continuously according to classical electrical network laws also change discontinuously due to switches opening or closing. Some biological systems behave similarly, with continuous change during normal operation and discontinuous change due to an impulsive stimulus. Similarly, velocities in a multibody system change continuously according to Newton's second law but undergo instantaneous changes in velocity and momentum due to collisions. Embedded systems and, more generally, systems involving both digital and analog components form another class of examples. Finally, modern control algorithms often lead to both kinds of behavior, due to either digital components used in implementation or logic and decision making encoded in the control algorithm. These examples fit into the class of hybrid dynamical systems, or simply hybrid systems.

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This article is a tutorial on modeling the dynamics of hybrid systems, on the elements of stability theory for hybrid systems, and on the basics of hybrid control. The presentation and selection of material is oriented toward the analysis of asymptotic stability in hybrid systems and the design of stabilizing hybrid controllers. Our emphasis on the robustness of asymptotic stability to data perturbation, external disturbances, and measurement error distinguishes the approach taken here from other

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approaches to hybrid systems. While we make some connections to alternative approaches, this article does not aspire to be a survey of the hybrid system literature, which is vast and multifaceted.

The interaction of continuous- and discrete-time dynamics in a hybrid system leads to rich dynamical behavior and phenomena not encountered in purely continuous-time systems. Consequently, several challenges are encountered on the path to a stability theory for hybrid systems and to a methodology for robust hybrid control design. The approach outlined in this article addresses these challenges, by using mathematical tools that go beyond classical analysis, and leads to a stability theory that unifies and extends the theories developed for continuous- and discrete-time systems. In particular, we give necessary and sufficient Lyapunov conditions for asymptotic stability in hybrid systems, show uniformity and robustness of asymptotic stability, generalize the invariance principle to the hybrid setting and combine it with Barabasin-Krasovskii techniques, and show the utility of such results for hybrid control design. Despite their necessarily more technical appearance, these results parallel what students of nonlinear systems are familiar with.

We now present some background leading up to the model of hybrid systems used in this article. A widely used model of a continuous-time dynamical system is the first-order differential equation $\dot{x} = f(x)$, with x belonging to an n-dimensional Euclidean space $\mathbb{R}^n$. This model can be expanded in two directions that are relevant for hybrid systems. First, we can consider differential equations with state constraints, that is, $\dot{x} = f(x)$ and $x \in C$, where C is a subset of $\mathbb{R}^n$. For example, the set C might indicate that the force of gravity cannot push a ball through the floor. Alternatively, the set C might indicate a set of physically meaningful initial conditions of the system. Second, we can consider the situation where the right-hand side of the differential equation is replaced by a set that may depend on x. For example, when the force applied to a particle varies with time in an unknown way in the interval $[a, b]$, we can model the derivative of the velocity as belonging to $[a, b]$. Another reason for considering set-valued right-hand sides is to account for the effect of perturbations, such as measurement error in a feedback control system, on a differential equation. Both situations lead to the differential inclusion $\dot{x} \in F(x)$, where F is a set-valued mapping.

The examples of hybrid control systems provided in this article only scratch the surface of what is possible using hybrid feedback control.

Combining the two generalizations leads to constrained differential inclusions $\dot{x} \in F(x), x \in C$.

A typical model of a discrete-time dynamical system is the first-order equation $x^+ = g(x)$, with $x \in \mathbb{R}^n$. The notation x^+ indicates that the next value of the state is given as a function of the current state x through the value $g(x)$. As for differential equations, it is a natural extension to consider constrained difference equations and difference inclusions, which leads to the model $x^+ \in G(x), x \in D$, where G is a set-valued mapping and D is a subset of $\mathbb{R}^n$.

Since a model of a hybrid dynamical system requires a description of the continuous-time dynamics, the discrete-time dynamics, and the regions on which these dynamics apply, we include both a constrained differential inclusion and a constrained difference inclusion in a general model of a hybrid system in the form

$$\dot{x} \in F(x), \quad x \in C, \tag{1}$$
$$x^+ \in G(x), \quad x \in D. \tag{2}$$

The model (1), (2) captures a wide variety of dynamic phenomena including systems with logic-based state components, which take values in a discrete set, as well as timers, counters, and other components. Examples in this article demonstrate how to cast hybrid automata and switched systems, as well as sampled-data and networked control systems, into the form (1), (2). We refer to a hybrid system in the form (1), (2) as $\mathcal{H}$. We call C the flow set, F the flow map, D the jump set, and G the jump map.

For many systems, the generality provided by the inclusions in (1), (2) is not needed. Thus, the reader may replace the set-valued mappings and the corresponding inclusions in (1), (2) with equations and proceed confidently. It is often the geometry of sets C and D that produces the rich dynamical phenomena in a hybrid system rather than the multivaluedness of the mappings F and G. However, this article does justify, beyond the sake of generality, the use of differential inclusions and difference inclusions.

We provide examples of hybrid models in the following section. Subsequently, we make precise the meaning of a solution to a hybrid dynamical system and describe basic mathematical properties of the space of solutions. Afterward, we present results on asymptotic stability in hybrid systems, with an emphasis on robustness. Initially, we focus on Lyapunov functions as the primary stability analysis tool and show how Lyapunov functions are used in hybrid control design. Finally, we present tools for stability analysis based on limited events in hybrid systems and show how these tools are related to hybrid feedback control algorithms.

The main developments of the article are complemented by several supporting discussions. "Hybrid Automata" and "Switching Systems" relate systems in the form (1), (2) to hybrid automata and switching systems, respectively. "Related Mathematical Frameworks" presents other mathematical descriptions of systems where features of both continuous- and discrete-time dynamical systems are present. "Existence, Uniqueness, and other Well-Posedness Issues" discusses basic properties of solutions to hybrid systems in the form (1), (2). "Set Convergence" and "Robustness and Generalized Solutions" introduce mathematical tools from beyond classical analysis and motivate the assumptions placed on the data of a hybrid system $\mathcal{H}$, as given by (C, F, D, G). "Motivating Stability of Sets" and "Why 'Pre'-Asymptotic Stability?" explain distinct features of the asymptotic stability concept used in the article. "Converse Lyapunov Theorems" and "Invariance" state and discuss main tools used in the stability analysis. "Zeno Solutions" describes a phenomenon unique to hybrid dynamical systems. "Simulation in Matlab/Simulink" presents an approach to simulation of hybrid systems. The notation used throughout this article is defined in "List of Symbols."

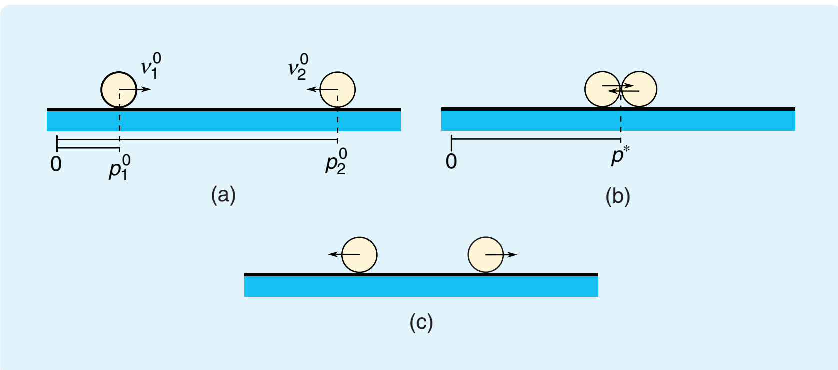

FIGURE 1 Collision between two particles. (a) Two particles are initialized to positions p_1^0 and p_2^0 and with velocities v_1^0 and v_2^0. (b) An impact between the particles occurs at the position p^*. (c) The direction of the motion of each particle is reversed after the impact.

HYBRID PHENOMENA AND MODELING

Colliding Masses

Many engineering systems experience impacts [8], [56]. Walking and jumping robots, juggling systems, billiards, and a bouncing ball are examples. Continuous-time equations of motion describe the behavior of these systems between impacts, whereas discrete dynamics approximate what happens during impacts.

Consider two particles that move toward each other, collide, and then move away from each other, as shown in Figure 1. Before and after the collision, the position and velocity of each particle are governed by Newton's second law. At impact, the velocity evolution is modeled by an instantaneous change in the velocities but no change in the positions of the particles.

The combined continuous and discrete behavior of the particles can be modeled as a hybrid system with a differential equation governing the continuous dynamics, constraints describing where the continuous dynamics apply, a difference equation governing the discrete dynamics, and constraints describing where the discrete dynamics apply. The state is the vector $x=(p,v)=(p_1,p_2,v_1,v_2)$, where p_1 and p_2 denote the particles' positions and v_1 and v_2 denote the particles' velocities. The state vector changes continuously if, as shown in Figure 1, the first particle's position is at or to the left of the second particle's position. This condition is described by the flow set $C \coloneqq \{(p,v) : p_1 \le p_2\}$. Assuming no friction, the flow map obtained applying Newton's second law and using (1) is $F(p,v)=(v_1,0,v_2,0)$.

An impact occurs when the positions of the particles are identical and their velocities satisfy $v_1 \ge v_2$. These conditions define the jump set $D \coloneqq \{(p,v) : p_1=p_2,\ v_1 \ge v_2\}$. Letting v_1^+ and v_2^+ indicate the velocities after an impact, we have the conservation of momentum equation

$$m_1 v_1^+ + m_2 v_2^+ = m_1 v_1 + m_2 v_2 \qquad (3)$$

and the energy dissipation equation

$$v_1^+ - v_2^+ = -\rho(v_1 - v_2), \qquad (4)$$

where m_1 and m_2 are the masses of the particles and the constant $\rho \in (0,1)$ is a restitution coefficient. Solving (3) and (4) for v_1^+ and v_2^+ and using (2) yields the jump map

$$G(p_1,p_2,v_1,v_2) = (p_1,p_2,v_1 - m_2\lambda(v_1-v_2), v_2 + m_1\lambda(v_1-v_2)),$$

where $\lambda = (1+\rho)/(m_1+m_2)$.

Impulsive Behavior in Biological Systems

Synchronization in groups of biological oscillators occurs in swarms of fireflies [10], groups of crickets [88], ensembles of neuronal oscillators [32], and groups of heart muscle cells [61]. Detailed treatments include [62] and [78]. The discussion below is related to [10] and [55], where models of a collection of nonlinear clocks with impulsive coupling are studied. A model of two linear clocks with impulsive coupling is used in [61] to analyze the synchronization of heart muscle cells.

List of Symbols

$\dot{x}$	The derivative, with respect to time, of the state of a hybrid system
x^+	The state of a hybrid system after a jump
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}_{\ge 0}$	The set of nonnegative real numbers, $\mathbb{R}_{\ge 0} = [0, \infty)$
$\mathbb{Z}$	The set of all integers
$\mathbb{N}$	The set of nonnegative integers, $\mathbb{N} = \{0, 1, \ldots\}$
$\mathbb{N}_{\ge k}$	$\{k, k+1, \ldots\}$ for a given $k \in \mathbb{N}$
$\varnothing$	The empty set
$\overline{\Sigma}$	The closure of the set Σ
$\operatorname{con}\Sigma$	The convex hull of the set Σ
$\overline{\operatorname{con}}\,\Sigma$	The closure of the convex hull of a set Σ
$\Sigma_1 \setminus \Sigma_2$	The set of points in Σ_1 that are not in Σ_2
$\Sigma_1 \times \Sigma_2$	The set of ordered pairs (σ_1, σ_2) with $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$
$x^\top$	The transpose of the vector x
(x, y)	Equivalent notation for the vector $[x^\top y^\top]^\top$
$\lvert x \rvert$	The Euclidean norm of a vector $x \in \mathbb{R}^n$
$\mathbb{B}$	The closed unit ball, of appropriate dimension, in the Euclidean norm
$\lvert x \rvert_\Sigma$	$\inf_{y\in\Sigma} \lvert x - y \rvert$ for a set $\Sigma \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$
$\mathbb{S}^n$	The set $\{x \in \mathbb{R}^{n+1} : \lvert x \rvert = 1\}$
$f : \mathbb{R}^m \to \mathbb{R}^n$	A function from $\mathbb{R}^m$ to $\mathbb{R}^n$
$F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$	A set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$
$R(\cdot)$	The rotation matrix $R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$
$F(\Sigma)$	$\bigcup_{x\in\Sigma} F(x)$ for the set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a set $\Sigma \subset \mathbb{R}^m$
$T_\Sigma(\eta)$	The *tangent cone* to the set $\Sigma \subset \mathbb{R}^n$ at $\eta \in \overline{\Sigma}$. $T_\Sigma(\eta)$ is the set of all vectors $w \in \mathbb{R}^n$ for which there exist $\eta_i \in \Sigma$, $\tau_i > 0$, for all $i = 1, 2, \ldots$ such that $\eta_i \to \eta$, $\tau_i \searrow 0$, and $(\eta_i - \eta)/\tau_i \to w$ as $i \to \infty$
$\mathcal{K}_\infty$	The class of functions from $\mathbb{R}_{\ge 0}$ to $\mathbb{R}_{\ge 0}$ that are continuous, zero at zero, strictly increasing, and unbounded
$L_V(\mu)$	The μ-level set of the function $V : \operatorname{dom} V \to \mathbb{R}$, which is the set of points $\{x \in \operatorname{dom} V : V(x) = \mu\}$

Hybrid Automata

Some models of hybrid systems explicitly partition the state of a system into a continuous state ξ and a discrete state q, the latter describing the mode of the system. For example, the values of q may represent modes such as "working" and "idle." In a temperature control system, q may stand for "on" or "off," while ξ may represent the temperature. By its nature, the discrete state can change only during a jump, while the continuous state often changes only during flows but sometimes may jump as well. These systems are called differential automata [82], hybrid automata [S2], [51], or simply hybrid systems [S1], [9]. All of these systems can be cast as a hybrid system of the form (1), (2).

The data of a hybrid automaton are usually given by

- a *set of modes* Q, which in most situations can be identified with a subset of the integers
- a *domain map* Domain: $Q \rightrightarrows \mathbb{R}^n$, which gives, for each $q \in Q$, the set Domain(q) in which the continuous state ξ evolves
- a *flow map* $f : Q \times \mathbb{R}^n \to \mathbb{R}^n$, which describes, through a differential equation, the continuous evolution of the continuous state variable ξ
- a *set of edges* Edges $\subset Q \times Q$, which identifies the pairs (q, q') such that a transition from the mode q to the mode q' is possible
- a *guard map* Guard : Edges $\rightrightarrows \mathbb{R}^n$, which identifies, for each edge $(q, q') \in$ Edges, the set Guard(q, q') to which the continuous state ξ must belong so that a transition from q to q' can occur
- a *reset map* Reset : Edges $\times \mathbb{R}^n \to \mathbb{R}^n$, which describes, for each edge $(q, q') \in$ Edges, the value to which the continuous state $\xi \in \mathbb{R}^n$ is set during a transition from mode q to mode q'. When the continuous state variable ξ remains constant at a jump from q to q', the map Reset$(q, q', \cdot)$ can be taken to be the identity.

Figure S1 depicts part of a state diagram for a hybrid automaton. The continuous dynamics of two modes are shown, together with the guard conditions and reset rules that govern transitions between these modes.

We now show how a hybrid automaton can be modeled as a hybrid system in the form (1), (2). First, we reformulate a hybrid automaton as a hybrid system with explicitly shown modes. For each $q \in Q$, we take

$$C_q = \mathrm{Domain}(q), \quad D_q = \bigcup_{(q,q') \in \mathrm{Edges}} \mathrm{Guard}(q, q'),$$
$$F_q(\xi) = f(q, \xi), \quad \text{for all } \xi \in C_q,$$
$$G_q(\xi) = \bigcup_{\{q' : \xi \in \mathrm{Guard}(q, q')\}} (\mathrm{Reset}(q, q', \xi), q'), \quad \text{for all } \xi \in D_q.$$

When ξ is an element of two different guard sets Guard(q, q') and Guard(q, q''), $G_q(\xi)$ is a set consisting of at least two points. Hence, G_q can be set valued. In fact, G_q is not necessarily a function even when every Reset$(q, q', \cdot)$ is the identity map. With C_q, F_q, D_q, and G_q defined above, we consider the hybrid system with state $(\xi, q) \in \mathbb{R}^n \times \mathbb{R}$ and representation

$$\dot{\xi} = F_q(\xi), \quad q \in Q, \ \xi \in C_q,$$
$$(\xi^+, q^+) \in G_q(\xi), \quad q \in Q, \ \xi \in D_q.$$

Example S1: Reformulation of a Hybrid Automaton

Consider the hybrid automaton shown in figures S2 and S3, with the set of modes $Q = \{1, 2\}$; the domain map given by

$$\mathrm{Domain}(1) = \mathbb{R}_{\le 0} \times \mathbb{R}, \quad \mathrm{Domain}(2) = \{0\} \times \mathbb{R};$$

the flow map, for all $\xi \in \mathbb{R}^2$, given by

$$f(1, \xi) = (1, 1), \quad f(2, \xi) = (0, -1);$$

the set of edges given by Edges $= \{(1, 1), (1, 2), (2, 1)\}$; the guard map given by

$$\mathrm{Guard}(1, 1) = \mathbb{R}_{\ge 0} \times \mathbb{R}_{\le 0},$$
$$\mathrm{Guard}(1, 2) = \mathbb{R}^2_{\ge 0},$$
$$\mathrm{Guard}(2, 1) = \{0\} \times \mathbb{R}_{\le 0};$$

and the reset map, for all $\xi \in \mathbb{R}^2$, given by

$$\mathrm{Reset}(1, 1, \xi) = (-5, 0),$$
$$\mathrm{Reset}(1, 2, \xi) = \xi,$$
$$\mathrm{Reset}(2, 1, \xi) = 2\xi.$$

The sets Guard$(1, 1)$ and Guard$(1, 2)$ overlap, indicating that, in mode 1, a reset of the state ξ to $(-5, 0)$ or a switch of the mode to 2 is possible from points

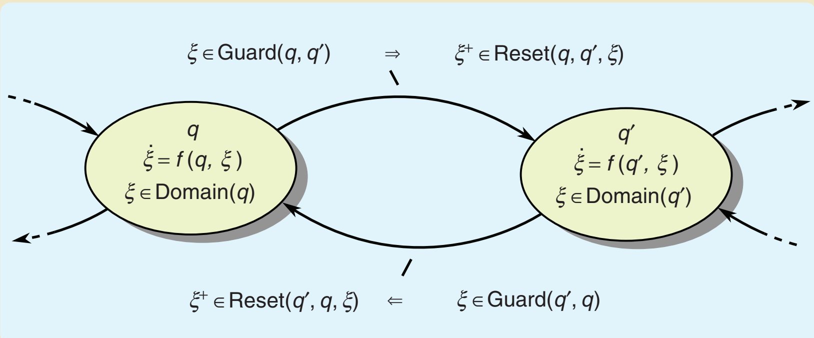

FIGURE S1 Two modes, q and q', of a hybrid automaton. In mode q, the state ξ evolves according to the differential equation $\dot{\xi} = f(q, \xi)$ in the set Domain(q). A transition from mode q to mode q' can occur when, in mode q, ξ is in the set Guard(q, q'). During the transition, ξ changes to a value ξ^+ in Reset(q, q', ξ). Transitions from mode q to other modes, not shown in the figure, are governed by similar rules.

where $\xi_1 \geq 0$ and $\xi_2 = 0$. Formulating this hybrid automaton as a hybrid system with explicitly shown modes leads to $D_1 = \text{Guard}(1, 1) \cup \text{Guard}(1, 2) = \mathbb{R}_{\geq 0} \times \mathbb{R}$ and the set-valued jump map G_1 given by

$$G_1(\xi) = \begin{cases} (-5, 0, 1), & \text{if } \xi_1 \geq 0, \xi_2 < 0, \\ (-5, 0, 1) \cup (\xi_1, \xi_2, 2), & \text{if } \xi_1 \geq 0, \xi_2 = 0, \\ (\xi_1, \xi_2, 2), & \text{if } \xi_1 \geq 0, \xi_2 > 0. \end{cases}$$

■

To formulate a hybrid automaton in the form (1), (2), we define $x = (\xi, q) \in \mathbb{R}^{n+1}$ and

$$C = \bigcup_{q \in Q} (C_q \times \{q\}), \quad F(x) = F_q(\xi) \times \{0\} \quad \text{for all } x \in C,$$

$$D = \bigcup_{q \in Q} (D_q \times \{q\}), \quad G(x) = G_q(\xi) \quad \text{for all } x \in D.$$

When the domains and guards are closed sets, the flow and jump sets C and D are also closed. Similarly, when the flow and reset maps are continuous, the flow map F and the jump map G satisfy the Basic Assumptions.

Example S1 Revisited: Solutions to a Hybrid Automaton

Consider the hybrid system modeling the hybrid automaton of Example S1. The initial condition $\xi = (3, 3)$, in mode $q = 1$, of the hybrid automaton corresponds to the initial condition (3, 3, 1) for the hybrid system. The maximal solution to the hybrid system starting from (3, 3, 1), denoted x_a, has domain $\text{dom}\, x_a = (0, 0) \cup (0, 1)$ and is given by $x_a(0, 0) = (3, 3, 1)$, $x_a(0, 1) = (3, 3, 2)$. The solution jumps once, the jump takes x_a outside of both the jump set and the flow set, and thus cannot be extended.

The hybrid system has multiple solutions from the initial point $(0, 0, 1)$. One maximal solution starting from $(0, 0, 1)$, denoted x_b, is complete and never flows. This solution has $\text{dom}\, x_b = \{0\} \times \mathbb{N}$ and is given by $x_b(0, j) = 1.5 - .5(-1)^j$. That is, the solution x_b switches back and forth between mode 1 and mode 2 infinitely many times. Another solution starting from $(0, 0, 1)$, denoted x_c, has $\text{dom}\, x_c = (0, 0) \cup ([0, 5], 1) \cup ([5, 10], 2) \cup (10, 3)$ and is given by

$$x_c(t, j) = \begin{cases} (0, 0, 1), & \text{if } t = 0, j = 0, \\ (-5 + t, t, 1), & \text{if } t \in [0, 5], j = 1, \\ (0, 5 - (t - 5), 2), & \text{if } t \in [5, 10], j = 2, \\ (0, 0, 1), & \text{if } t = 10, j = 3. \end{cases}$$

In the language of hybrid automata, this solution undergoes a reset of the state without a switch of the mode, flows for five units of time until it hits a guard, switches the mode without resetting the state, flows for another five units of time until it hits another guard, and switches the mode without resetting the state. This solution is not maximal since it can be extended in several ways. One way is by concatenating x_c with x_b, that is, by setting $x_c(t + 10, j + 3) = x_b(t, j)$ for $(t, j) \in \text{dom}\, x_b$. In other words, x_c can be extended by back and forth switches between modes. The solution x_c can be also extended to be periodic. We can consider $x_c(t, j) = (-5 + (t - 10), t - 10, 1)$ for $t \in [10, 15]$, $j = 4$, $x_c(t, j) = (0, 5 - (t - 15), 2)$ for $t \in [15, 20]$, $s = 5$, $x_c(20, 6) = (0, 0, 1)$, and repeat.

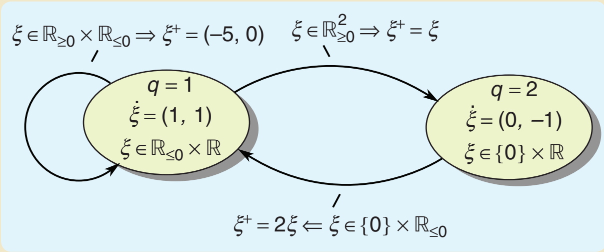

FIGURE S2 Two modes of the hybrid automaton in Example S1. In mode 1, the state ξ evolves according to the differential equation $\dot{\xi} = (1, 1)$ in the set $\text{Domain}(1) = \mathbb{R}_{\leq 0} \times \mathbb{R}$. A transition from mode 1 to mode 2 occurs when ξ is in the guard set $\text{Guard}(1, 2) = \mathbb{R}^2_{\geq 0}$ but not in the guard set $\text{Guard}(1, 1) = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}$. During the transition, ξ does not change its value. Also in mode 1, a jump in ξ to the value $(-5, 0)$ occurs when ξ is in the guard set $\text{Guard}(1, 1) = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}$ but not in the guard set $\text{Guard}(1, 2) = \mathbb{R}^2_{\geq 0}$. When ξ belongs to Guard(1, 2) and Guard(1, 1), either the transition to mode 2 or the jump of ξ can occur. In mode 2, the state ξ evolves according to the differential equation $\dot{\xi} = (0, -1)$ in the set $\text{Domain}(2) = \{0\} \times \mathbb{R}$. A transition from mode 2 to mode 1 can occur when ξ is in the guard set $\text{Guard}(2, 1) = \{0\} \times \mathbb{R}_{\leq 0}$. During the transition, ξ changes to the value 2ξ.

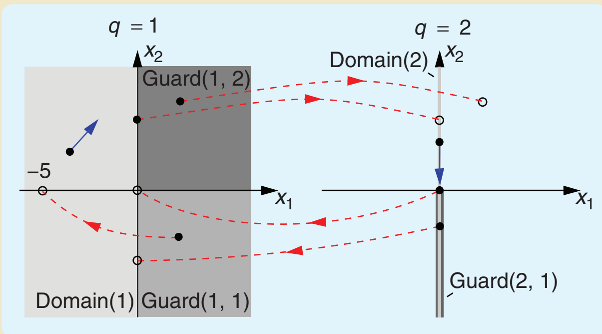

FIGURE S3 Data for the hybrid automaton in Example S1. Solid arrows indicate the direction of flow in Domain(1) and Domain(2). Dashed arrows indicate jumps from Guard(1, 1) and Guard(2, 1).

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Switching Systems

A switching system is a differential equation whose right-hand side is chosen from a family of functions based on a switching signal [49], [S4]. For each switching signal, the switching system is a time-varying differential equation. As in [S3], we study the properties of a switching system not under a particular switching signal but rather under various classes of switching signals.

In the framework of hybrid systems, information about the class of switching signals often can be embedded into the system data by using timers and reset rules, which can be viewed as an autonomous, that is, time-invariant hybrid subsystem. Results for switching systems, including converse Lyapunov theorems and invariance principles [25], can be then deduced from results obtained for hybrid systems.

A switched system can be written as

$$\dot{\xi} = f_q(\xi), \tag{S1}$$

where, for each $q \in Q = \{1, 2, \ldots, q_{\max}\}$, $f_q : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. A *complete solution* to the system (S1) consists of a locally absolutely continuous function $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and a function $q : \mathbb{R}_{\geq 0} \to Q$ that is piecewise constant, has a finite number of discontinuities in each compact time interval, and satisfies $\dot{\xi}(t) = f_{q(t)}(\xi(t))$ for almost all $t \in \mathbb{R}_{\geq 0}$. In what follows, given a complete solution (ξ, q) to (S1), let l be the number of discontinuities of q, with the possibility of $l = \infty$, and let $t_0 = 0$ and $\{t_i\}_{i=1}^{l}$ be the increasing sequence of times at which q is discontinuous. For simplicity, we discuss complete solutions only.

A solution (ξ, q) to (S1) is a *dwell-time solution with dwell time* $\tau_D > 0$ if $t_{i+1} - t_i \geq \tau_D$ for all $i = 1, 2, \ldots, l-1$. That is, switches are separated by at least an amount of time τ_D. Each dwell-time solution can be generated as part of a solution to the hybrid system with state $x = (\xi, q, \tau) \in \mathbb{R}^{n+2}$ given by

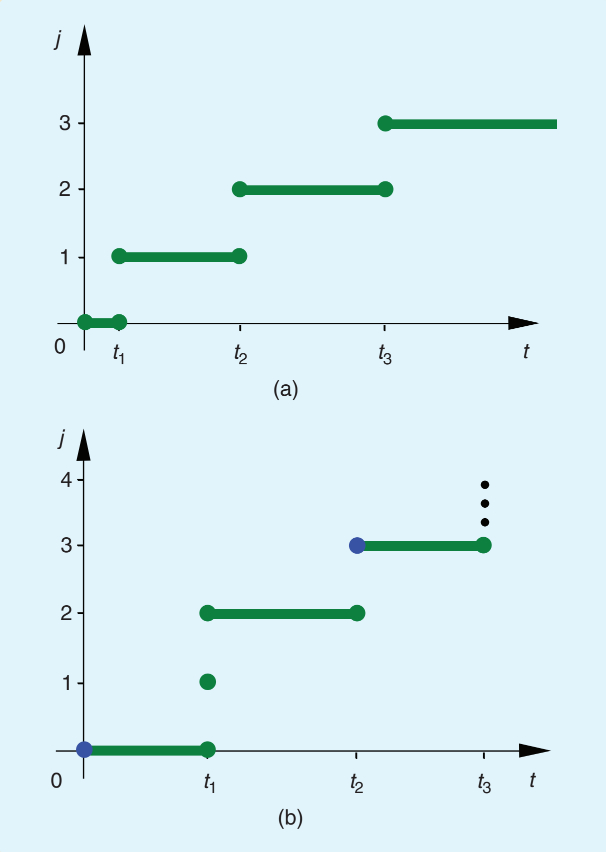

FIGURE S4 Hybrid time domain for a solution under dwell-time switching and average dwell-time switching. (a) Hybrid time domain for a dwell-time solution with dwell-time constant τ_D larger or equal than $\min\{t_2 - t_1, t_3 - t_2, \ldots\}$. (b) Hybrid time domain for an average dwell-time solution for parameters (δ, N) satisfying the average dwell-time condition in (S2). For example, parameters $(\delta, N) = (4/\min\{t_1, t_2 - t_1\}, 2)$ and $(\delta, N) = (4/t_2, 4)$ satisfy (S2). The domain repeats periodically, as denoted by the blue dot.

Consider a group of fireflies, each of which has an internal clock state. Suppose each firefly's clock state increases monotonically until it reaches a positive threshold, assumed to be the same for each firefly. When a firefly's clock reaches its threshold, the clock resets to zero and the firefly flashes, which causes the other fireflies' clocks to jump closer to their thresholds. In this way, the flash of one firefly affects the internal clocks of the other fireflies.

Figure 2 depicts the evolution of the internal clocks of two fireflies coupled through flashes. The time units are normalized so that each firefly's internal clock state takes values in the interval $[0, 1]$, and thus the threshold for flashing is one for each firefly.

A hybrid model for a system of n fireflies, each with the same clock characteristics, has the state $x = (x_1, \ldots, x_n)$, flow map $F(x) = (f(x_1), \ldots, f(x_n))$, where $f : [0, 1] \to \mathbb{R}_{>0}$ is continuous, and flow set $C = [0, 1]^n$, where $[0, 1]^n$ indicates the set of points x in $\mathbb{R}^n$ for which each component x_i belongs to the interval $[0, 1]$. The function f governs the rate at which each clock state evolves in the interval $[0, 1]$.

Since jumps in the state of the system are allowed when any one of the fireflies' clocks reaches its threshold, the jump set is $D = \{x \in C : \max_i x_i = 1\}$. One way to model the impulsive changes in the clock states is through a rule that instantaneously advances a clock state by a factor $1 + \varepsilon$, where $\varepsilon > 0$, when this action does not take the clock state past its threshold. Otherwise, the clock state is reset to zero, just as if it had reached its threshold. The corresponding jump map G does not satisfy the regularity condition (A3) of the Basic Assumptions imposed in the section "Basic Mathematical Properties." The algorithm for defining generalized solutions in "Robustness and

$$\left.\begin{array}{rcl}\dot{\xi} & = & f_q(\xi)\\ \dot{q} & = & 0\\ \dot{\tau} & \in & [0, 1/\tau_D]\end{array}\right\} =: F(x),\ x \in C := \mathbb{R}^n \times Q \times [0, 1],$$

$$\left.\begin{array}{rcl}\xi^+ & = & \xi\\ q^+ & \in & Q\\ \tau^+ & = & 0\end{array}\right\} =: G(x), x \in D := \mathbb{R}^n \times Q \times \{1\}.$$

Note that it takes at least an amount of time τ_D for the timer state τ to increase from zero to one with velocity $\dot{\tau} \in [0, 1/\tau_D]$. Therefore, τ ensures that jumps of this hybrid system occur with at least τ_D amount of time in between them. In fact, there is a one-to-one correspondence between dwell-time solutions to (S1) and solutions to the hybrid system initialized with $\tau = 1$ for which $\dot{\tau} = 1$ when $\tau \in [0, 1)$ and $\dot{\tau} = 0$ when $\tau = 1$.

A solution (ξ, q) to (S1) is an *average dwell-time solution* if the number of switches in a compact interval is bounded by a number that is proportional to the length of the interval, with proportionality constant $\delta \geq 0$, plus a positive constant N [34]. In the framework of hybrid systems, each average dwell-time solution has a hybrid time domain E such that, for each pair (s, i) and (t, i) belonging to E and satisfying with $s \leq t$ and $i \leq j$,

$$j - i \leq (t - s)\delta + N. \tag{S2}$$

Dwell-time solutions are a special case, corresponding to $\delta = 1/\tau_D$ and $N = 1$. Every hybrid time domain that satisfies (S2) can be generated by the hybrid subsystem with compact flow and jump sets given by

$$\dot{\tau} \in [0, \delta], \quad \tau \in [0, N], \tag{S3}$$

$$\tau^+ = \tau - 1, \quad \tau \in [1, N]. \tag{S4}$$

The time domain for each solution of this hybrid system satisfies the constraint (S2). Furthermore, for every hybrid time domain E satisfying (S2) there exists a solution of (S3), (S4), starting at $\tau = N$, and defined on E [S5], [14]. In turn, switching systems under an average dwell-time constraint with parameters (δ, N) are captured by the hybrid system with state $x = (\xi, q, \tau) \in \mathbb{R}^{n+2}$ given by

$$\left.\begin{array}{rcl}\dot{\xi} & = & f_q(\xi)\\ \dot{q} & = & 0\\ \dot{\tau} & \in & [0, \delta]\end{array}\right\} =: F(x), \quad x \in C := \mathbb{R}^n \times Q \times [0, N],$$

$$\left.\begin{array}{rcl}\xi^+ & = & \xi\\ q^+ & \in & Q\\ \tau^+ & = & \tau - 1\end{array}\right\} =: G(x), \quad x \in D := \mathbb{R}^n \times Q \times [1, N].$$

Figure S4 depicts a hybrid time domain for a dwell-time solution and an average dwell-time solution to a switching system.

More elaborate families of solutions to switching systems can be modeled by means of hybrid systems. We briefly mention one such family. A solution (x, q) to (S1) is a *persistent dwell-time solution with persistent dwell time* $\tau_D > 0$ *and period of persistence* $T > 0$ if there are infinitely many intervals of length at least τ_D on which no switches occur, and such intervals are separated by at most an amount of time T [S3]. A hybrid system that models such solutions involves two timers. One timer ensures that periods with no switching last at least an amount of time τ_D; the other timer ensures that periods of arbitrary switching do not last more than an amount of time T. The hybrid system also involves a differential inclusion $\dot{\xi} \in \Phi(\xi)$, where $\Phi(\xi) := \overline{\mathrm{con}} \bigcup_{q \in Q} f_q(\xi)$, to describe solutions to the switching system during periods of arbitrary switching.

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Generalized Solutions" motivates the modified jump map $G(x) = (g(x_1), \ldots, g(x_n))$, where

$$g(x_i) = \begin{cases} (1+\varepsilon)x_i, & \text{when } (1+\varepsilon)x_i < 1,\\ 0, & \text{when } (1+\varepsilon)x_i > 1,\\ \{0, 1\}, & \text{when } (1+\varepsilon)x_i = 1,\end{cases}$$

which does satisfies the regularity condition (A3). This jump map advances the clock state x_i by the factor $1 + \varepsilon$ when this action keeps x_i below the threshold, and it resets x_i to zero when multiplying x_i by $1 + \varepsilon$ produces a value greater than the threshold value. Either resetting the clock state to zero or advancing the clock state by $1 + \varepsilon$ can occur when $(1 + \varepsilon)x_i = 1$.

A group of fireflies can exhibit almost global synchronization, meaning that, from almost every initial condition, the state vector tends to the set where all of the clock states are equal [55]. Synchronization analysis for the case $n = 2$ is given in Example 25.

Explicit Zero-Crossing Detection

Zero-crossing detection (ZCD) algorithms for sinusoidal signals are crucial for estimating phase and frequency as well as power factor in electric circuits. ZCD algorithms employ a discrete state, which remembers the most recent sign of the signal and is updated when the signal crosses zero, as indicated in Figure 3.

We cast a simple ZCD algorithm for a sinusoidal signal in terms of a hybrid system. Let the sinusoid be generated as the output of the linear system $\dot{\xi}_1 = \omega\xi_2$, $\dot{\xi}_2 = -\omega\xi_1$, $y = \xi_1$, where $\omega > 0$, and let q denote a discrete state taking values in $Q := \{-1, 1\}$ corresponding to the sign of ξ_1. The state of the hybrid system is $x = (\xi, q)$, while the flow map is $F(x) = (\omega\xi_2, -\omega\xi_1, 0)$.

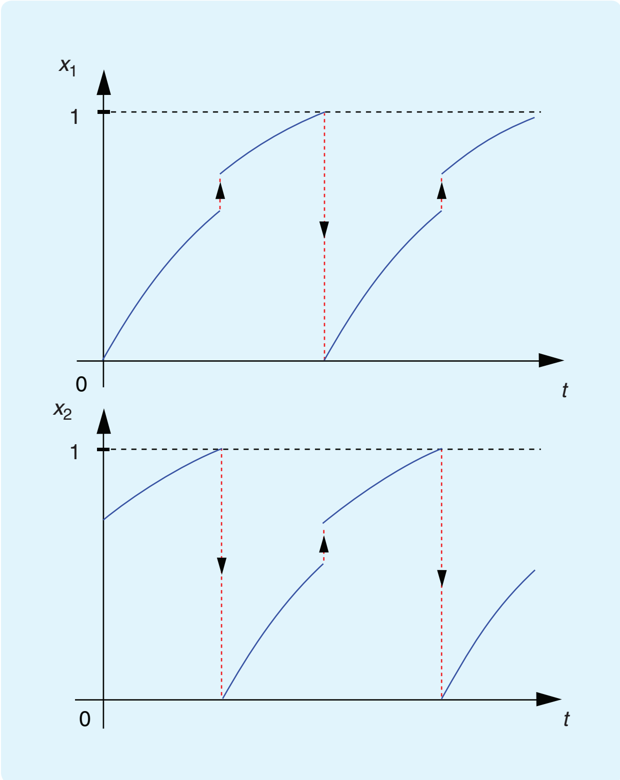

FIGURE 2 Trajectories of the internal clocks of two fireflies with impulsive coupling. When either clock state x_1 or x_2 reaches the unit threshold, both states experience a jump. When a state reaches the threshold, it is reset to zero. At the same time, the other state is increased by a factor $1+\varepsilon$ if this increase does not push the state past the threshold; otherwise, this state is also reset to zero.

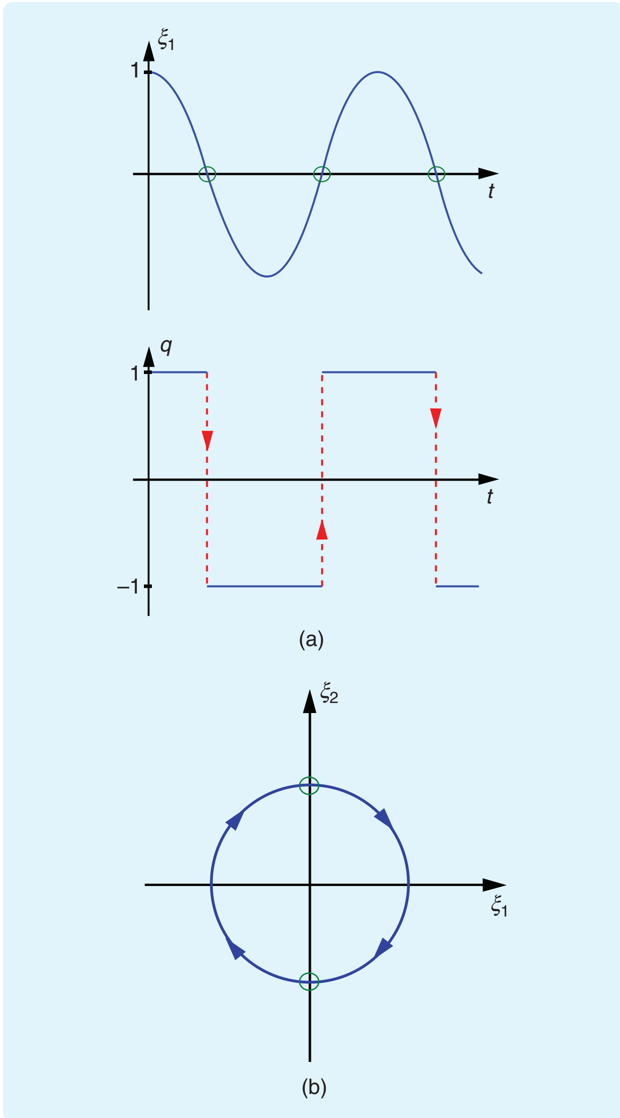

FIGURE 3 Detection of zero crossings of a sinusoidal signal. The sinusoidal signal ξ_1 is the output of the linear system $\dot{\xi}_1 = \omega\xi_2$, $\dot{\xi}_2 = -\omega\xi_1$, where $\omega > 0$. (a) The discrete state q is toggled at every zero crossing of the sinusoidal signal ξ_1. (b) The signal evolves in the $\xi = (\xi_1, \xi_2)$ plane.

As ξ_1 changes sign, a zero-crossing event occurs. We model the detection of this event as a toggling of the state q through the rule $q^+ = -q$. In a more elaborate model, either a counter that keeps track of the number of zero-crossing events can be incremented, or a timer state that monitors the amount of time between zero-crossing events can be reset. The state ξ does not change during jumps. The jump map is thus $G(x) = (\xi, -q)$.

When q and ξ_1 have the same sign, that is, $q\xi_1 \geq 0$, flows are allowed. This behavior corresponds to the flow set $C = \bigcup_{q\in Q}(C_q \times \{q\})$, where $C_q := \{\xi \in \mathbb{R}^2 : q\xi_1 \geq 0\}$. In other words, the flow set C is the union of two sets. One set corresponds to points where $q = 1$ and $\xi_1 \geq 0$, while the other set corresponds to points where $q = -1$ and $\xi_1 \leq 0$.

When $\xi_1 = 0$ and the sign of q is opposite to the sign of the derivative of ξ_1, that is, $q\xi_2 \leq 0$, the value of q is toggled. This behavior corresponds to the jump set $D = \bigcup_{q\in Q}(D_q \times \{q\})$, where $D_q := \{\xi \in \mathbb{R}^2 : \xi_1 = 0, q\xi_2 \leq 0\}$. Thus, the jump set D is the union of two sets. One set corresponds to points where $q = 1$, $\xi_1 = 0$, and $\xi_2 \leq 0$, while the other set corresponds to points where $q = -1$, $\xi_1 = 0$, and $\xi_2 \geq 0$.

Figure 4 shows the flow and jump sets of the hybrid system. The figure also depicts the sinusoidal signal ξ_1 and the discrete state q obtained for initial conditions with ξ_1 starting at one, ξ_2 starting at zero, and q starting at one.

Sample-and-Hold Control Systems

In a typical sample-and-hold control scenario, a continuous-time plant is controlled by a digital controller. The controller samples the plant's state, computes a control signal, and sets the plant's control input to the computed value. The controller's output remains constant between updates. Sample-and-hold devices perform analog-to-digital and digital-to-analog conversions.

As noted in [59], the closed-loop system resulting from this control scheme can be modeled as a hybrid system. Sampling, computation, and control updates in sample-and-hold control are associated with jumps that occur when one or more timers reach thresholds defining the update rates. When these operations are performed synchronously, a single timer state and threshold can be used to trigger their execution. In this case, a sample-and-hold implementation of a control law samples the state of the plant and updates its input when a timer reaches the threshold $T > 0$, which defines the sampling period. During this update, the timer is reset to zero.

For the static, state-feedback law $u = \kappa(\xi)$ for the plant $\dot{\xi} = f(\xi, u)$, a hybrid model uses a memory state z to store the samples of u, as well as a timer state τ to determine when each sample is stored. The state of the resulting closed-loop system, which is depicted in Figure 5, is taken to be $x = (\xi, z, \tau)$.

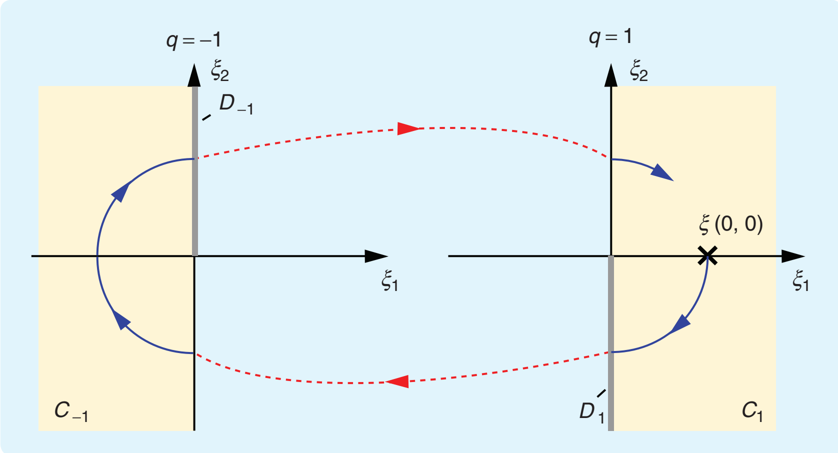

FIGURE 4 Flow and jump sets for each $q \in Q$ and trajectory to the hybrid system in "Explicit Zero-Crossing Detection." The trajectory starts from the initial condition at $(t, j) = (0, 0)$ given by $\xi_1(0, 0) = 1$, $\xi_2(0, 0) = 0$, $q(0, 0) = 1$. The jumps occur on the ξ_2 axis and toggle q. Flows are permitted in the left-half plane for $q = -1$ and in the right-half plane for $q = 1$.

During flow, which occurs until τ reaches the threshold T, the state of the plant evolves according to $\dot{\xi} = f(\xi, z)$, the value of z is kept constant, and τ grows at the constant rate of one. In other words, $\dot{z} = 0$ and $\dot{\tau} = 1$. This behavior corresponds to the flow set $C = \mathbb{R}^n \times \mathbb{R}^m \times [0, T]$, while the flow map is given by $F(x) = (f(\xi, z), 0, 1)$ for all $x \in C$.

When the timer reaches the threshold T, the timer state τ is reset to zero, the memory state z is updated to $\kappa(\xi)$, but the plant state ξ does not change. This behavior corresponds to the jump set $D := \mathbb{R}^n \times \mathbb{R}^m \times \{T\}$ and the jump map $G(x) := (\xi, \kappa(\xi), 0)$ for all $x \in D$.

Hybrid Controllers for Nonlinear Systems

Hybrid dynamical systems can model a variety of closed-loop feedback control systems. In some hybrid control applications the plant itself is hybrid. Examples include juggling [70], [73] and robot walking control [63]. In other applications, the plant is a continuous-time system that is controlled by an algorithm employing discrete-valued states. This type of control appears in a broad class of industrial applications, where programmable logic controllers and microcontrollers are employed for automation. In these applications, discrete states, as well as other variables in software, are used to implement control logic that incorporates decision-making capabilities into the control system.

Consider a plant described by the differential equation

$$\dot{x}_p = f_p(x_p, u), \tag{5}$$

where $x_p \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, and f_p is continuous. A hybrid controller for this plant has state $x_c \in \mathbb{R}^m$, which can contain logic states, timers, counters, observer states, and other continuous-valued and discrete-valued states. A hybrid controller is defined by a flow set $C_c \subset \mathbb{R}^{n+m}$, flow map $f_c : C_c \to \mathbb{R}^n$, jump set $D_c \subset \mathbb{R}^{n+m}$, and a possibly set-valued jump map $G_c : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^m$, together with a feedback law $\kappa_c : C_c \to \mathbb{R}^r$ that specifies the control signal u. Figure 6 illustrates this setup.

During continuous-time evolution, which can occur when the composite closed-loop state $x = (x_p, x_c)$ belongs to the set C_c, the controller state satisfies $\dot{x}_c = f_c(x)$ and the control signal is generated as $u = \kappa_c(x)$. At jumps, which are allowed when the closed-loop state belongs to D_c, the state of the controller is reset using the rule $x_c^+ \in G_c(x)$. The closed-loop system is a hybrid system with state $x = (x_p, x_c)$, flow set $C = C_c$, jump set $D = D_c$, flow map

$$F(x) = \begin{bmatrix} f_p(x_p, \kappa_c(x)) \\ f_c(x) \end{bmatrix} \quad \text{for all } x \in C, \tag{6}$$

and jump map

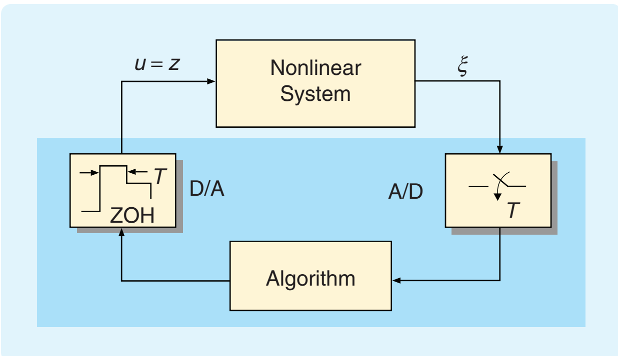

FIGURE 5 Digital control of a continuous-time nonlinear system with sample-and-hold devices performing the analog-to-digital (A/D) and digital-to-analog (D/A) conversions. Samples of the state ξ of the plant and updates of the control law $\kappa(\xi)$ computed by the algorithm are taken after each amount of time T. The controller state z stores the values of $\kappa(\xi)$.

The interaction of continuous- and discrete-time dynamics in a hybrid system leads to rich dynamical behavior and phenomena not encountered in purely continuous-time systems.

$$G(x) = \begin{bmatrix} x_p \\ G_c(x) \end{bmatrix} \quad \text{for all } x \in D. \tag{7}$$

One way hybrid controllers arise is through supervisory control. A supervisor oversees a collection of controllers and makes decisions about which controller to insert into the closed-loop system based on the state of the plant and the controllers. The supervisor associates to each controller a region of operation and a region where switching to other controllers is possible. These regions are subsets of the state space. In the region where changes between controllers are allowed, the supervisor specifies the controllers to which authority can be switched. In supervisory control, it is possible for the individual controllers to be hybrid controllers. Through this degree of flexibility, it is possible to generate hybrid control algorithms through a hierarchy of supervisors.

The following example features a supervisor for two state-feedback control laws.

Example 1: Dual-Mode Control for Disk Drives

Control of read/write heads in hard disk drives requires precise positioning on and rapid transitioning between tracks on a disk drive. To meet these dual objectives, some commercial hard disk drives use mode-switching control [27], [81], [87], which combines a track-seeking controller and a track-following controller. The track-seeking controller rapidly steers the magnetic head to a neighborhood of the desired track, while the track-following controller regulates position and velocity, precisely and robustly, to enable read/write operations. Mode-switching control uses the track-seeking controller to steer the magnetic head's state to a point where the track-following controller is applicable, and then switches the control input to the track-following controller. The control strategy results in a hybrid closed-loop system.

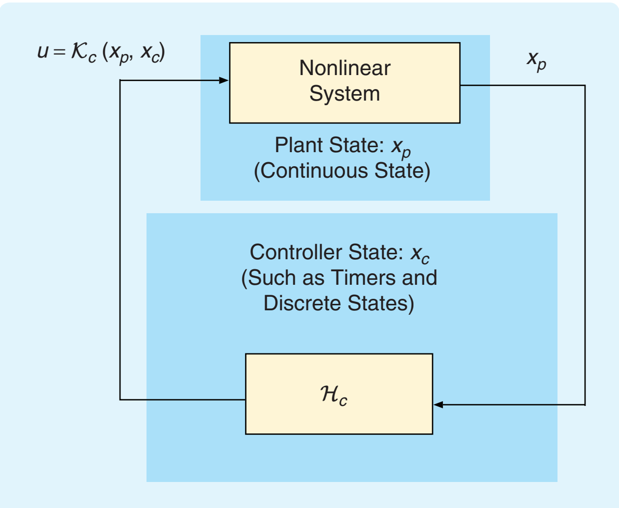

FIGURE 6 Closed-loop system consisting of a continuous-time nonlinear system and a hybrid controller. The nonlinear system has state x_p, which is continuous, and input u. The hybrid controller has state x_c, which has continuous state variables, such as timer states, and discrete state variables, such as logic modes. The control input $u = \kappa_c(x_p, x_c)$ to the nonlinear system is a function of the plant state x_p and the controller state x_c.

Let $p \in \mathbb{R}$ be the position and $v \in \mathbb{R}$ the velocity of the magnetic head in the disk drive. The dynamics can be approximated by the double integrator system $\dot{p} = v$, $\dot{v} = u$ [27], [87].

The hybrid controller for the magnetic head supervises both the track-seeking control law $u = \kappa_1(p, v, p^*)$ and the track-following control law $u = \kappa_2(p, v, p^*)$, where p^* is the desired position. We assume that the track-seeking control law globally asymptotically stabilizes the point $(p^*, 0)$, while the track-following control law locally asymptotically stabilizes the point $(p^*, 0)$. Let C_2 be a compact neighborhood of $(p^*, 0)$ that is contained in the basin of attraction for $(p^*, 0)$ when using the track-following control law, and let D_1 be a compact neighborhood of $(p^*, 0)$ such that solutions using the track-following control law that start in D_1 do not reach the boundary of C_2. Also define $C_1 = \overline{\mathbb{R}^2 \backslash D_1}$ and $D_2 = \overline{\mathbb{R}^2 \backslash C_2}$. Figure 7 illustrates these sets.

Let the controller state $q \in Q := \{1, 2\}$ denote the operating mode. The track-seeking mode corresponds to $q = 1$, while the track-following mode corresponds to $q = 2$. The mode-switching strategy uses the track-seeking controller when $(p, v) \in C_1$ and the track-following controller when $(p, v) \in C_2$. Figure 7 indicates the intersection of C_1 and C_2, where either controller can be used. To prevent chattering between the two controllers in the intersection of C_1 and C_2, the supervisor allows mode switching when $(p, v) \in D_q$. In other words, a switch from the track-seeking mode to the track-following mode can occur when $(p, v) \in D_1$, while a switch from the track-following mode to the track-seeking mode can occur when $(p, v) \in D_2$.

The hybrid controller executing this logic has the flow set $C_c = \bigcup_{q \in Q}(C_q \times \{q\})$, flow map $f_c(p, v, q) = 0$, jump set

$D_c := \bigcup_{q\in Q}(D_q \times \{q\})$, and jump map $G_c(p, v, q) = 3 - q$, which toggles q in the set $Q = \{1, 2\}$.

The idea behind this control construction applies to arbitrary nonlinear control systems and state-feedback laws [66]. ■

CONCEPT OF A SOLUTION

Generalized Time Domains

Solutions to continuous-time dynamical systems are parameterized by $t \in \mathbb{R}_{\geq 0}$, whereas solutions to discrete-time dynamical systems are parameterized by $j \in \mathbb{N}$. Parameterization by $t \in \mathbb{R}_{\geq 0}$ is possible for a continuous-time system even when solutions experience jumps, as long as at most one jump occurs at each time t. For example, parameterization by $t \in \mathbb{R}_{\geq 0}$ is used for switched systems [49] as well as for impulsive dynamical systems [43], [30]. However, parameterization by $t \in \mathbb{R}_{\geq 0}$ of discontinuous solutions to a dynamical system may be an obstacle for establishing sequential compactness of the space of solutions. For example, sequential compactness may require us to admit a solution with two jumps at the same time instant to represent the limit of a sequence of solutions for which the time between two consecutive jumps shrinks to zero. By considering a generalized time domain, we can overcome such obstacles and, furthermore, treat continuous- and discrete-time systems in a unified framework.

A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* [23], [26] if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_0 \leq t_1 \leq t_2 \leq \cdots$, or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$. An example of a hybrid time domain is shown in Figure 8.

A hybrid time domain is called a hybrid time set in [17] and is equivalent to a generalized time domain [51] defined as a sequence of intervals, some of which may consist of one point. The idea of a hybrid time domain is present in the concept of a solution given in [4], which explicitly includes a nondecreasing sequence of jump times in the solution description.

More general time domains are sometimes considered. For details, see [54], [18], or the discussion of time scales in "Related Mathematical Frameworks." Some time domains make it possible to continue solutions past infinitely many jumps. For an initial exposition of hybrid system and for the analysis of many hybrid control algorithms, domains with this feature are not necessary.

Solutions

A solution to a hybrid system is a function, defined on a hybrid time domain, that satisfies the dynamics and constraints given by the data of the hybrid system. The data in (1), (2) has four components, which are the flow set, the flow map, the jump set, and the jump map. For a hybrid system (1), (2) on $\mathbb{R}^n$, the flow set C is a subset of $\mathbb{R}^n$, the flow map is a set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the jump set D is a subset of $\mathbb{R}^n$, and the jump map is a set-valued mapping $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. A set-valued mapping on $\mathbb{R}^n$ associates, to each $x \in \mathbb{R}^n$, a set in $\mathbb{R}^n$. A function is a set-valued mapping whose values can be viewed as sets that consist of one point.

A *hybrid arc* is a function $x: \operatorname{dom} x \to \mathbb{R}^n$, where $\operatorname{dom} x$ is a hybrid time domain and, for each

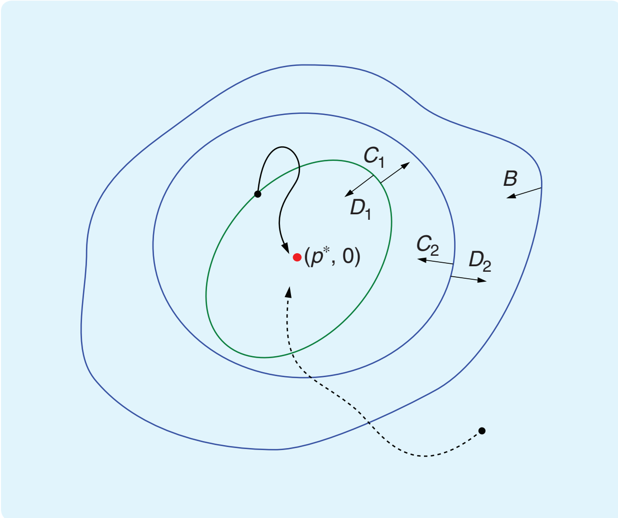

FIGURE 7 Sets of the hybrid controller for dual-mode control of disk drives. The flow and jump sets for the track-seeking mode $q = 1$ and the track-following mode $q = 2$ are constructed from the sets C_1, D_1 and C_2, D_2, respectively. The set $\mathcal{B}$ is the basin of attraction for $(p^*, 0)$ when the track-following controller is applied. In addition, this set contains the compact set D_1, from which solutions with the track-following controller do not reach the boundary of C_2, a compact subset of $\mathcal{B}$. This property is illustrated by the solid black solution starting from D_1. The dashed black solution is the result of applying the track-seeking controller, which steers solutions to the set D_1 in finite time.

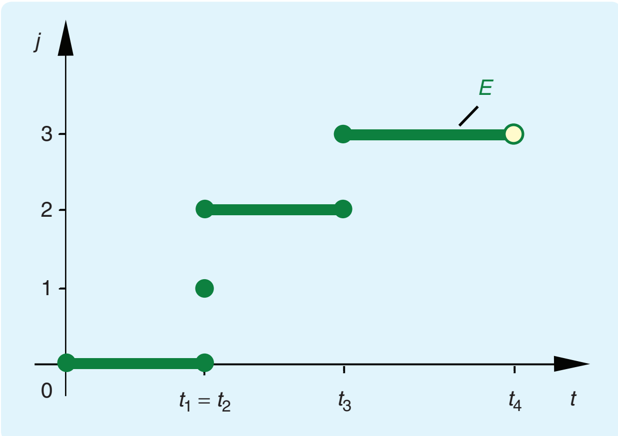

FIGURE 8 A hybrid time domain. The hybrid time domain, which is denoted by E, is given by the union of $[0, t_1] \times \{0\}$, $[t_2, t_2] \times \{1\}$, $[t_2, t_3] \times \{2\}$, and $[t_3, t_4) \times \{3\}$.

Related Mathematical Frameworks

Interest in hybrid systems grew rapidly in the 1990s with computer scientists and control systems researchers coming together to organize several international workshops. See [S8] and similar subsequent collections. Additional books dedicated to hybrid systems include [86] and [S10]. The legacy of the cooperative initiative with computer science is the successful conference "Hybrid Systems: Computation and Control (HSCC)," now a part of the larger "cyber-physical systems week," which includes real-time and embedded systems and information processing in sensor networks. At the same time, many mathematical frameworks related to hybrid systems have also appeared in the literature. Some of those frameworks are discussed below. Additional ideas appear in the concept of a discontinuous dynamical system, described in [S11] and [S12].

IMPULSIVE DIFFERENTIAL EQUATIONS

Impulsive differential equations consist of the classical differential equation $\dot{x}(t) = f(x(t))$, which applies at all times except the impulse times, and the equation $\Delta x(t_i) = I_i(x(t_i))$, which describes the impulsive behavior at impulse times. The impulse times are often fixed a priori for each particular solution and form an increasing sequence $t_1, t_2, \ldots$. In other words, a solution with the state $x(t_i)$ before the ith jump has the value $x(t_i) + I_i(x(t_i))$ after the jump. Solutions to impulsive differential equations are piecewise differentiable or piecewise absolutely continuous functions parameterized by time t. These functions cannot model multiple jumps at a single time instant.

An impulsive differential equation can be recast as a hybrid system in the case where the impulse times are determined by the condition $x(t) \in D$ for some set D. This situation requires some conditions on D and I_i to ensure that $x(t_i) + I_i(x(t_i)) \notin D$. For simplicity, consider the case where I_i is the same map I for each i. Then the corresponding hybrid system has the flow map f, the jump map $x \mapsto x + I(x)$, the jump set D, and the flow set C given by the complement of D.

Natural generalizations of impulsive differential equations include impulsive differential inclusions, where either f or I may be replaced by a set-valued mapping. For details, see [S6], [43], [S15], and [30].

MEASURE-DRIVEN DIFFERENTIAL EQUATIONS

The classical differential equation $\dot{x}(t) = f(x(t))$ can be rewritten as

$$dx(t) = f(x(t))\,dt.$$

Measure-driven differential equations are formulated as

$$dx(t) = f_1(x(t))\,dt + f_2(x(t))\,\mu(dt),$$

where f_1, f_2 are functions and μ is a nonnegative scalar or vector-valued Borel measure. Solutions to measure-driven differential equations are given by functions of bounded variation parameterized by t and are not necessarily differentiable, absolutely continuous, or even continuous. The discontinuous behavior is due to the presence of atoms in the measure μ. In control situations, the driving measure μ, in particular, the atoms of μ, can approximate time-dependent controls that take large values on short intervals.

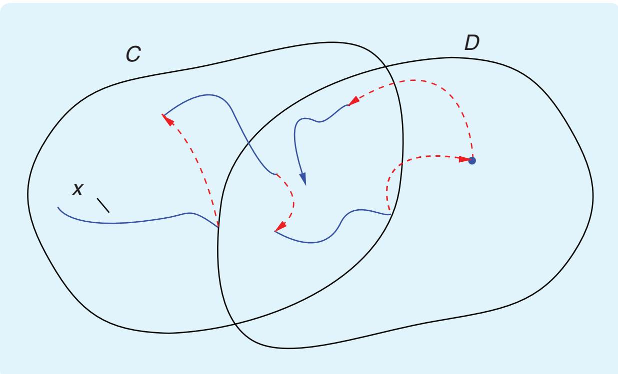

FIGURE 9 Evolution of a solution to a hybrid system. Flows and jumps of the solution x are allowed only on the flow set C and from the jump set D, respectively. The solid blue curves indicate flow. The dashed red arcs indicate jumps. The solid curves must belong to the flow set C. The dashed arcs must originate from the jump set D.

fixed j, $t \mapsto x(t, j)$ is a locally absolutely continuous function on the interval

$$I_j = \{t : (t, j) \in \operatorname{dom} x\}.$$

The hybrid arc x is a *solution to the hybrid system* $\mathcal{H} = (C, F, D, G)$ if $x(0,0) \in C \cup D$ and the following conditions are satisfied.

Flow Condition

For each $j \in \mathbb{N}$ such that I_j has nonempty interior,

$$\begin{aligned} \dot{x}(t,j) &\in F(x(t,j)) && \text{for almost all } t \in I_j, \\ x(t,j) &\in C && \text{for all } t \in [\min I_j, \sup I_j). \end{aligned}$$

Jump Condition

For each $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j+1) \in G(x(t,j)),$$

$$x(t,j) \in D.$$

If the flow set C is closed and I_j has nonempty interior, then the requirement $x(t,j) \in C$ for all $t \in [\min I_j, \sup I_j)$ in the flow condition is equivalent to $x(t,j) \in C$ for

Natural generalizations, needed to analyze mechanical systems with friction or impacts [S13], include measure-driven differential inclusions, where f_1, f_2 are replaced by set-valued mappings. Formulating a robust notion of a solution to measure-driven differential inclusions is technically challenging [S9], [S14].

DYNAMICAL SYSTEMS ON TIME SCALES

A framework for unifying the classical theories of differential and difference equations is that of dynamical systems on time scales [S7]. Given a time scale $\mathbb{T}$, which is a nonempty closed subset of $\mathbb{R}$, a generalized derivative of a function $\phi:\mathbb{T}\to\mathbb{R}$ relative to $\mathbb{T}$ can be defined. This generalized derivative reduces to the standard derivative when $\mathbb{T}=\mathbb{R}$, and to the difference $\phi(n+1)-\phi(n)$ when evaluated at n and for $\mathbb{T}=\mathbb{N}$. As special cases, classical differential and difference equations can be written as systems on time scales. Systems on time scales can also be used to model populations that experience a repeated pattern consisting of continuous evolution followed by a dormancy [S7, Ex. 1.39].

Consider a time scale $\mathbb{T}$ that is unbounded to the right, and, for $t\in\mathbb{T}$, define $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$. The generalized derivative of $\phi:\mathbb{T}\to\mathbb{R}$ at $t\in\mathbb{T}$ is the number $\phi^{\Delta}(t)$, if it exists, such that, for each $\varepsilon>0$ and each $s\in\mathbb{T}$ sufficiently close to t,

$$|[\phi(\sigma(t))-\phi(s)]-\phi^{\Delta}(t)[\sigma(t)-s]|\le\varepsilon|\sigma(t)-s|.$$

The function ϕ is differentiable if ϕ^{Δ} exists at every $t\in\mathbb{T}$. A dynamical system on the time scale $\mathbb{T}$ has the form

$$x^{\Delta}(t)=f(x(t))\quad \text{for every } t\in\mathbb{T}.$$

One advantage of the framework of dynamical systems on time scales is the generality of the concept of a time scale. A drawback, especially for control engineering purposes, is that a time scale is chosen a priori, and all solutions to a system are defined on the same time scale.

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all $t\in I_j$ and is also equivalent to $x(t,j)\in C$ for almost all $t\in I_j$.

The solution x to a hybrid system is *nontrivial* if dom x contains at least one point different from $(0,0)$; *maximal* if it cannot be extended, that is, the hybrid system has no solution x' whose domain dom x' contains dom x as a proper subset and such that x' agrees with x on dom x; and complete if dom x is unbounded. Every complete solution is maximal.

Figure 9 shows a solution to a hybrid system flowing, as solutions to continuous-time systems do, at points in the flow set C, and jumping, as solutions to discrete-time systems do, from points in the jump set D. At points where D overlaps with the interior of C, solutions can either flow or jump. Thus, the jump set D enables rather than forces jumps. To force jumps from D, the flow set C can be replaced by either the set $C\setminus D$ or the set $\overline{C\setminus D}$.

The parameterization of a solution x by $(t,j)\in$ dom x means that $x(t,j)$ represents the state of the hybrid system after t time units and j jumps. Figure 10 shows a

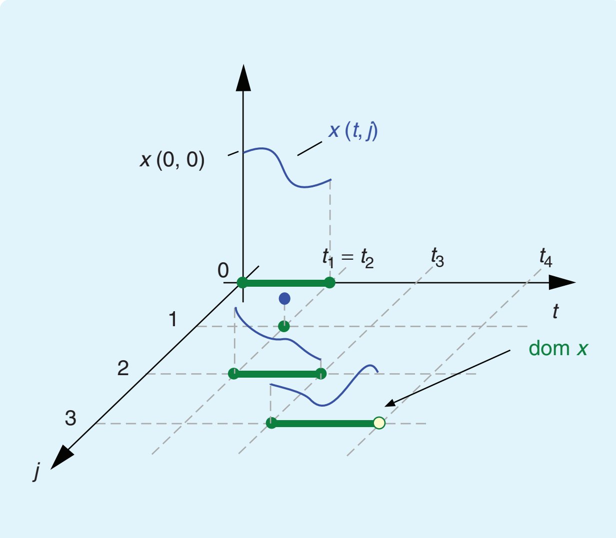

FIGURE 10 A solution to a hybrid system. The solution, which is denoted by x, has initial condition $x(0,0)$, is given by a hybrid arc, and has hybrid time domain dom x. The hybrid time domain E in Figure 8 is equal to dom x.

solution to a hybrid system and illustrates the parameterization by (t, j).

Every solution x to a hybrid system has a hybrid time domain $\operatorname{dom} x$ associated with it. However, for a given hybrid system, not every hybrid time domain is the domain of a solution to this system. This phenomenon goes beyond what can happen in unconstrained continuous-time systems, where solutions may blow up in finite time, and, hence, may be defined on only a bounded subset of $[0, \infty)$. For example, for a hybrid system in which two jumps cannot occur at the same time instant, a hybrid time domain that contains (t, j), $(t, j+1)$, and $(t, j+2)$ for some $t \in \mathbb{R}_{\geq 0}$, $j \in \mathbb{N}$ is not a domain of any solutions. Hence, we do not pick a hybrid time domain and then look for a solution to a hybrid system on that time domain. Rather, the domain must be generated along with the solution. Example 2 shows how a hybrid system can have complete solutions with different domains.

Example 2: Solutions to a Hybrid System

Consider the hybrid system in $\mathbb{R}^2$ given by $D = \{x \in \mathbb{R}^2 : x_1 \leq x_2 \leq 3\}$, $C = \mathbb{R}^2 \setminus D$, $f(x) = (-1, 1)$ for all $x \in C$, and $g(x) = (2x_1^2, 0)$, as depicted in Figure 11. Solutions flow in C with velocity $(-1, 1)$ and jump from points $x = (x_1, x_2) \in D$ to $(2x_1^2, 0)$. We consider maximal solutions from points of the form $(z, 0)$ with $z \in \{8, 4, 2, 1, 0\}$.

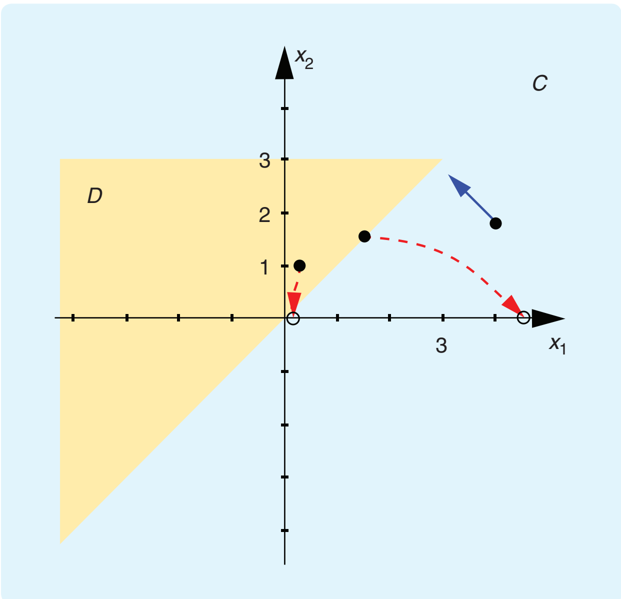

FIGURE 11 Data for the hybrid system in Example 2. The jump set $D = \{x \in \mathbb{R}^2 : x_1 \leq x_2 \leq 3\}$ is the shaded region. The flow set $C = \mathbb{R}^2 \setminus D$ is the complement of the jump set D. The flow map is $f(x) = (-1, 1)$ for all $x \in C$; hence solutions flow, when in C, with velocity $(-1, 1)$. The jump map is $g(x) = (2x_1^2, 0)$, where $x = (x_1, x_2)$; hence solutions jump from points $x \in D$ to $(2x_1^2, 0)$. A solution that starts in C with $x_2 \geq 3$ or with $x_1 + x_2 \geq 6$ flows and never jumps. Otherwise, a solution starting in C flows toward D, reaches D on the line $x_1 = x_2$, and then jumps to a point on the nonnegative x_1-axis. A solution that starts in D jumps to a point on the nonnegative x_1-axis.

The maximal solution starting from $(8, 0)$, denoted by x_a, flows and never hits the jump set D. More precisely, $\operatorname{dom} x_a = \mathbb{R}_{\geq 0} \times \{0\}$ and $x_a(t, 0) = (8 - t, t)$ for $t \geq 0$.

The maximal solution starting from $(4, 0)$, denoted by x_b, hits the jump set at the point $(2, 2)$, jumps to $(8, 0)$, and then flows without hitting the jump set D again. More precisely, $\operatorname{dom} x_b = [0, 2] \times \{0\} \cup [2, \infty) \times \{1\}$ and

$$x_b(t, j) = \begin{cases} (4 - t, t) & \text{for} \quad 0 \leq t \leq 2, j = 0, \\ (8 - (t - 2), t - 2) & \text{for} \quad 2 \leq t, j = 1. \end{cases}$$

The maximal solution starting from $(2, 0)$, denoted by x_c, hits the jump set at $(1, 1)$, jumps to $(2, 0)$, and repeats this behavior infinitely many times. More precisely, $\operatorname{dom} x_c = \bigcup_{j=0}^{\infty} [j, j+1] \times \{j\}$ and

$$x_c(t, j) = (2 - (t - j), t - j) \text{ for } j = 0, 1, 2, \ldots, t \in [j, j+1].$$

The maximal solution starting from $(1, 0)$, denoted by x_d, jumps infinitely many times and converges to the origin. More precisely, $\operatorname{dom} x_d = \bigcup_{j=0}^{\infty} [t_j, t_{j+1}] \times \{j\}$, where $a_0 = 1$, $a_j = a_{j-1}^2/2$ for $i = 1, 2, \ldots$, $t_j = \sum_{i=0}^{j-1} a_i/2$ and

$$x_d(t, j) = (a_j - (t - t_j), t - t_j) \\ \text{for } j = 0, 1, 2, \ldots, t \in [t_j, t_{j+1}].$$

Finally, the maximal solution starting from $(0, 0)$, denoted by x_e, never flows, has $\operatorname{dom} x_e = \{0\} \times \mathbb{N}$ and is given by $x_e(0, j) = (0, 0)$ for all $j \in \mathbb{N}$.

The solutions x_a, x_b, x_c, x_d, and x_e are complete. Note that solutions x_d and x_e are complete even though $\operatorname{dom} x_d$ and $\operatorname{dom} x_e$ do not contain points (t, j) with arbitrarily large t. ■

BASIC MATHEMATICAL PROPERTIES

Basic questions about solutions to dynamical systems concern existence, uniqueness, and dependence on initial conditions and other parameters. Existence and uniqueness questions for hybrid systems are addressed in "Existence, Uniqueness, and Other Well-Posedness Issues." The dependence of solutions on initial conditions, and compactness of the space of solutions to hybrid systems, are essential tools for developing a stability theory for hybrid systems. These tools depend on the concept of graphical convergence of hybrid arcs.

The main results of this section are proven in [26]. Closely related work includes [17], regarding compactness of the solution space, and [82], [9], and [12], regarding the dependence of solutions on initial conditions.

Basic Assumptions

The Basic Assumptions are the following three conditions on the data (C, F, D, G) of a hybrid system:

A1) C and D are closed sets in $\mathbb{R}^n$.

A2) $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on C, and such that $F(x)$ is nonempty and convex for each $x \in C$.

A3) $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on D, and such that $G(x)$ is nonempty for each $x \in D$.

A set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graph $\{(x, y) : x \in \mathbb{R}^n, y \in F(x)\} \subset \mathbb{R}^{2n}$ is closed. In terms of set convergence (see "Set Convergence"), F is outer semicontinuous if and only if, for each $x \in \mathbb{R}^n$ and each sequence $x_i \to x$, the outer limit $\limsup_{i\to\infty} F(x_i)$ is contained in $F(x)$. The mapping F is *locally bounded* on a set C if, for each compact set $K \subset C$, $F(K)$ is bounded. If F is locally bounded and $F(x)$ is closed for each $x \in \mathbb{R}^n$, then F is outer semicontinuous if and only if, for each x and $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x + \delta\mathbb{B}) \subset F(x) + \varepsilon\mathbb{B}$. A continuous function $f : C \to \mathbb{R}^n$, where C is closed, can be viewed as a set-valued mapping whose values on C consist of one point and are the empty set outside of C. Then, f is locally bounded on C and outer semicontinuous.

The Basic Assumptions combine what is typically assumed, in continuous- and in discrete-time systems, to obtain the continuous- and discrete-time versions of the results we state below for hybrid systems. "Robustness and Generalized Solutions" provides further motivation for introducing these assumptions.

Graphical Convergence and Sequential Compactness of the Space of Solutions

Solutions to differential equations and inclusions are continuous functions parameterized by $t \in \mathbb{R}_{\geq 0}$, and, thus, uniform distance and uniform convergence are adequate tools for analyzing them. To analyze solutions to hybrid systems, the more elaborate yet intuitive concepts of graphical distance and graphical convergence are needed. Before discussing these concepts, we briefly illustrate difficulties in using uniform distance for discontinuous functions.

Example 3: Bouncing Ball and the Uniform Distance Between Solutions

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \{x : x_1 \geq 0\}, \qquad f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \text{for all } x \in C,$$

$$D := \{x : x_1 = 0, x_2 \leq 0\}, \qquad g(x) := -\rho x \quad \text{for all } x \in D,$$

which is the bouncing ball system considered in Example S4 of "Existence, Uniqueness, and Other Well-Posedness Issues." We use $\gamma = 1$ and $\rho = 1/2$ in the calculations of this example.

Suppose that we abandon hybrid time domains and view the trajectories of the bouncing ball system as piecewise continuous functions parameterized by t only. For example, given $\delta \in [0, 1)$, the velocity resulting from dropping the ball from height $1 + \delta$ with initial velocity zero is given on the interval $t \in [0, 2]$ by

$$x_2^\delta(t) = \begin{cases} -t, & t \in [0, \sqrt{2(1+\delta)}), \\ -t + 3\sqrt{(1+\delta)/2}, & t \in [\sqrt{2(1+\delta)}, 2]. \end{cases}$$

See Figure 14. The uniform, that is, L_∞, distance between the velocities of two balls, one dropped from height 1 and the other from $1 + \delta$ with $\delta > 0$, is given by

$$\sup_{t\in[0,2]} |x_2^0(t) - x_2^\delta(t)| = 3\sqrt{2}/2$$

and does not decrease to zero as δ decreases to zero. In particular, the velocity of a ball dropped from height $1 + \delta$ does not converge uniformly, as δ decreases to zero, to the velocity of the ball dropped from height one. In other words, the velocity of the ball does not depend continuously on initial conditions, when uniform distance is used. On the intuitive level though, velocities and in fact the whole trajectories of balls dropped from nearby initial conditions appear close to one another. ■

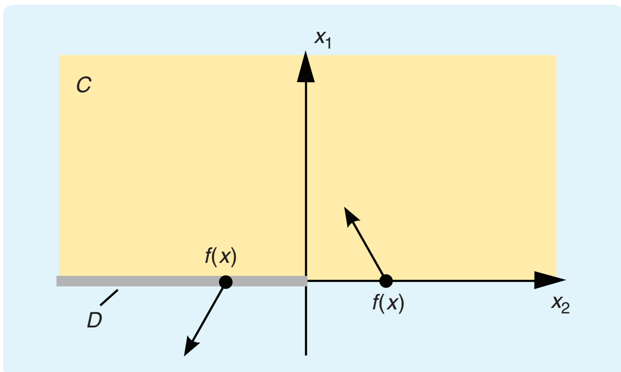

FIGURE 12 Flow and jump sets for the bouncing ball system in Example 3. The state x_1, represented on the vertical axis, is the height of the ball. The state x_2, represented on the horizontal axis, is the velocity of the ball. The flow set C is the closed upper half-plane. The flow map is shown at two points on the x_2-axis. Flow is possible from the point where the flow map is directed into the flow set but is not possible from the point where the flow map is directed out of the flow set. At the latter point, the flow map and the tangent cone to the flow set do not intersect, which makes flow impossible. Since the jump set D contains the latter point, a jump, that is, a bounce of the ball, is possible.

Existence, Uniqueness, and Other Well-Posedness Issues

The term "well-posed" for a mathematical problem usually implies that a solution exists, is unique, and depends continuously on the data of the problem. For hybrid dynamical systems, there are reasons to consider models that do not have solutions from some initial conditions, that do not have unique solutions, and that exhibit only a semicontinuous dependence on the data of the problem. Issues related to semicontinuous dependence on data are discussed in the main text. Here we discuss issues related to existence and uniqueness.

EXISTENCE OF SOLUTIONS AND BEHAVIOR OF MAXIMAL SOLUTIONS

At a point on the boundary of the flow set, if the flow map points out of the flow set then the hybrid system can fail to have a nontrivial solution, that is, a solution x such that dom x contains at least one point different from (0, 0). In the main text, we consider systems where existence of a nontrivial solution can fail at some points since this flexibility can be helpful in stability analysis. Indeed, for system (15), there are points on the boundary of C_i that do not admit a nontrivial solution. Nevertheless, the behavior of the nontrivial solutions to (15) are used to draw stability conclusions about mode-switching control for a disk drive system. For a further discussion of the role of existence in stability analysis, see "Why 'Pre'-Asymptotic Stability?"

To establish the existence of solutions, the following theorem points out that the existence of nontrivial solutions from a point $\xi \in C \cup D$ amounts to the existence of a discrete-time nontrivial solution or a continuous-time nontrivial solution.

Proposition S2

Suppose that $\mathcal{H}$ satisfies the Basic Assumptions and that $\xi \in \mathbb{R}^n$ is such that either $\xi \in D$ or there exists a nontrivial solution z to $\dot{z} \in F(z)$, that is, an absolutely continuous function $z:[0,\varepsilon] \to \mathbb{R}^n$ with $\varepsilon > 0$ satisfying $\dot{z}(t) \in F(z(t))$ for almost all $t \in [0,\varepsilon]$, such that $z(0) = \xi$ and $z(t) \in C$ for all $t \in (0,\varepsilon]$. Then there exists a nontrivial solution x to $\mathcal{H}$, with $x(0,0) = \xi$.

Indeed, if $\xi \in D$, then $G(\xi)$ is nonempty and $x(0,0) = \xi$, $x(0,1) \in G(\xi)$ provides a nontrivial solution to $\mathcal{H}$ with dom $x = (0,0) \cup (0,1)$. If there exists a nontrivial solution z to $\dot{z} \in F(z)$, as described in the assumption of Proposition S2, then $x(t,0) = z(t)$ provides a nontrivial solution to $\mathcal{H}$ with dom $x = [0,\varepsilon] \times \{0\}$.

Viability theory for differential inclusions provides sufficient conditions on F and C for flowing solutions to exist. One simple condition involves tangent cones to the set C at points near ξ. If C is closed, $\xi \in C$, and there exists a neighborhood U of ξ such that, for each $\eta \in U \cap C$, $F(\eta) \cap T_C(\eta) \neq \emptyset$, then there exist $\varepsilon > 0$ and $z:[0,\varepsilon] \to \mathbb{R}^n$ such that $z(0) = \xi$, $\dot{z}(t) \in F(z(t))$, and $z(t) \in C$ for almost all $t \in (0,\varepsilon]$. For details, see [S16, Prop. 3.4.2]. Figure S5 depicts a flow map F and the tangent cone to a given set C at several points $\xi \in C$. Additional discussion about existence of solutions to hybrid systems appears in [S17] and [51].

Existence of nontrivial solutions from each initial condition in $C \cup D$ has bearing on the structure of maximal solutions.

Theorem S3

Suppose that $\mathcal{H}$ satisfies the Basic Assumptions and, for every $\xi \in C \cup D$, there exists a nontrivial solution to $\mathcal{H}$ starting from ξ. Let x be a maximal solution to $\mathcal{H}$. Then exactly one of of the following three cases holds:

a) x is complete.

b) x blows up in finite (hybrid) time. In other words, $J = \max\{j : \text{there exists } t \text{ such that } (t,j) \in \operatorname{dom} x\}$ and $T = \sup\{t : (t,J) \in \operatorname{dom} x\}$ are both finite, the interval $\{t : (t,J) \in \operatorname{dom} x\}$ has nonempty interior, is open to the right so that $(T,J) \notin \operatorname{dom} x$, and $|x(t,J)| \to \infty$ when $t \to T$.

c) x eventually jumps out of $C \cup D$. In other words, $(T,J) \in \operatorname{dom} x$ and $x(T,J) \notin C \cup D$, where T and J defined in (b) are finite.

Note the lack of symmetry between continuous time and discrete time in b) and c) above. Finite-time blowup of a solution to a hybrid system cannot result from jumping, while a solution ending up outside $C \cup D$ cannot result from flowing. Consequently, no solutions can leave $C \cup D$ when $G(D) \subset C \cup D$. Finite-time blowup is excluded when, for example, C is bounded

We now consider the concept of graphical convergence of hybrid arcs along with the related concept of distance, which focus not just on the values of the hybrid arcs but on their graphs. One benefit of this approach is that different hybrid time domains can be handled easily. Note that bouncing balls dropped from different initial heights lead to different hybrid time domains of the hybrid arcs representing their heights and velocities. For example, the hybrid time domain of the trajectory of the bouncing ball dropped from height $1+\delta$, where $\delta \in [0,1)$, depends on δ since the time at which the jump occurs depends on the initial height. In fact, for $t \le 2$, the hybrid time domain is given by $[0, \sqrt{2(1+\delta)}] \times \{0\} \cup [\sqrt{2(1+\delta)}, 2] \times \{1\}$.

The *graph* of a hybrid arc x is the set

$$\operatorname{gph} x = \{(t,j,\xi) : (t,j) \in \operatorname{dom} x, \xi = x(t,j)\}.$$

The sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid arcs *converges graphically* if the sequence $\{\operatorname{gph} x_i\}_{i=1}^{\infty}$ of graphs converges in the sense of set convergence; see "Set Convergence." The *graphical limit* of a graphically convergent sequence or there exists $c \in \mathbb{R}_{\geq 0}$ such that, for each $x \in C$ and each $f \in F(x)$, $|f| \leq c(|x|+1)$.

Example S4: Bouncing Ball and Existence of Solutions

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \{x : x_1 \geq 0\}, \quad f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \text{for all } x \in C,$$
$$D := \{x : x_1 = 0,\, x_2 \leq 0\}, \quad g(x) := -\rho x \quad \text{for all } x \in D,$$

where $\rho \in [0, 1)$ is the restitution coefficient and $\gamma > 0$ is the gravity constant. This data models a ball bouncing on a floor. The state x_1 corresponds to height above the floor and x_2 corresponds to vertical velocity. Figure 12 depicts the flow and jump sets as well as the flow map at two points. The Basic Assumptions are satisfied. To verify sufficient conditions for the existence of nontrivial solutions from each point in $C \cup D$, it is enough to show that $f(\xi) \in T_C(\xi)$ for each $\xi \in C \setminus D$. For all $\xi \in C$ such that $\xi_1 > 0$, $T_C(\xi) = \mathbb{R}^2$. Consequently, for $\xi \in C \setminus D$ with $\xi_1 > 0$, $f(\xi) \in T_C(\xi)$ trivially holds. For all $\xi \in C$ with $\xi_1 = 0$, $T_C(\xi) = \mathbb{R}_{\geq 0} \times \mathbb{R}$, that is, the tangent cone is the right-half plane. For $\xi \in C \setminus D$ with $\xi_1 = 0$ we also have $\xi_2 > 0$, and consequently $f(\xi) \in T_C(\xi)$ holds. In summary, the assumption of Proposition S2 holds for every point $\xi \in C \cup D$, and nontrivial solutions to the hybrid system exist from each such point. Note though that $f(\xi) \notin T_C(\xi)$ for $\xi \in C \cap D$.

Additional arguments are needed to show that all maximal solutions are complete. Since $g(D) \subset C \cup D$, solutions do not jump out of $C \cup D$. Additional arguments, carried out when Example S4 is revisited in the section "Asymptotic Stability," show that all solutions are bounded and, hence, they do not blow up in finite time. Consequently, all maximal solutions are complete.

Solutions to the bouncing ball model exhibit Zeno behavior, as discussed in "Zeno Solutions." Simulations for the model are given in "Simulation in Matlab/Simulink." ■

UNIQUENESS OF SOLUTIONS

In dynamical systems, nonunique solutions can arise. One physically-motivated model that exhibits nonunique solutions is the differential equation corresponding the reverse-time evolution of a leaky bucket [S21, Example 4.2.1].

Nonuniqueness can occur in mathematical models that are designed to generate all possible solutions that meet certain conditions. For example, all Lipschitz continuous functions with Lipschitz constant equal to 1 are generated by the differential inclusion $\dot{x} \in [-1, 1]$ where $x \in \mathbb{R}$.

Additionally, consider a nonlinear control system $\dot{x} = f(x, u)$, where f is Lipschitz continuous and $u = \kappa(x)$ is a possibly discontinuous feedback. The family of locally absolutely continuous functions that arise as the limit of a sequence of solutions x_i to $\dot{x}_i = f(x_i, \kappa(x_i + e_i))$, $x_i(0) = x_0$ with e_i measurable and $\lim_{i\to\infty} \sup_{t\geq 0} |e_i(t)| = 0$ is equivalent to the family of solutions to the differential inclusion $\dot{x} \in F(x)$, $x(0) = x_0$ where $F(x) = \bigcap_{\delta>0} \overline{\text{con}} f(x, \kappa(x + \delta\mathbb{B}))$ [S20]. The signals e_i can be associated with arbitrarily small measurement noise in the control system. A version of this result for hybrid systems is

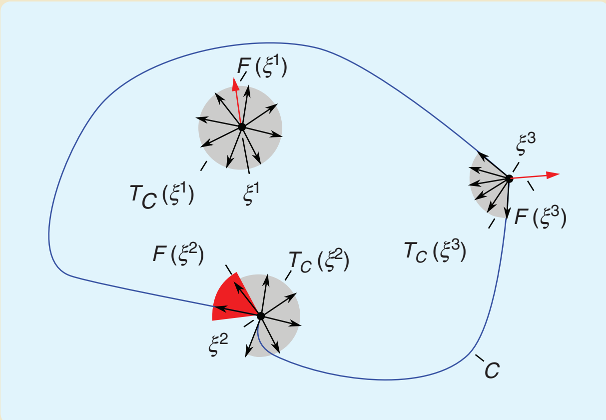

FIGURE S5 A flow map F and the tangent cone to a set C represented at several points $\xi \in C$. Directions in the flow map, single-valued at ξ^1, ξ^3 and set-valued at ξ^2, are shown in red. Tangent cones are represented in gray, sample directions in the tangent cones are in black. At points in the interior of C, such as ξ^1, the tangent cone is the entire space. At ξ^1 and ξ^2, the intersection between the flow map and the tangent cone is nonempty, whereas at ξ^3 this intersection is empty.

of hybrid arcs $\{x_i\}_{i=1}^{\infty}$, defined as the set-valued mapping whose graph is the limit of graphs of arcs x_i, need not be a hybrid arc. However, if the graphically convergent sequence is locally bounded and consists of solutions to a hybrid system that satisfies the Basic Assumptions, then the graphical limit is always a hybrid arc, and in fact, a solution to the hybrid system. The following result, given in [69, Thm. 5.36] and [26, Lem. 4.3], states the sequential compactness of the space of solutions to a hybrid system that satisfies the Basic Assumptions.

Theorem 4

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of solutions to a hybrid system $\mathcal{H}$ meeting the Basic Assumptions. Suppose that the sequence $\{x_i\}_{i=1}^{\infty}$ is locally uniformly bounded in the sense that, for each $\tau > 0$, there exists a compact set $K_\tau \subset \mathbb{R}^n$ such that, for each $i = 1, 2, \ldots$ and each $(t, j) \in \text{dom}\, x_i$ with $t + j \leq \tau$, it follows that $x_i(t, j) \in K_\tau$. Then the sequence $\{x_i\}_{i=1}^{\infty}$ has a graphically convergent subsequence; moreover, if the sequence $\{x_i\}_{i=1}^{\infty}$ is graphically convergent, then its graphical limit is a hybrid arc that is a solution to the hybrid system $\mathcal{H}$.

discussed in "Robustness and Generalized Solutions." The solutions to $\dot{x} \in F(x)$, $x(0) = x_0$ can be nonunique. For example, for a system with $x \in \mathbb{R}$, $u \in \mathbb{R}$, $f(x, u) = u$, $\kappa(x) = 1$ for $x \geq 0$, and $\kappa(x) = -1$ for $x < 0$, we get $F(0) = [-1, 1]$, $F(x) = 1$ for $x > 0$, and $F(x) = -1$ for $x < 0$. Thus, for $x_0 = 0$, there are multiple solutions. One solution is $x(t) = 0$ for all $t \geq 0$. Additional solutions are $x(t) = 0$ for $t \in [0, \bar{t}]$, $x(t) = (t - \bar{t})$ for $t > \bar{t} \geq 0$ or $x(t) = -(t - \bar{t})$ for $t > \bar{t} \geq 0$.

Nonunique solutions arise when developing a model for switched systems that generates all solutions under arbitrary switching among a finite set of locally Lipschitz vector fields. This set of solutions is equivalent to the set of solutions to a differential inclusion where the set-valued right-hand side at a point is equal to the union of the vector fields at that point. See [S18, Chapter 4]. For this differential inclusion, solutions are not unique. Each vector field produces a solution. Additional solutions are produced by following one vector field for some amount of time, switching to a different vector field for some amount of time, and so on.

A similar situation arises when modeling switched systems under a restricted class of switching signals. In this case, the switched system can be modeled by an autonomous hybrid system that produces all of the possible solutions produced by switching signals that belong to the class. In this setup, for each initial condition of the hybrid system, there are many solutions, each generated by a particular switching signal. See "Switching Systems."

Similarly, when addressing networked control systems, an autonomous hybrid model is used that generates all solutions that can occur for a class of transmission time sequences. See Example 27.

Sufficient conditions for uniqueness can be invoked, if desired. Several sufficient conditions for uniqueness of solutions to ordinary differential equations are given in the literature. The simplest condition is that the differential equation's vector field is locally Lipschitz continuous. Relaxed conditions also exist. See [S19] for further discussion.

The following result characterizes uniqueness of solutions to hybrid systems. Formally, *uniqueness of solutions* holds

The first conclusion of Theorem 4 is a property of set convergence. The second conclusion is specific to the hybrid system setting. In summary, Theorem 4 states that, from each locally uniformly bounded sequence of solutions to a hybrid system that satisfies the Basic Assumptions, we can extract a graphically convergent subsequence whose graphical limit is a solution to the hybrid system. Consequences of Theorem 4 for asymptotic stability in hybrid systems are discussed in the "Asymptotic Stability" section.

Graphical convergence of hybrid arcs to a hybrid arc has an equivalent pointwise description. Consider a sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid arcs and a hybrid arc x. Then $\{x_i\}_{i=1}^{\infty}$ converges graphically to x if and only if the following two conditions hold:

i) For every sequence of points $(t_i, j) \in \operatorname{dom} x_i$ such that the sequences $\{t_i\}_{i=1}^{\infty}$ and $\{x_i(t_i, j)\}_{i=1}^{\infty}$ are convergent, it follows that $(t, j) \in \operatorname{dom} x$, where $t = \lim_{i\to\infty} t_i$ and $x(t, j) = \lim_{i\to\infty} x_i(t_i, j)$.

ii) For every $(t, j) \in \operatorname{dom} x$ there exists a sequence $\{(t_i, j)\}_{i=1}^{\infty}$, where $(t_i, j) \in \operatorname{dom} x_i$, such that $t = \lim_{i\to\infty} t_i$ and $x(t, j) = \lim_{i\to\infty} x_i(t_i, j)$.

Figure 13 depicts several solutions from a graphically convergent sequence of solutions to the bouncing ball system in Example 3.

Dependence of Solutions on Initial Conditions

One of the consequences of Theorem 4 is the semicontinuous dependence of solutions to a hybrid system on initial conditions. To state this result rigorously, in Theorem 5, we define a concept of distance between hybrid arcs that is closely related to graphical convergence.

Given $\tau \geq 0$ and $\varepsilon > 0$, the hybrid arcs x and y are (τ, ε)-*close* if the following conditions are satisfied:

a) For each $(t, j) \in \operatorname{dom} x$ with $t + j \leq \tau$ there exists $s \in \mathbb{R}_{\geq 0}$ such that $(s, j) \in \operatorname{dom} y$, $|t - s| < \varepsilon$, and

$$|x(t, j) - y(s, j)| < \varepsilon.$$

b) For each $(t, j) \in \operatorname{dom} y$ with $t + j \leq \tau$ there exists $s \in \mathbb{R}_{\geq 0}$ such that $(s, j) \in \operatorname{dom} x$, $|t - s| < \varepsilon$, and

$$|y(t, j) - x(s, j)| < \varepsilon.$$

The concept of (τ, ε)-closeness provides an equivalent characterization of graphical convergence of hybrid arcs. Consider a locally uniformly bounded sequence of hybrid arcs $\{x_i\}_{i=1}^{\infty}$ and a hybrid arc x. Then the sequence $\{x_i\}_{i=1}^{\infty}$ converges graphically to x if and only if, for every $\tau \geq 0$ and $\varepsilon > 0$, there exists i_0 such that, for all $i > i_0$, the hybrid arcs x_i and x are (τ, ε)-close.

Equipped with the concept of (τ, ε)-closeness, we again revisit Example 3.

Example 3 Revisited: Bouncing Ball and (τ, ε)-Closeness

Consider the bouncing ball model with $\gamma = 1$ and $\rho = 1/2$. Given $\delta \in [0, 1)$, the hybrid arc representing the velocity of the ball dropped from height $1 + \delta$ with velocity zero, for times $t \leq 2$, is given by

$$x_2^{\delta}(t, j) = \begin{cases} -t, & t \in [0, \sqrt{2(1+\delta)}],\ j = 0, \\ -t + 3\sqrt{(1+\delta)/2}, & t \in [\sqrt{2(1+\delta)}, 2],\ j = 1. \end{cases}$$

Consider the hybrid arcs, x_2^0 and x_2^{δ}, where $\delta \in (0, 1)$. These arcs are (τ, ε)-close, with any $\tau \geq 0$ and $\varepsilon = 3\sqrt{2}(\sqrt{1+\delta} - 1)/2$. To show this, it is sufficient to for the hybrid system $\mathcal{H}$ if, for any two maximal solutions x_1 and x_2 to $\mathcal{H}$, if $x_1(0,0) = x_2(0,0)$, then $\operatorname{dom} x_1 = \operatorname{dom} x_2$ and $x_1(t,j) = x_2(t,j)$ for all $(t,j) \in \operatorname{dom} x_1$.

Proposition S5

Uniqueness of solutions holds for a hybrid system with data (C, F, D, G) if and only if the following conditions hold:

1) For each initial point $\xi \in C$ there exists a unique maximal solution to the differential inclusion $\dot{z}(t) \in F(z(t))$ satisfying $z(0) = \xi$ and $z(t) \in C$.
2) For each initial point $\xi \in D$, $G(\xi)$ is a singleton.
3) For each initial point $\xi \in C \cap D$, there are no nontrivial solutions to $\dot{z}(t) \in F(z(t))$ satisfying $z(0) = \xi$ and $z(t) \in C$.

The first condition is ensured when F is a locally Lipschitz continuous function but can hold when F is set valued. In contrast, set-valuedness of G at a point in D immediately leads to non-unique solutions. Hence, the second condition cannot be weakened. The third condition is ensured when for each $\xi \in C \cap D$, $T_C(\xi) \cap F(\xi) = \emptyset$, where $T_C(\xi)$ denotes the tangent cone to C at ξ. This condition indicates that, roughly speaking, the vector field given by F should point to the outside of C at points in $C \cap D$.

The bouncing ball model in Example S4 satisfies the three conditions of Proposition S5 and thus generates unique solutions from all initial conditions.

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consider $\tau = 3$. Indeed, since we are considering arcs with domains restricted to (t, j) such that $t + j \le 3$, $(3, \varepsilon)$-closeness implies (τ, ε)-closeness for all $\tau \ge 0$.

To verify condition a) of (τ, ε)-closeness, note that, for each $(t, 0) \in \operatorname{dom} x_2^0$ with $t \le 3$, we have $(t, 0) \in \operatorname{dom} x_2^\delta$ and $x_2^0(t,0) = x_2^\delta(t,0)$. In practical terms, the velocities of the balls are the same until the ball dropped from the lower height bounces. For each $(t,1) \in \operatorname{dom} x_2^0$ with $t \le 2$, that is, with $t + 1 \le 3$, there exists $(s, 1) \in \operatorname{dom} x_2^\delta$ with $|t - s| \le \sqrt{2(1+\delta)} - \sqrt{2}$ and $|x_2^0(t,j) - x_2^\delta(s,j)| \le 3\sqrt{2}(\sqrt{1+\delta} - 1)/2$. In fact, for $t \le \sqrt{2(1+\delta)}$, we can take $s = \sqrt{2(1+\delta)}$, while, for the remaining t's, we can take $s = t$. Consequently, for each $(t,j) \in \operatorname{dom} x_2^0$ with $t + j \le \tau = 3$ there exists $s \ge 0$ such that $(s,j) \in \operatorname{dom} x_2^\delta$, $|t - s| < \varepsilon$ and $|x(t,j) - y(s,j)| < \varepsilon$, where $\varepsilon = 3\sqrt{2}(\sqrt{1+\delta} - 1)/2$. A similar calculation can be carried out for condition b) of (τ, ε)-closeness.

Thanks to the equivalent characterization of graphical convergence in terms of (τ, ε)-closeness, the $(\tau, 3\sqrt{2}(\sqrt{1+\delta} - 1)/2)$-closeness of x_2^0 and x_2^δ, for each $\tau \ge 0$, implies that x_2^δ converge graphically, as $\delta \to 0$, to x_2^0. ∎

For the bouncing ball, whose maximal solution is unique for each initial condition, we can establish that the arc x_2^δ depends continuously on δ at zero, for an appropriately defined concept of continuous dependence. In the absence of uniqueness of solutions, continuous dependence cannot be expected. Continuous dependence fails in simpler settings, too. For the differential equation $\dot{x} = 2\sqrt{|x|}$, solutions $(t + \sqrt{\delta})^2$ from initial points $x(0) = \delta > 0$ converge uniformly on compact time intervals to t^2, as $\delta \to 0$. But from the initial point zero, there also exists a solution, identically equal to zero, which is not a limit of any sequence of solutions from positive initial points. This simple setting already illustrates what outer semicontinuous dependence on initial conditions is, namely,

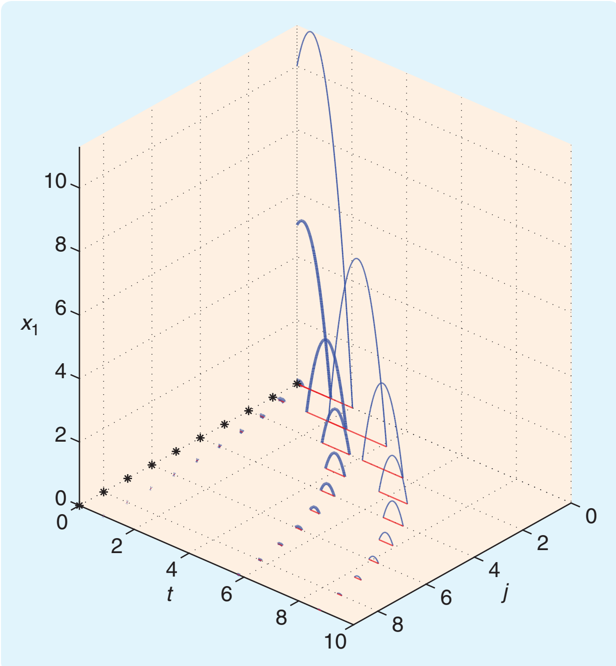

FIGURE 13 The height x_1 of solutions to the bouncing ball system in Example 3. The height of three solutions with nonzero initial height and velocity given by (10, 5), (5, 2), and (0.1, 0.5), are shown in blue. The solution from (0, 0), denoted in black with $*$ marks, is a solution that jumps and does not flow. Initial conditions closer to the origin result in the blue graphs that more closely resemble the black graph. In other words, solutions with initial conditions close to (0, 0) are graphically close to the solution from (0, 0).

that each solution from the initial point $\delta > 0$ close to zero is close to some solution from zero. In the language of sequences, the limit of a sequence of solutions from initial points close to zero is a solution from zero, even though some other solutions from zero are not limits of any solutions from nearby initial points. We stress here, again, that continuous dependence on initial points is not needed to develop fundamental stability theory results, such as converse Lyapunov theorems and invariance principles.

In a hybrid system, even the domains of solutions with close initial points can differ significantly. An example illustrating this phenomenon is the system on $\mathbb{R}$ with $C = \mathbb{R}_{\geq 0}$, $f(x) = -x$, $D = (-\infty, 0]$, and $g(x) = x/2$. For all $\tau > \varepsilon > 0$, the solution $x(t, 0) = 0$ for all $t \geq 0$, from the initial point 0, is not (τ, ε)-close to any of the solutions $y(0, j) = -\delta/2^j$, $j = 1, 2, \ldots$, independently of how small $\delta > 0$ is. Note, however, that the solutions $y(0, j) = -\delta/2^j$, $j = 1, 2, \ldots$ converge to another solution from zero, namely $z(0, j) = 0$, $j = 1, 2, \ldots$.

The following result, proven in [26, Cor. 4.8], concerns the outer semicontinuous dependence of solutions to hybrid systems on initial conditions.

Theorem 5

Suppose that the hybrid system $\mathcal{H}$ meets the Basic Assumptions and $\xi \in \mathbb{R}^n$ is such that each maximal solution to $\mathcal{H}$ from ξ is either complete or bounded. Then, for every $\tau \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for each solution x^δ to $\mathcal{H}$ with $|x^\delta(0, 0) - \xi| \leq \delta$, there exists a solution x to $\mathcal{H}$ with $x(0, 0) = \xi$ such that x^δ and x are (τ, ε)-close.

Additional Consequences of Sequential Compactness

Sequential compactness of the space of solutions of a hybrid system, stated in Theorem 4, results in uniformity of various properties. We illustrate this fact with a property related to the lack of complete solutions that jump but do not flow. The result below, and the phrase "uniformly non-Zeno," are taken from [17]. See "Zeno Solutions" for a further discussion of the Zeno phenomenon.

Proposition 6

Consider the hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, and a compact set $K \subset \mathbb{R}^n$ that is forward invariant, in other words, such that all solutions x to $\mathcal{H}$ with $x(0, 0) \in K$ satisfy $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. Then, exactly one of the following conditions is satisfied:

a) There exists a complete solution x to $\mathcal{H}$ with $x(0, 0) \in K$ and $\operatorname{dom} x = \{0\} \times \mathbb{N}$.

b) The set of all solutions with initial points in K is uniformly non-Zeno, that is, there exist $T > 0$ and $J \in \mathbb{N}$ such that, for each solution x to $\mathcal{H}$ with $x(0, 0) \in K$, each $(t, j), (t', j') \in \operatorname{dom} x$ with $|t - t'| \leq T$ satisfies $|j - j'| \leq J$.

For illustration purposes, we outline a proof. The conditions a) and b) are mutually exclusive. Negating b) yields a sequence of solutions $\{x_i\}_{i=1}^{\infty}$, with $x_i(0, 0) \in K$, and, for each $i = 1, 2, \ldots$, $(t_i, j_i), (t_i', j_i') \in \operatorname{dom} x_i$ with $|t_i - t_i'| \leq 1/i$ and $|j_i - j_i'| > i$. Without loss of generality, we can assume that $t_i \leq t_i'$ and $j_i < j_i'$, for all $i = 1, 2, \ldots$. Define a sequence of hybrid arcs y_i by $y_i(t, j) = x_i(t + t_i, j + j_i)$, which implicitly defines $\operatorname{dom} y_i$ to be the tail of $\operatorname{dom} x_i$. Forward invariance of K implies that, for every i, $y_i(t, j) \in K$ for all $(t, j) \in \operatorname{dom} y_i$. Thus, the sequence $\{y_i\}_{i=1}^{\infty}$ is locally uniformly bounded in the sense of Theorem 4. Part a) of that theorem implies that there exists a graphically convergent subsequence of the sequence y_i. Part b) implies that the graphical limit of the subsequence, denoted y, is a solution to $\mathcal{H}$. Clearly, $y(0, 0) \in K$. It remains to conclude that y is complete and never flows. This conclusion comes out from the fact that the point $(t_i' - t_i, j_i' - j_i)$ is an element of $\operatorname{dom} y_i$, which says that y_i jumps at least i times in at most $1/i$ time units, and from the definition of graphical convergence.

Similar, yet even simpler arguments, are used in the following result to establish the compactness of reachable sets.

Proposition 7 [26, Cor. 4.7]

Suppose that the hybrid system $\mathcal{H}$ satisfies the Basic Assumptions. Consider a compact set K such that every solution to $\mathcal{H}$ starting in K is either complete or bounded, and $m > 0$. Then, the reachable set

$$\mathcal{R}_{\leq m}(K) := \{x(t, j) \mid x(0, 0) \in K, (t, j) \in \operatorname{dom} x, t + j \leq m\}$$

is compact.

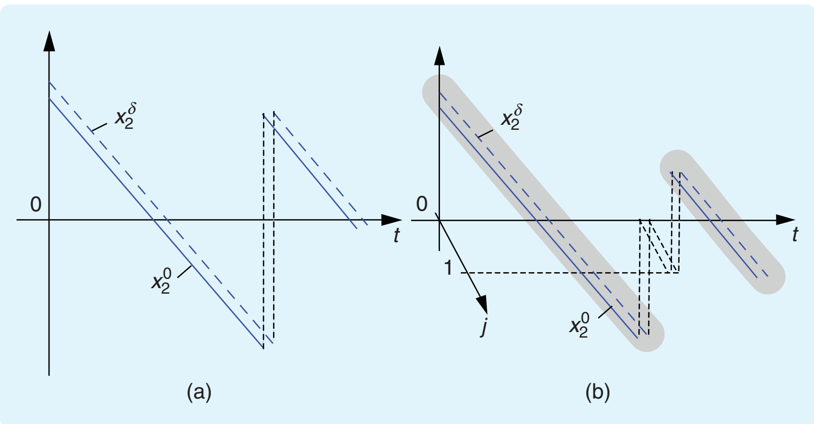

FIGURE 14 Components x_2^0, x_2^δ of solutions from "Example 3 Revisited," representing the velocities of bouncing balls dropped from heights 1 and $1 + \delta$, respectively. (a) Velocities parameterized by t. The velocities are not close in the uniform distance since their difference, at times after the first ball bounces and before the second ball bounces, is large. (b) Velocities on hybrid time domains. The shaded neighborhoods of their graphs indicate that the velocities are graphically close.

This article is a tutorial on modeling the dynamics of hybrid systems, on the elements of stability theory for hybrid systems, and on the basics of hybrid control.

Outer semicontinuous dependence of solutions to a hybrid system on initial conditions, stated in Theorem 5, can be generalized to allow state perturbations of the hybrid system. Given a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^n$, a continuous function $\sigma: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $\delta \geq 0$, consider a hybrid system $\mathcal{H}_{\delta\sigma}$ with the state $x \in \mathbb{R}^n$ and data

$$C_{\delta\sigma} := \{x: (x + \delta\sigma(x)\mathbb{B}) \cap C \neq \emptyset\}, \tag{8}$$

$$F_{\delta\sigma}(x) := \overline{\text{con}}\, F((x + \delta\sigma(x)\mathbb{B}) \cap C) + \delta\sigma(x)\mathbb{B} \quad \text{for all } x \in C_{\delta\sigma}, \tag{9}$$

$$D_{\delta\sigma} := \{x: (x + \delta\sigma(x)\mathbb{B}) \cap D \neq \emptyset\}, \tag{10}$$

$$G_{\delta\sigma}(x) := \{v: v \in g + \delta\sigma(g)\mathbb{B},\ g \in G((x + \delta\sigma(x)\mathbb{B}) \cap D)\} \quad \text{for all } x \in D_{\delta\sigma}. \tag{11}$$

Perturbations, of the system $\mathcal{H}$, of this kind are used in the analysis of robustness of asymptotic stability. Figure 15 illustrates the idea behind perturbations of the sets C and D.

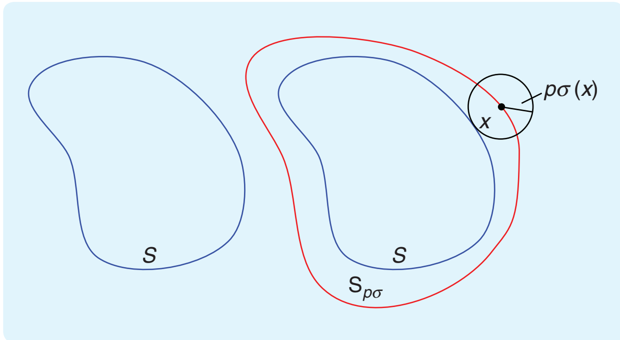

FIGURE 15 Enlargement of a set S due to a state-dependent perturbation of size $\rho\sigma$. The perturbed set $S_{\rho\sigma}$ contains all points x with Euclidean distance from the unperturbed set S no larger than $\rho\sigma(x)$.

Theorem 8 [26, Corollary 5.5]

Suppose that the hybrid system $\mathcal{H}$ satisfies the Basic Assumptions and $\xi \in \mathbb{R}^n$ is such that each maximal solution to $\mathcal{H}$ from ξ is either complete or bounded. Let $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous function. Then, for every $\tau \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for each solution x^δ to $\mathcal{H}_{\delta\sigma}$ with $|x^\delta(0,0) - \xi| \leq \delta$, there exists a solution x to $\mathcal{H}$ with $x(0,0) = \xi$ such that x^δ and x are (τ, ε)-close.

ASYMPTOTIC STABILITY

This section addresses asymptotic stability in hybrid dynamical systems, including basic definitions and equivalent characterizations of asymptotic stability. Examples are provided to illustrate the main concepts. Some of the definitions and results have a formulation that is slightly different from classical stability theory for differential equations. This difference is mainly due to the fact that existence of solutions is a more subtle issue for hybrid systems than it is for classical systems. See "Existence, Uniqueness, and Other Well-Posedness Issues." Otherwise, the stability theory results that are available for hybrid systems typically parallel the results that are available for classical systems.

Definition and Examples

As discussed in "Motivating Stability of Sets," the solutions of a dynamical system sometimes converge to a set rather than to an equilibrium point. Thus, we study asymptotic stability of sets. The scope is limited to compact sets for simplicity.

Roughly speaking, a compact set $\mathcal{A}$ is asymptotically stable if solutions that start close to $\mathcal{A}$ stay close to $\mathcal{A}$, and complete solutions that start close to $\mathcal{A}$ converge to $\mathcal{A}$. We now make this concept precise. A compact set $\mathcal{A}$ is *stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0,0)|_{\mathcal{A}} \leq \delta$ implies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all solutions x to $\mathcal{H}$ and all $(t,j) \in \text{dom}\, x$. The notation $|x|_{\mathcal{A}} = \min\{|x - y| : y \in \mathcal{A}\}$ indicates the distance of the vector x to the set $\mathcal{A}$. If $\mathcal{A}$ is the origin then $|x|_{\mathcal{A}} = |x|$. A compact set $\mathcal{A}$ is *pre-attractive* if there exists a neighborhood of $\mathcal{A}$ from which each solution is bounded and the complete solutions converge to $\mathcal{A}$, that is, $|x(t,j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, where $(t,j) \in \text{dom}\, x$. The prefix "pre-" is used since it is not a requirement that maximal solutions starting near $\mathcal{A}$ be complete. See "Why 'Pre'-Asymptotic Stability?" for additional reasons to consider asymptotic stability without insisting on completeness of solutions. A compact set $\mathcal{A}$ is *pre-asymptotically stable* if it is stable and pre-attractive.

For a pre-asymptotically stable compact set $\mathcal{A} \subset \mathbb{R}^n$, its *basin of pre-attraction* is the set of points in $\mathbb{R}^n$ from which each solution is bounded and the complete solutions converge to $\mathcal{A}$. By definition, the basin of pre-attraction contains a neighborhood of $\mathcal{A}$. In addition, each point in $\mathbb{R}^n \backslash (C \cup D)$ belongs to the basin of pre-attraction since no solution starts at a point in $\mathbb{R}^n \backslash (C \cup D)$. If the basin of pre-attraction is $\mathbb{R}^n$ then the set $\mathcal{A}$ is *globally pre-asymptotically stable*. We drop "pre" when all solutions starting in the basin of pre-attraction are complete.

Set Convergence

The concept of convergence of sets, as well as many other elements of set-valued analysis, are important ingredients of modern analysis. For example, in optimization and optimal control, set convergence helps in the study of how sets of optimal solutions or optimal controls, when these solutions or controls are not unique, depend on parameters or initial conditions. The brief exposition of set convergence given below follows the terminology and definitions of [69].

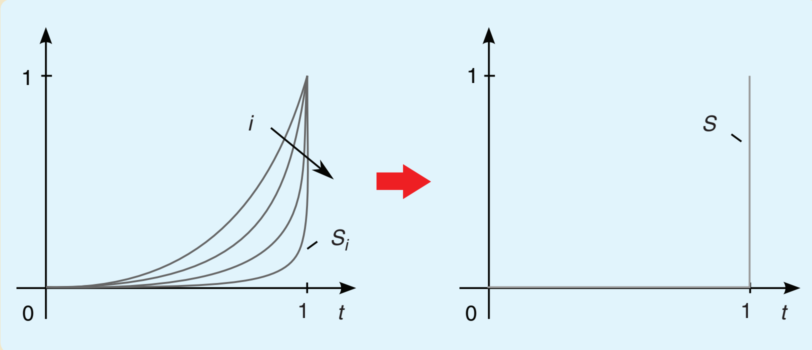

FIGURE S6 A sequence of sets S_i converging to the reflected L-shaped set S. Equivalently, the sequence of functions with graphs given by S_is converges graphically to a set-valued mapping with graph given by S.

Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^n$. The *outer limit* of this sequence, denoted $\limsup_{i\to\infty} S_i$, is the set of all accumulation points of sequences of points $x_i \in S_i$; more precisely, it is the set of all points $x \in \mathbb{R}^n$ or which there exists a sequence of points x_k, $k = 1, 2, \ldots$, and a subsequence $\{S_{i_k}\}_{k=1}^{\infty}$ of the sequence $\{S_i\}_{i=1}^{\infty}$ such that $x_k \in S_{i_k}$ and $x_k \to x$ as $k \to \infty$. The *inner limit* of this sequence, denoted $\lim \mathrm{int}_{i\to\infty} S_i$, is the set of all points $x \in \mathbb{R}^n$ for which there exists a sequence of points $x_i \in S_i$, $i = 1, 2, \ldots$, such that $x_i \to x$ as $i \to \infty$. The *limit* of the sequence $\{S_i\}_{i=1}^{\infty}$, denoted $\lim_{i\to\infty} S_i$, exists if the inner and outer limits are equal, in which case $\lim_{i\to\infty} S_i = \limsup_{i\to\infty} S_i = \liminf_{i\to\infty} S_i$.

The outer and inner limit always exist, but these limits may be empty. Furthermore, the outer limit is nonempty when the sequence S_i does not escapes to infinity, in the sense that, for each $r > 0$, there exists $i_r \in \mathbb{N}$ such that, for all $i > i_r$, the intersection of S_i with a ball of radius r is empty. Finally, the inner and outer limits—and thus the limit, if it exists—are closed, independently of whether or not each of the S_is is closed.

Some basic examples of set convergence are the following:

a) If each S_i is a singleton, that is, $S_i = \{s_i\}$, where $s_i \in \mathbb{R}^n$, then the outer limit of the sequence $\{S_i\}_{i=1}^{\infty}$ is the set of accumulation points of the sequence s_i; the inner limit is nonempty if and only if the sequence s_i is convergent, in which case $\liminf_{i\to\infty} S_i = \lim_{i\to\infty} S_i = \lim_{i\to\infty} s_i$.

b) Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of nonnegative numbers. For the sequence of closed balls of radius r_i, $S_i = r_i\mathbb{B}$, it follows that $\limsup_{i\to\infty} S_i = \bar{r}\mathbb{B}$, where $\bar{r}$ is the upper limit of the sequence $\{r_i\}_{i=1}^{\infty}$, and $\liminf_{i\to\infty} S_i = \underline{r}\mathbb{B}$, where $\underline{r}$ is the lower limit of the sequence $\{r_i\}_{i=1}^{\infty}$. When $\bar{r} = \infty$, $\bar{r}\mathbb{B}$ corresponds to all of $\mathbb{R}^n$.

c) Let $S_i = \{(t, t/i) : t \in \mathbb{R}\}$, in other words, let S_i be the graph of the linear function $t \mapsto t/i$. Then the sequence $\{S_i\}_{i=1}^{\infty}$ is convergent, and the limit is the graph of the function $t \mapsto 0$ defined on $\mathbb{R}$.

d) Let $S_i = \{(t, t^i) : t \in [0, 1]\}$, in other words, let S_i be the graph of the function $t \mapsto t^i$ on $[0, 1]$. Then the sequence $\{S_i\}_{i=1}^{\infty}$ is convergent, and the limit S has the reflected L shape, that is, $S = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$. Figure S6 shows S_i and S.

Example c) suggests that a sequence $\{S_i\}_{i=1}^{\infty}$ of sets can converge to S even though the Hausdorff distance between S_i and S is infinite for all $i = 1, 2, \ldots$. The Hausdorff distance can be used to characterize set convergence of bounded

Example 9: A Planar System

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \{x : x_1 \geq 0\}, \quad f(x) := \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} x \quad \text{for all } x \in C,$$
$$D := \{x : x_1 = 0, x_2 \leq 0\}, \quad g(x) := -\gamma x \quad \text{for all } x \in D,$$

where $\gamma > 0$, $\omega > 0$, and $\alpha \in \mathbb{R}$. During flows, a solution rotates in the clockwise direction through the set C until reaching the negative x_2-axis. The maximum amount of time spent in C before a jump occurs is π/ω units of time. The sign of α determines whether the norm of a solution increases or decreases during flows. At points on the negative x_2-axis, a solution jumps to the positive x_2-axis. The sign of $\gamma - 1$ determines whether the norm of a solution increases or decreases during jumps. Figure 16 illustrates different possibilities. If $\exp(\alpha\pi/\omega)\gamma < 1$ then the norm of a solution decreases over one cycle from the positive x_2-axis and back again. In this case, each solution x that starts close to the origin remains close to the origin and tends toward the origin as $t + j \to \infty$, where $(t, j) \in \operatorname{dom} x$. Thus, the origin is globally asymptotically stable. ■

Example 1 Revisited: Dual-Mode Control for Disk Drives

We start with a preliminary observation about a hybrid system that has no discrete-time dynamics and has the continuous-time dynamics sequences of sets. More precisely, a uniformly bounded sequence $\{S_i\}_{i=1}^{\infty}$ of closed sets converges to a closed set S if and only if the Hausdorff distance between S and S_i converges to zero.

Examples c) and d) also illustrate the concept of graphical convergence of functions or set-valued mappings. A sequence $\{F_i\}_{i=1}^{\infty}$ of set-valued mappings $F:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *converges graphically* to $F:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ if the sequence of graphs of F_is, which are subsets of $\mathbb{R}^{m+n}$, converges to the graph of F, in the sense of set convergence. In c), the sequence of functions $t \mapsto t/i$ converges graphically, and pointwise, to the function $t \mapsto 0$. In d), the sequence of functions $t \mapsto t^i$ on $[0, 1]$ converges graphically not to a function but to a set-valued mapping. More precisely, the graphical limit of functions $t \mapsto t^i$ on $[0, 1]$ is equal to zero for $t \in [0, 1)$ and $[0, 1]$ for $t = 1$, as can be seen in Figure S6. Note that the sequence of functions $t \mapsto t^i$ on $[0, 1]$ converges pointwise to the function that is equal to zero for $t \in [0, 1)$ and equal to one at $t = 1$; however, this convergence is not uniform on $[0, 1]$.

A natural example of the outer limit of a sequence of sets is provided by omega limits of solutions to dynamical systems. To illustrate omega limits in a continuous time setting, let $x:\mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be a function. Although x may be a solution to a differential equation, continuity properties of x are irrelevant for the following discussion. The omega limit of x, denoted $\omega(x)$, is the set of all points $\xi \in \mathbb{R}^n$ for which there exists a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_i \to \infty$ such that $x(t_i) \to \xi$. Then

$$\omega(x) = \limsup_{i\to\infty} S_i$$

where $S_i = \{x(t) : t \geq i\}$. See Figure S7. Since the sequence S_i is nonincreasing, the outer limit is in fact the limit. A property of set limits, as noted above, implies that $\omega(x)$ is closed. Another property implies that if $|x(t)|$ does not diverge to infinity then $\omega(x)$ is nonempty. In fact, properties of set convergence imply that if x is bounded, then $\omega(x)$ is nonempty and compact and x converges to $\omega(x)$. The convergence of x to $\omega(x)$ follows from the following property [69, Thm. 4.10] of set convergence: given a sequence of sets $\{S_i\}_{i=1}^{\infty}$ and a closed set S, $\lim_{i\to\infty} S_i = S$ if and only if, for all $\varepsilon > 0$ and $\rho > 0$, there exists $i_0 \in \mathbb{N}$ such that, for all $i > i_0$,

$$S \cap \rho\mathbb{B} \subset S_i + \varepsilon\mathbb{B}, \qquad S_i \cap \rho\mathbb{B} \subset S + \varepsilon\mathbb{B}. \tag{S5}$$

The second inclusion (S5), with ρ such that $x(t) \in \rho\mathbb{B}$ for all $t \in \mathbb{R}_{\geq 0}$, implies that x converges to $\omega(x)$. The characterization of set convergence in (5) is behind the relationship between graphical convergence of hybrid arcs and the concept of (τ, ε)-closeness between hybrid arcs, as used in the "Basic Mathematical Properties" section.

For references using set-valued analysis in dynamical systems, in particular, in differential inclusions, see [S22], [S16], and [S24]; for relevance to optimal control, see [S23].

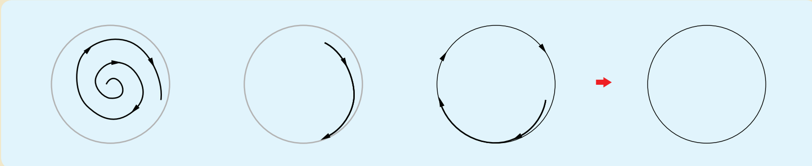

FIGURE S7 A solution to a differential equation approaching a periodic solution that covers a circle. The circle is the omega-limit of the solution. Tails of the solution converge, in the sense of set convergence, to the circle.

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[S23] F. H. Clarke, *Optimization and Nonsmooth Analysis.* Philadelphia: SIAM, 1990.

[S24] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, vol. 41, *Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 2002.

$$\dot{p} = v, \quad \dot{v} = \kappa(p, v, p^*), \quad (p, v) \in C. \tag{12}$$

Assume that if $C = \mathbb{R}^2$ then the point $(p^*, 0)$ is locally asymptotically stable with basin of attraction $\mathcal{B}$. Now let $C \subset \mathcal{B}$ and notice that this choice eliminates each solution of (12) with $C = \mathbb{R}^2$ that does not start in $\mathcal{B}$. It follows, for the system (12) with $C \subset \mathcal{B}$, that the point $(p^*, 0)$ is globally pre-asymptotically stable. A generalization of this observation appears in Theorem S10 of "Why 'Pre'-Asymptotic Stability?"

Now consider the hybrid system from Example 1, with state $x = (p, v, q)$ satisfying

$$\dot{p} = v, \quad \dot{v} = \kappa_q(p, v, p^*), \quad \dot{q} = 0, \quad (p, v) \in C_q, \tag{13}$$

$$p^+ = p, \quad v^+ = v, \quad q^+ = 3 - q, \quad (p, v) \in D_q. \tag{14}$$

According to the assumptions of Example 1 and the conclusions drawn above about the system (12), for each $i \in \{1, 2\}$ the point $(p^*, 0)$ is globally pre-asymptotically stable for the system

$$\dot{p} = v, \quad \dot{v} = \kappa_i(p, v, p^*) \quad (p, v) \in C_i. \tag{15}$$

Moreover, each solution of (15) with $i = 2$ that starts in D_1, which contains a neighborhood of $(p^*, 0)$ and is contained in C_2, does not reach the boundary of C_2. Also, $C_1 = \overline{\mathbb{R}^2 \setminus D_1}$ and $D_2 = \overline{\mathbb{R}^2 \setminus C_2}$.

Global asymptotic stability of the point $(p^*, 0, 2)$ for the system (13), (14) then follows from the global pre-asymptotic stability of $(p^*, 0)$ for (15) with $i \in \{1, 2\}$ together with the

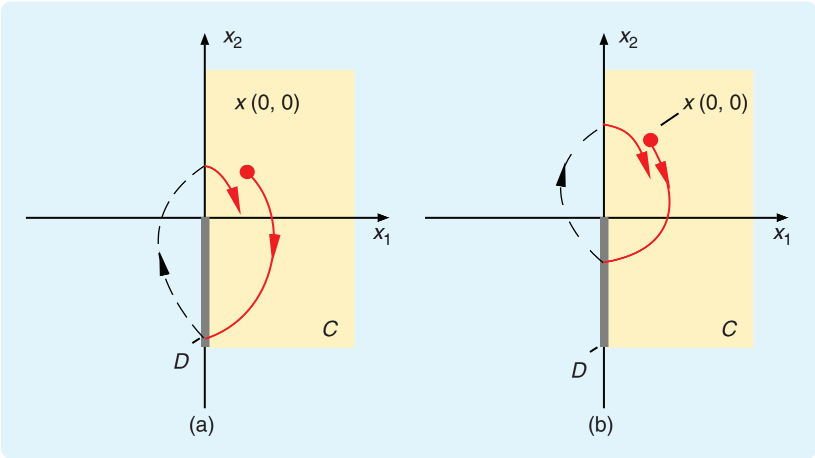

FIGURE 16 Flow and jump sets for the hybrid system in Example 9, with typical solutions to the system. Solutions flow clockwise in the right-half plane and jump when x_1 is zero and x_2 is nonpositive. In (a), the norm of the solution increases during flows and decreases at jumps. In (b), the norm decreases during flows and increases at jumps. In both cases, the origin is globally asymptotically stable.

fact that the maximum number of jumps a solution of the system (13), (14) experiences is two. To see the latter property, note that if there is more than one jump for a solution x of (13), (14) then one of the first two jumps must be from $q = 1$ to $q = 2$. At this jump, we must have $(p, v) \in D_1$ and, since p and v do not change during a jump, $(p, v) \in D_1$ after the jump. Since a solution of (15) with $i = 2$ that starts in D_1 does not reach D_2, the solution x of (13), (14) does not jump again after a jump from $q = 1$ to $q = 2$.

The principle behind this asymptotic stability result is generalized in the section "Stability Analysis Through Limited Events." ■

Example 2 Revisited: Asymptotic Stability

Consider the hybrid system in Example 2. We study asymptotic stability of the origin. The origin is not globally asymptotically stable since the solution starting at $(2, 0)$ does not converge to the origin. On the other hand, the solution starting at $(1, 0)$ satisfies $|x(t, j)| \leq |x(0, 0)|$ for all $(t, j) \in \operatorname{dom} x$ and $\lim_{t+j\to\infty} |x(t, j)| = 0$. More generally, constructing solutions as in Example 2, it follows that the origin is asymptotically stable with basin of attraction given by

$$\{x \in C: x_2 < 3,\ x_1 + x_2 \in (-2, 2)\} \cup \{x \in D: x_1 \in (-1, 1)\}. \qquad ■$$

A consequence of Theorem 4 appears in Theorem 10, which states that pre-asymptotic stability of a compact set is implied by forward invariance, as defined in Proposition 6, together with uniform pre-attractivity. A compact set $\mathcal{A}$ is *uniformly pre-attractive* from a set K if for each $\varepsilon > 0$ there exists $T > 0$ such that $x(0, 0) \in K$, $(t, j) \in \operatorname{dom} x$, and $t + j \geq T$ imply $|x(t, j)|_{\mathcal{A}} \leq \varepsilon$. The next section establishes the implication that is opposite to the one stated in Theorem 10.

Theorem 10 [26, Prop. 6.1]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, if the compact set $\mathcal{A}$ is forward invariant and uniformly pre-attractive from a compact set containing a neighborhood of $\mathcal{A}$, then the set $\mathcal{A}$ is pre-asymptotically stable.

Example 11: An Impulsive Observer with Finite-Time Convergence

This example comes from [67], which addresses impulsive observers for linear systems. Consider a linear, continuous-time system $\dot{\xi} = F\xi + v$, where ξ belongs to a compact set $K_1 \subset \mathbb{R}^n$ and v belongs to a compact, convex set $K_2 \subset \mathbb{R}^n$. Let $H \in \mathbb{R}^{r\times n}$ and assume we have measurements of the output vector $H\xi$ and the input vector v. The pair (H, F) is *observable* if $\dot{\xi} = F\xi$ and $H\xi(t) \equiv 0$ imply $\xi(t) \equiv 0$. This property enables assigning the spectra of the matrix $F - LH$ arbitrarily through the matrix L. In particular, a classical dynamical system with state $\hat{\xi}$ can be constructed so that $\xi(t) - \hat{\xi}(t)$ approaches zero as $t \to \infty$. Such a dynamical system is called an observer. A classical observer has the form $\dot{\hat{\xi}} = (F - LH)\hat{\xi} + LH\xi + v$, where L is chosen so that $F - LH$ is Hurwitz, meaning that each eigenvalue of $F - LH$ has negative real part. This choice gives the observation error equation $\dot{e} = (F - LH)e$, where $e{:} = \xi - \hat{\xi}$. Since $F - LH$ is Hurwitz, the error e converges to zero exponentially.

We consider a hybrid observer that reconstructs the state ξ in finite time. The first thing to note is that the observability of the pair (H, F) permits finding matrices L_1 and L_2 such that, for almost all $\delta > 0$, the matrix $I - \exp((F - L_2H)\delta)\exp(-(F - L_1H)\delta)$ is invertible [67, Remark 1]. Define $F_i := F - L_iH$ and henceforth assume that $\delta > 0$, L_1 and L_2 are such that $I - \exp(F_2\delta)\exp(-F_1\delta)$ is invertible.

Consider a hybrid system with state $x = (\xi, \hat{\xi}_1, \hat{\xi}_2, \tau)$, flow set $C := K_1 \times \mathbb{R}^n \times \mathbb{R}^n \times [0, \delta]$, jump set $D := K_1 \times \mathbb{R}^n \times \mathbb{R}^n \times \{\delta\}$, flow map

$$F(x) = \left\{ \begin{bmatrix} F\xi + v \\ F_1\hat{\xi}_1 + (F - F_1)\xi + v \\ F_2\hat{\xi}_2 + (F - F_2)\xi + v \\ 1 \end{bmatrix} : v \in K_2 \right\},$$

and jump map

$$G(x) = \begin{bmatrix} \xi \\ G_1\hat{\xi}_1 + G_2\hat{\xi}_2 \\ G_1\hat{\xi}_1 + G_2\hat{\xi}_2 \\ 0 \end{bmatrix},$$

where

Many engineering systems experience impacts; walking and jumping robots, juggling systems, billiards, and a bouncing ball are examples.

$$[G_1 \quad G_2] := (I - \exp(F_2\delta)\exp(-F_1\delta))^{-1} \times [-\exp(F_2\delta)\exp(-F_1\delta) \;\; I].$$

This hybrid system contains two different continuous-time observers, of the form $\dot{\hat{\xi}}_i = (F - L_i H)\hat{\xi}_i + L_i H\xi + v$, the states of which make jumps every δ seconds according to the rule specified by the jump map.

We show that the compact set $\mathcal{A} := \{(\xi, \hat{\xi}_1, \hat{\xi}_2) \in K_1 \times \mathbb{R}^n \times \mathbb{R}^n : \xi = \hat{\xi}_1 = \hat{\xi}_2\} \times [0, \delta]$ is globally asymptotically stable using Theorem 10. The set $\mathcal{A}$ is forward invariant since $G(\mathcal{A} \cap D) \subset \mathcal{A}$ and, during flows, the errors $e_i := \xi - \hat{\xi}_i$ satisfy $\dot{e}_i = F_i e_i$. Moreover, the set $\mathcal{A}$ is globally uniformly attractive. In particular, $(t, j) \in \operatorname{dom} x$ and $t \geq 2\delta$ imply $j \geq 2$ and $x(t, j) \in \mathcal{A}$. The condition on j follows from the nature of the data of the hybrid system. The condition $x(t, j) \in \mathcal{A}$ follows from the fact that, when there exists w such that ξ, $\hat{\xi}_1$, and $\hat{\xi}_2$ satisfy

$$\hat{\xi}_1 = \xi + \exp(F_1\delta)w, \quad \hat{\xi}_2 = \xi + \exp(F_2\delta)w,$$

the jump map sends the state to $\mathcal{A}$. The given relations are satisfied after one jump followed by a flow interval of length δ. In this case, w is equal to the difference between ξ and $\hat{\xi}_1$ immediately after the jump. We conclude from Theorem 10 that the set $\mathcal{A}$ is globally asymptotically stable. Moreover, the analysis above shows that the convergence to $\mathcal{A}$ is in finite time $t \leq 2\delta$. ■

Equivalence with Uniform Asymptotic Stability

In this section, we describe uniform pre-asymptotic stability on compact subsets of the basin of pre-attraction and point out that this property is equivalent to pre-asymptotic stability. The first step in stating this characterization is to make an observation about the basin of pre-attraction that extends classical results [42, pp. 69–71] for differential and difference equations to the hybrid setting. This result, and all of the subsequent results in this section, depend on Theorem 4.

Theorem 12 [26, Prop. 6.4], [14, Thm. 3.14]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, the basin of pre-attraction for a compact, pre-asymptotically stable set $\mathcal{A}$ is an open, forward invariant set containing a neighborhood of $\mathcal{A}$.

This theorem helps in developing some concepts used to express the fact that pre-asymptotic stability is equivalent to uniform pre-asymptotic stability on compact subsets of the basin of pre-attraction. In that direction, the next result states that excursions away from and convergence toward a pre-asymptotically stable compact set are uniform over compact subsets of the basin of pre-attraction.

Theorem 13 [26, Prop. 6.3]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, let the compact set $\mathcal{A}$ be pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_{\mathcal{A}}$. For each compact set $K_0 \subset \mathcal{B}_{\mathcal{A}}$, the compact set $\mathcal{A}$ is uniformly pre-attractive from K_0, and there exists a compact set $K_1 \subset \mathcal{B}_{\mathcal{A}}$ such that solutions starting in K_0 satisfy $x(t, j) \in K_1$ for all $(t, j) \in \operatorname{dom} x$.

The properties established in Theorem 13, of uniform overshoot and uniform convergence from compact subsets of the basin of pre-attraction, can be expressed in terms of a single bound on solutions starting in the basin of pre-attraction. To that end, let $\mathcal{A}$ be compact and let $\mathcal{O}$ be an open set containing $\mathcal{A}$. A continuous function $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is called a *proper indicator for* $\mathcal{A}$ *on* $\mathcal{O}$ if $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and also $\omega(x_i)$ tends to infinity when x_i tends to infinity or tends to the boundary of $\mathcal{O}$. Every open set $\mathcal{O}$ and compact set $\mathcal{A} \subset \mathcal{O}$ admit a proper indicator. Thus, using Theorem 12, for each pre-asymptotically stable set $\mathcal{A}$ there exists a proper indicator for $\mathcal{A}$ on its basin of pre-attraction. The function $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $\omega(x) := |x|_{\mathcal{A}}$ for all $x \in \mathbb{R}^n$ is a proper indicator for $\mathcal{A}$ on $\mathbb{R}^n$. For a general open set $\mathcal{O}$, it is always possible to take $\omega(x) = |x|_{\mathcal{A}}$ for x sufficiently close to $\mathcal{A}$. The concept of a proper indicator function first appears in conjunction with stability theory in [42]. A typical formula for a proper indicator is given in that work and in [39, (C.14)]. A sublevel set of a proper indicator on $\mathcal{O}$ is a compact subset of $\mathcal{O}$.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if it is continuous; for each $s \geq 0$, $r \mapsto \beta(r, s)$ is nondecreasing and zero at zero; and, for each $r \geq 0$, $s \mapsto \beta(r, s)$ is nonincreasing and tends to zero when s tends to infinity. Class-$\mathcal{KL}$ functions are featured prominently in [31] and have become familiar to the nonlinear systems and control community through their use in the input-to-state stability property [76]. The next result is a generalization to hybrid systems of a result contained in [42] for continuous differential equations.

Robustness and Generalized Solutions

When designing control systems, engineers must ensure that the closed-loop behavior is robust to reasonable levels of measurement noise, plant uncertainty, and environmental disturbances.

For linear systems, eigenvalues of matrices change continuously with parameters, and asymptotic stability of an equilibrium point is an open-set condition on eigenvalues. It follows that sufficiently small perturbations to the system matrices do not change stability properties. Similarly, sufficiently small additive disturbances lead to small excursions from the equilibrium point.

Classical results for ordinary differential and difference equations with nonlinear but continuous right-hand sides establish that small perturbations to the right-hand side result in small changes to the solutions on compact time intervals. This property is described for hybrid systems in theorems 5 and 8. If an equilibrium point or, more generally, a compact set is asymptotically stable for the nominal system, then these compact time intervals can be stitched together to establish that sufficiently small disturbances lead to small excursions from the asymptotically stable compact set. This idea is behind the results in theorems 15 and 17. This behavior is the essence of the total stability property described in [31, Sec. 56], and also contained in the local version of the input-to-state stability property [76].

For differential equations and difference equations with a discontinuous right-hand side, asymptotic stability of a compact set is not necessarily robust to arbitrary small perturbations [S26], [S27], [S25]. The lack of robustness motivates generalized notions of solutions. For differential equations, [S20] establishes the connection between the generalized notion of solution in [S26], expressed in terms of small state perturbations, and the generalized notion of solution given in [S28], expressed in terms of set-valued dynamics. The set-valued dynamics arise when a discontinuous right-hand side of a differential, or difference, equation is converted into an inclusion. The resulting system is the regularization of the original system, that is, it is a *regularized system*.

The conversion of a differential, or difference, equation with a discontinuous right-hand side to an inclusion is done by considering a closure of the graph of the discontinuous right-hand side and, in the differential equation case, by taking the convex hull of the values of the right-hand side. For an illustration, see Figure S8. The price paid for such a conversion is the introduction of additional solutions, some of which may not behave well. However, these extra solutions are meaningful since they arise from arbitrarily small state perturbations that converge to zero asymptotically. Moreover, asymptotic stability in the original, possibly discontinuous, system is robust if and only if asymptotic stability holds for the regularized system.

Anticipating that a similar result holds for hybrid systems and desiring robustness in asymptotically stable hybrid systems, we consider the stability properties of regularized systems. Following the lead of discontinuous continuous-time and discrete-time systems, we convert discontinuous flow and jump equations to inclusions. Interpreting these inclusions as closure operations on the discontinuous flow map and jump map as Figure S8 indicates, we also take the closures of the flow and jump sets. As for continuous- and discrete-time systems, these operations may introduce new solutions. However, the regularized hybrid system satisfies the Basic Assumptions, and therefore, the results of the main text are applicable. In particular, if the regularized system has an asymptotically stable compact set $\mathcal{A}$, then $\mathcal{A}$ is robustly asymptotically stable for the regularized system, and consequently, $\mathcal{A}$ is robustly asymptotically stable for the original system.

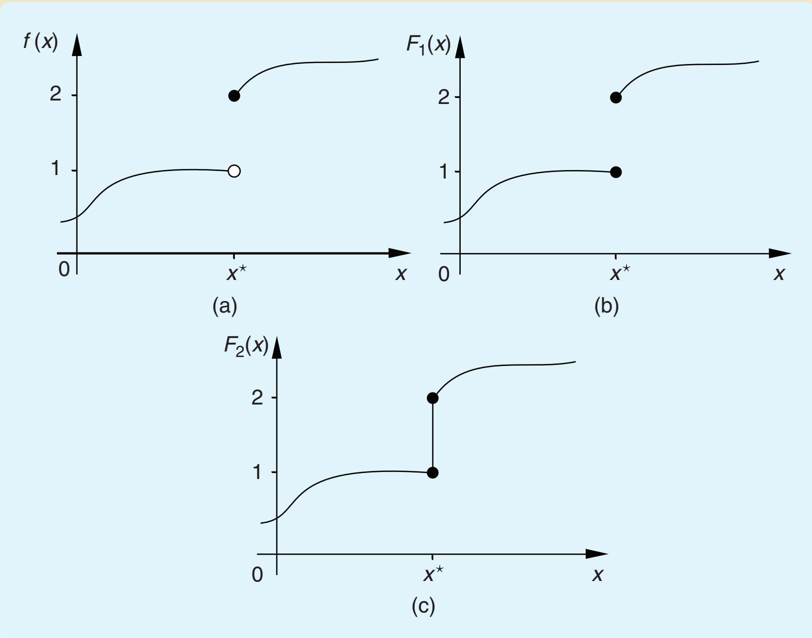

FIGURE S8 Regularization of a discontinuous function. (a) The function $f:\mathbb{R}\to\mathbb{R}$ has a discontinuity at $x = x^*$. (b) The set-valued mapping $F_1:\mathbb{R}\rightrightarrows\mathbb{R}$ is the mapping whose graph is the closure of the graph of f. Therefore, $F_1(x^*)$ is the set $\{1, 2\}$. (c) The set-valued mapping $F_2:\mathbb{R}\rightrightarrows\mathbb{R}$ is the mapping obtained by taking the convex hull of the values of the mapping in (b), at each $x\in\mathbb{R}$. Therefore, $F_2(x^*)$ is the interval $[1, 2]$.

Example S6: A Frictionless Ball and Two Rooms Separated by a Zero-Width Wall

Consider a particle that moves, with no friction, in one of two rooms separated by a thin wall. A hybrid system describing this situation has the state variable $x = (\xi, q) \in \mathbb{R}^2$ and data

$$\begin{aligned} C &= ([-1, 1] \setminus \{0\}) \times \{-1, 1\}, \\ F(x) &= (q, 0) \text{ for all } x \in C, \\ D &= \{(-1, -1)\} \cup (\{0\} \times \{-1, 1\}) \cup \{(1, 1)\}, \\ G(x) &= (\xi, -q) \text{ for all } x \in D. \end{aligned}$$

That is, the particle, whose position is denoted by ξ and velocity by q, moves with speed one in the interval $[-1, 0]$ or $[0, 1]$, and experiences a reversal of the velocity when at any of the boundary points: -1, 0, or 1.

Each of the sets $[-1, 0] \times \{-1, 1\}$ and $[0, 1] \times \{-1, 1\}$ is forward invariant. (When the solution definition is modified so that multiple jumps at the same ordinary time instant are not allowed and the flow constraint is relaxed to $x(t, j) \in C$ for all t in the interior of I_j, each maximal solution is complete and its time domain is unbounded in the ordinary time direction.) However, this forward invariance behavior is not robust, even approximately, to an arbitrarily small inflation of the set C. In particular, when C is changed to $\overline{C} = [-1, 1] \times \{-1, 1\}$, there exist solutions starting from $x = (1, -1)$ that reach $x = (-1, -1)$ in two time units.

At least two ways can be used to induce the desired behavior while using closed flow and jump sets. One natural approach can be interpreted as thickening the wall between the two rooms. In particular, letting $\varepsilon \in (0, 1)$, the data

$$\begin{aligned} C &= ([-1, 1] \setminus (-\varepsilon, \varepsilon)) \times \{-1, 1\}, \\ F(x) &= (q, 0) \text{ for all } x \in C, \\ D &= \{(-1, -1)\} \cup \{(-\varepsilon, 1)\} \cup \{(\varepsilon, -1)\} \cup \{(1, 1)\}, \\ G(x) &= (\xi, -q) \text{ for all } x \in D, \end{aligned}$$

results in solutions, as defined in the main text, rendering each of the sets $[-1, -\varepsilon] \times \{-1, 1\}$ and $[\varepsilon, 1] \times \{-1, 1\}$ robustly forward invariant.

When we insist on a wall of zero width, an extra state variable should be added that gives the system information about how it arrived at the point $(0, q)$, whether from the room on the left or the one on the right. For example, we can consider the hybrid system with state $(\xi, q, r) \in \mathbb{R}^3$ with data

$$\begin{aligned} C &= \{(\xi, q, r) \in [-1, 1] \times \{-1, 1\} \times \{-1, 1\} : \xi r \geq 0\}, \\ F(x) &= (q, 0, 0) \text{ for all } x \in C, \\ D &= \{(-1, -1, -1)\} \cup \{(1, 1, 1)\} \\ &\quad \cup \{(0, 1, -1)\} \cup \{(0, -1, 1)\}, \\ G(x) &= (\xi, -q, r) \text{ for all } x \in D. \end{aligned}$$

The hybrid system with this data is such that the set C is forward invariant. In particular, since r is constant and belongs to the set $\{-1, 1\}$, the state ξ cannot change sign. ■

The next example is discussed in [14].

Example S7: Asymptotic Stability Without Robustness

Consider the hybrid system with data

$$\begin{aligned} &C = [0, 1],\ F(x) = -x \text{ for all } x \in C, \\ &D = (1, 2],\ G(x) = 1 \text{ for all } x \in D. \end{aligned}$$

Solutions that start in $(1, 2]$ jump to one and then, not being in the jump set D, flow according to the differential equation $\dot{x} = -x$ toward the origin. The origin is thus globally asymptotically stable. But notice that when the jump map is replaced by $G(x) = 1 + \varepsilon$, where $\varepsilon > 0$, the point $x = 1 + \varepsilon$ is an equilibrium. In this sense, the global asymptotic stability of the origin is not robust. This lack of robustness can be seen in the regularized system $(C, F, \overline{D}, G)$, which exhibits a countable number of solutions from the initial condition $x = 1$. These solutions remain at the value 1 for n jumps, where n is any nonnegative integer, and then flow toward the origin. One additional solution remains at one through an infinite number of jumps. This behavior is depicted in Figure S9.

Example S8: Robust Asymptotic Stability

Consider the hybrid system with state $x = (\xi, q) \in \mathbb{R}^2$, data

$$\begin{aligned} C &:= \{(\xi, q) \in \mathbb{R} \times \{-1, 1\} : 2\xi q \geq -1\}, \\ F(\xi, q) &:= \begin{bmatrix} -\xi + q \\ 0 \end{bmatrix} \text{ for all } (\xi, q) \in C, \\ D &:= \{(\xi, q) \in \mathbb{R} \times \{-1, 1\} : 2\xi q < -1\}, \\ G(\xi, q) &:= \begin{bmatrix} \xi \\ -q \end{bmatrix} \text{ for all } (\xi, q) \in D. \end{aligned}$$

This system can be associated with a hybrid control algorithm for the continuous-time, linear system $\dot{\xi} = u$, which uses an internal state $q \in \{-1, 1\}$ and aims to globally asymptotically stabilize the two point set $\mathcal{A} := \{(-1, -1), (1, 1)\}$. This set is globally asymptotically stable since $\mathcal{A}$ is contained in C, the flow stabilizes the point $\xi = q$, $\mathcal{A}$ is disjoint from D, the quantity $2\xi q$ increases along flows when it has the value -1, and jumps from D are mapped to points in C. Note, however, that the jump set D is not closed. Consider the effect of closing D. For the hybrid system with data $(C, F, \overline{D}, G)$, solutions starting from points where $2x_1x_2 = -1$ are no longer unique. From such points, flowing is possible, as is a single jump followed by flow. Nevertheless, the set $\mathcal{A}$ is still globally asymptotically stable for $(C, F, \overline{D}, G)$. Thus, the asymptotic stability of $\mathcal{A}$ in the original system is robust. The lack of unique solutions in the regularized system can be associated with the fact that very different solutions arise when starting from $2\xi q = -1$ and considering arbitrarily small measurement noise on the state ξ in the closed-loop control system. ■

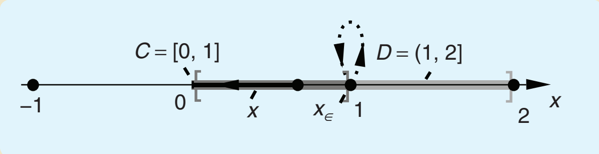

FIGURE S9 The effect of state perturbations in Example S7 showing that the global asymptotic stability of the origin is not robust. The solution x starts in $(-\infty, 1)$ and flows according to the differential equation $\dot{x} = -x$ toward the origin. The solution x_ε starts at $x_\varepsilon(0, 0) = 1$ and is obtained under the presence of a perturbation e of size ε. The perturbation is such that $x_\varepsilon(t, j) + e(t, j) \in D$ for all $(t, j) \in \mathrm{dom}\, x_\varepsilon$. Hence, as denoted with dotted line, the solution x_ε jumps from 1 to 1 infinitely many times, indicating that the origin is not robustly asymptotically stable to small perturbations.

A mathematical description of the phenomenon displayed in examples S6 and S7 is summarized as follows. Consider a hybrid system for which the data (C, f, D, g) do not meet all of the Basic Assumptions. That is, the flow set C is possibly not closed, the flow map $f : C \to \mathbb{R}^n$ is possibly not continuous, the jump set D is possibly not closed, and the jump map $g : D \to \mathbb{R}^n$ is possibly not continuous. Single-valued mappings f and g are considered here for simplicity; a more general result involving set-valued mappings is possible.

We consider two notions of generalized solutions to hybrid systems. One notion uses a hybrid system with the data $(\overline{C}, F, \overline{D}, G)$, which is obtained from the data (C, f, D, g) by taking the closures of C and D and defining

$$F(x) = \bigcap_{\delta>0} \overline{\mathrm{con}}\, f\left((x + \delta\mathbb{B}) \cap C\right) \quad \text{for all } x \in \overline{C}, \tag{S6}$$

$$G(x) = \bigcap_{\delta>0} \overline{g\left((x + \delta\mathbb{B}) \cap D\right)} \quad \text{for all } x \in \overline{D}. \tag{S7}$$

The mapping G defined in (S7) is the mapping whose graph is the closure of the graph of the function g. When the function f is locally bounded, the mapping F in (S6) is obtained by first considering the closure of the graph of f and then taking the pointwise convex hull. Figure S8 illustrates these two constructions. An alternative interpretation of $(\overline{C}, F, \overline{D}, G)$ is that it represents the smallest set of data that meets the Basic Assumptions and contains the data (C, f, D, g). The other notion of generalized solutions considers the effects of vanishing perturbations on the state. More precisely, it considers the graphical limits of sequences of solutions generated with state perturbations, as the perturbation size decreases to zero. The two notions of generalized solutions turn out to be equivalent, as the following result states.

Theorem 14 [26, Thm. 6.5], [14, Prop. 7.3]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, if the compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction given by $\mathcal{B}_\mathcal{A}$ then, for each function ω that is a proper indicator for $\mathcal{A}$ on $\mathcal{B}_\mathcal{A}$, there exists $\beta \in \mathcal{KL}$ such that each solution x starting in $\mathcal{B}_\mathcal{A}$ satisfies $\omega(x(t,j)) \le \beta(\omega(x(0,0)), t+j)$ for all $(t,j) \in \mathrm{dom}\, x$.

Theorem 14 contains the result in Theorem 13 that excursions away from and convergence toward a pre-asymptotically stable compact set are uniform over compact subsets of the basin of pre-attraction. In other words, pre-asymptotic stability for compact sets is equivalent to uniform pre-asymptotic stability, which sometimes is called $\mathcal{KL}$-stability. Thus, when we provide sufficient conditions for pre-asymptotic stability, we in fact give sufficient conditions for $\mathcal{KL}$-stability.

When discussing global pre-asymptotic stability, where $\mathcal{B}_\mathcal{A} = \mathbb{R}^n$, we can use $\omega(x) = |x|_\mathcal{A}$. Thus, for global pre-asymptotic stability, the bound in Theorem 14 becomes $|x(t,j)|_\mathcal{A} \le \beta(|x(0,0)|_\mathcal{A}, t+j)$ for all $(t,j) \in \mathrm{dom}\, x$. When $\omega(x) = |x|_\mathcal{A}$ and $\beta(s,r) = \gamma s \exp(-\lambda r)$ for some positive real numbers γ and λ, the set $\mathcal{A}$ is globally pre-exponentially stable.

Robustness

A feature of hybrid systems satisfying the Basic Assumptions is that pre-asymptotic stability is robust.

One way to characterize the robustness of pre-asymptotic stability of a compact set $\mathcal{A}$ is to study the effect of state-dependent perturbations on the hybrid system data and show that, when the perturbations are small enough, the pre-asymptotic stability of $\mathcal{A}$ and the basin of pre-attraction are preserved, as in Theorem 15. Typically, these state-dependent perturbations must decrease in size as the state approaches the pre-asymptotically stable set and also as the state approaches the boundary of the basin of pre-attraction.

Another way to characterize robustness is to consider constant perturbation levels and show that these perturbations lead to "practical" pre-asymptotic stability from arbitrarily large subsets of the basin of attraction, as in Theorem 17.

In some cases the nominal system can tolerate state-dependent perturbations that grow without bound when the state grows unbounded. These systems are closely related to hybrid systems having inputs and possessing the input-to-state stability (ISS) property [76]. ISS for hybrid dynamical systems is studied in [13].

For a given hybrid system $\mathcal{H}$ with data (C, F, D, G) and a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, we define the σ-perturbation $\mathcal{H}_\sigma$ of $\mathcal{H}$ through the data

$$C_\sigma := \{x : (x + \sigma(x)\mathbb{B}) \cap C \neq \emptyset\}, \tag{16}$$

$$F_\sigma(x) := \overline{\mathrm{con}} F((x + \sigma(x)\mathbb{B}) \cap C) + \sigma(x)\mathbb{B} \quad \text{for all } x \in C_\sigma, \tag{17}$$

$$D_\sigma := \{x : (x + \sigma(x)\mathbb{B}) \cap D \neq \emptyset\}, \tag{18}$$

$$G_\sigma(x) := \{v : v \in g + \sigma(g)\mathbb{B},\ g \in G((x + \sigma(x)\mathbb{B}) \cap D)\} \quad \text{for all } x \in D_\sigma. \tag{19}$$

Theorem S9 [S29, Thm. 3.1, Rem. 5.4]

Suppose that the functions $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are locally bounded on $\mathbb{R}^n$. Let $x : \operatorname{dom} x \to \mathbb{R}^n$ be a hybrid arc such that $\operatorname{dom} x$ is compact. Then, the following statements are equivalent:

a) x is a solution to the hybrid system $(\overline{C}, F, \overline{D}, G)$;

b) there exist hybrid arcs x_i and functions $e_i : \operatorname{dom} x_i \to \mathbb{R}^n$, $i \in \{1, 2, \dots\}$, such that $\lim_{i\to\infty} x_i(0,0) = x(0,0)$, the sequence $\{x_i\}_{i=1}^{\infty}$ converges graphically to x, $\lim_{i\to\infty} \sup_{(t,j)\in \operatorname{dom} x_i} |e_i(t,j)| = 0$, and, for every $i \in \{1, 2, \dots\}$ the following hold:

– For each fixed j, $t \mapsto e_i(t,j)$ is measurable.

– For all $j \in \mathbb{N}$ such that $I_{i,j} := \{t : (t,j) \in \operatorname{dom} x_i\}$ has nonempty interior,

$$\begin{aligned} \dot{x}_i(t,j) &= f(x_i(t,j) + e_i(t,j)) \quad \text{for almost all } t \in I_{i,j}, \\ x_i(t,j) + e_i(t,j) &\in C \quad \text{for almost all } t \in [\min I_{i,j}, \sup I_{i,j}). \end{aligned}$$

– For all $(t,j) \in \operatorname{dom} x_i$ such that $(t, j+1) \in \operatorname{dom} x_i$,

$$\begin{aligned} x_i(t, j+1) &= g_i(x_i(t,j) + e_i(t,j)), \\ x_i(t,j) + e_i(t,j) &\in D. \end{aligned}$$

A result corresponding to Theorem S9 is given in [S25, Thm. 3.2] for hybrid systems for which the perturbations e_i enter the closed-loop system through feedback, and do not affect all of the dynamics. This result considers equations $\dot{x}_i = f'(x_i, u_c)$, $x_i^+ = g'(x_i, u_d)$ in (b) above with state-feedback laws $u_c = k_c(x_i + e_i)$ and $u_d = k_d(x_i + e_i)$, and poses stronger continuity assumptions on the functions f' and g', but allows the functions k_c and k_d to be discontinuous.

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These definitions match the definitions (8)–(11) with $\delta = 1$. Figure 15 illustrates the idea behind perturbations of the sets C and D. Note that $C \subset C_\sigma$, $D \subset D_\sigma$, $F(x) \subset F_\sigma(x)$ for all $x \in C$ and $G(x) \subset G_\sigma(x)$ for all $x \in D$. Since the data of $\mathcal{H}_\sigma$ contain the data of $\mathcal{H}$, the solutions of $\mathcal{H}$ are also solutions to $\mathcal{H}_\sigma$. On the other hand, $\mathcal{H}_\sigma$ typically exhibits solutions that are not solutions to $\mathcal{H}$.

The extra solutions of $\mathcal{H}_\sigma$ can be linked to solutions that arise due to parameter variations, measurement noise in control systems, and external disturbances. For a link to solutions that arise from parameter variations, see Example 18. For the case of external disturbances d that are bounded in norm by a value $M > 0$, observe that the solutions of $\dot{x} = F(x) + d$ are contained in the set of solutions of $\dot{x} \in F(x) + M\mathbb{B}$ and, taking $\sigma(x) = M$ for all x, $F(x) + M\mathbb{B} \subset F_\sigma(x)$. The same link between external disturbances and system perturbations holds for $x^+ = G(x) + d$.

Now consider the case of measurement noise in a hybrid control system of the form discussed in the section "Hybrid Controllers for Nonlinear Systems," given as

$$\left.\begin{aligned} \dot{x}_p &= f_p(x_p, \kappa_c(x_p + e, x_c)) \\ \dot{x}_c &= f_c(x_p + e, x_c) \end{aligned}\right\} \quad (x_p + e, x_c) \in C, \tag{20}$$

$$\left.\begin{aligned} x_p^+ &= x_p \\ x_c^+ &\in G_c(x_p + e, x_c) \end{aligned}\right\} \quad (x_p + e, x_c) \in D, \tag{21}$$

where e represents measurement noise, assumed to be bounded in norm by $M > 0$. When the function f_p is continuous, there exists a continuous function $\tilde{\sigma}$ such that $\tilde{\sigma}(0, x) = 0$ for all $x \in \mathbb{R}^{n+m}$ and, for all $e \in \mathbb{R}^n$ satisfying $|e| \le M$ and all $x = (x_p, x_c) \in \mathbb{R}^{n+m}$,

$$\begin{aligned} |f_p(x_p, \kappa_c(x_p + e, x_c)) - f_p(x_p + e, \kappa_c(x_p + e, x_c))| \\ \le \tilde{\sigma}(M, x). \end{aligned}$$

In this case, the hybrid control system (20)–(21) can be written as

$$\begin{aligned} \dot{x} &= F(x + d_1) + d_2, & x + d_1 &\in C, \\ x^+ &\in G(x + d_1), & x + d_1 &\in D, \end{aligned}$$

where F and G are defined in (6) and (7), respectively, $|d_1| \le M$, and $|d_2| \le \tilde{\sigma}(M, x)$. Therefore, the solutions of the hybrid control system (20)–(21) are contained in the solutions of $(C_\sigma, F_\sigma, D_\sigma, G_\sigma)$ where $\sigma(x) := \max\{M, \tilde{\sigma}(M, x)\}$ for all $x \in \mathbb{R}^{n+m}$.

The mappings F_σ and G_σ may be set valued at points x where $\sigma(x) > 0$, even when F and G are single-valued mappings. Also, when $x \in C \cap D$ and $\sigma(x) > 0$, the point x belongs to the interior of both C_σ and D_σ. Thus, at such points, the system $\mathcal{H}_\sigma$ has solutions that initially flow and also solutions that initially jump.

For a hybrid system $\mathcal{H}$ having a compact set $\mathcal{A}$ that is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_\mathcal{A}$, Theorem 15 below asserts the existence of a continuous function σ that is positive on $\mathcal{B}_\mathcal{A} \backslash \mathcal{A}$ so that, for the hybrid system $\mathcal{H}_\sigma$, the compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_\mathcal{A}$. In other words, pre-asymptotic stability

Motivating Stability of Sets

Intuition built on the theory of linear systems conditions us to think of asymptotic stability as a property of an equilibrium point. However, it is also natural to consider the asymptotic stability of a set. Conceptually, set stability places many apparently different phenomena under one umbrella. Set stability is needed for systems that include timers, counters, and other discrete states that do not converge. In addition, set stability is helpful for characterizing systems that exhibit complicated asymptotic behavior.

Asymptotic stability of a set is defined in the main text. Roughly speaking, a set is globally asymptotically stable if each solution that starts close to the set remains close to the set, each solution is bounded, and each solution with an unbounded time domain converges to the set.

To illustrate set stability, we consider a linear, sampled-data control system. We group the plant state, hold state, and controller state into the state $\xi \in \mathbb{R}^n$. Between sampling events, the state ξ evolves according to a closed-loop, linear differential equation $\dot{\xi} = F\xi$ for some matrix F. For each state component that remains constant between sampling events, the corresponding row of F is filled with zeros. At sampling events, the state ξ is updated according to the equation $\xi^+ = J\xi$ for some matrix J. For each state component that does not change at sampling events, the corresponding row of J is filled with zeros except for a one in the appropriate column. The interaction between the continuous evolution $\dot{\xi} = F\xi$ and the discrete evolution $\xi^+ = J\xi$ is scheduled by a timer state τ that evolves according to $\dot{\tau} = 1$ between sampling events and is reset to zero at sampling events. Sampling events occur whenever the timer state reaches the value $T > 0$, which denotes the sampling period of the system. Thus, the sampled-data control system can be written as a hybrid system with state $x = (\xi, \tau)$ satisfying

$$\left.\begin{array}{rcl} \dot{\xi} & = & F\xi \\ \dot{\tau} & = & 1 \end{array}\right\} \quad x \in C := \mathbb{R}^n \times [0, T],$$

$$\left.\begin{array}{rcl} \xi^+ & = & J\xi \\ \tau^+ & = & 0 \end{array}\right\} \quad x \in D := \mathbb{R}^n \times \{T\}.$$

The matrices F and J are designed so that ξ converges to zero. However, the timer state τ does not converge. Letting $\tau_0 \in [0, T]$ denote the initial value of τ, the solution satisfies $\tau(t, j) = t - (jT - \tau_0)$ for all positive integers j and all $t \in [jT - \tau_0, (j+1)T - \tau_0]$. Consequently, τ revisits every point in the interval $[0, T]$. For this reason, the hybrid system does not possess an asymptotically stable equilibrium point. On the other hand, the compact set $\mathcal{A} := \{0\} \times [0, T]$ is globally asymptotically stable. In particular, each solution to the hybrid system converges to the compact set $\mathcal{A}$. This property captures the fact that the state ξ converges to zero.

As another example of set stability, consider a system switching among a family of asymptotically stable linear systems under an average dwell-time constraint, as discussed in "Switching Systems." The corresponding hybrid system has the form

$$\left.\begin{array}{rcl} \dot{\xi} & = & F_q\xi \\ \dot{q} & = & 0 \\ \dot{\tau} & \in & [0, \delta] \end{array}\right\} \quad C := \mathbb{R}^n \times Q \times [0, N],$$

$$\left.\begin{array}{rcl} \xi^+ & = & \xi \\ q^+ & \in & Q \\ \tau^+ & = & \tau - 1 \end{array}\right\} \quad D := \mathbb{R}^n \times Q \times [1, N],$$

where $Q \subset \mathbb{R}$ is compact and the eigenvalues of F_q have negative real part for each $q \in Q$. When the average dwell-time parameter $\delta > 0$ is small enough, the compact set $\{0\} \times Q \times [0, N]$ is asymptotically stable. In particular, the state ξ converges to zero. The states q and τ may or may not converge, depending on the particular solution considered.

In some situations, hybrid systems admit complicated, asymptotically stable compact sets that can be characterized through the concept of an Ω-limit set [S33]. Ω-limit sets are exploited for nonlinear control design, for example for nonlinear output regulation [S30]. Results pertaining to Ω-limit sets for hybrid systems are given in [S32] and [S31]. In [S32, Cor. 3] and [S31, Thm. 1], conditions are given under which the Ω-limit set is an asymptotically stable compact set. In particular, suppose there exist $T > 0$ and compact sets K_0 and K_T, with K_T contained in the interior of K_0, such that each solution x starting in K_0 is bounded and, for all $(t, j) \in \operatorname{dom} x$ satisfying $t + j \geq T$, we have $x(t, j) \in K_T$. Then, either the Ω-limit set from K_0 is empty or it is the smallest compact, asymptotically stable set contained in K_T.

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and the corresponding basin of pre-attraction are preserved under the perturbation σ. This result provides a key step in the proof of the existence of a smooth Lyapunov function for a hybrid system with a pre-asymptotically stable compact set. For more information, see "Converse Lyapunov Theorems."

Theorem 15 [14, Theorem 7.9]

For the hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, suppose that a compact set $\mathcal{A} \subset \mathbb{R}^n$ is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_{\mathcal{A}}$. Then there exists a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying $\sigma(x) > 0$ for

Why "Pre"-Asymptotic Stability?

In engineered hybrid systems, it is reasonable to insist that each maximal solution that starts sufficiently near an asymptotically stable set has an unbounded time domain. Recall that a solution with an unbounded time domain is called a complete solution. However, a complete solution does not necessarily have a time domain that is unbounded in the t direction; see "Zeno Solutions" for more information on systems with complete solutions having domains that are bounded in the t direction.

The definition of global asymptotic stability used in the main text, which we call global pre-asymptotic stability, does not stipulate completeness of each maximal solution. A compact set $\mathcal{A}$ is globally pre-asymptotically stable if each solution that starts close to $\mathcal{A}$ remains close to $\mathcal{A}$, each solution is bounded, and each complete solution converges to $\mathcal{A}$. Thus, solutions do not need to be complete, but complete solutions must converge to $\mathcal{A}$. A set is globally asymptotically stable when it is globally pre-asymptotically stable and all solutions are complete.

Completeness of solutions is not required in the pre-asymptotic stability definition as a matter of convenience. For example, since completeness does not need to be verified, sufficient conditions for pre-asymptotic stability are simpler than they would be otherwise. When completeness is important, it can be established by verifying pre-asymptotic stability together with local existence of solutions.

Conceptually, pre-asymptotic stability offers advantages over asymptotic stability. On the one hand, pre-asymptotic stability is all that is needed for the existence of smooth Lyapunov functions. See "Converse Lyapunov Theorems" for more details. Pre-asymptotic stability also makes it easy to express local pre-asymptotic stability in terms of global pre-asymptotic stability, as in the next theorem [14], by considering a reduced system that is contained in the original system. The system $(C_\odot, F_\odot, D_\odot, G_\odot)$ is contained in the system (C, F, D, G) if $C_\odot \subset C$, $F_\odot(x) \subset F(x)$ for all $x \in C_\odot$, $D_\odot \subset D$ and $G_\odot(x) \subset G(x)$ for all $x \in D_\odot$. Completeness of solutions for (C, F, D, G) does not guarantee completeness of solutions for $(C_\odot, F_\odot, D_\odot, G_\odot)$.

Theorem S10

Suppose that, for the hybrid system (C, F, D, G), the compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_{\mathcal{A}}$. Then, for each hybrid system $(C_\odot, F_\odot, D_\odot, G_\odot)$ that is contained in (C, F, D, G), the set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction containing $\mathcal{B}_{\mathcal{A}}$. In particular, if there exists a compact set $K \subset \mathcal{B}_{\mathcal{A}}$ such that $C_\odot \cup D_\odot \subset K$, then the set $\mathcal{A}$ is globally pre-asymptotically stable for $(C_\odot, F_\odot, D_\odot, G_\odot)$.

One particular application of Theorem S10 says that if the hybrid system $\mathcal{H}$ has the compact set $\mathcal{A}$ pre-asymptotically stable, then so does the hybrid system that uses only the continuous-time data (C, F) and so does the hybrid system that uses only the discrete-time data (D, G). The converse of this assertion holds only when the separate systems admit a common Lyapunov function. See "Converse Lyapunov Theorems" for more details.

It is reasonable to question the utility of studying systems for which no solutions are complete. One motivation comes from the results discussed in the section "Hybrid Feedback Control Based on Limited Events", where it is established that pre-asymptotic stability of a compact set for a system with events inhibited implies pre-asymptotic stability of the compact set for the system with events allowed as long as events are not too frequent. In this statement, pre-asymptotic stability is useful since the system with inhibited events may not exhibit complete solutions, whereas the system with events allowed may exhibit complete solutions.

all $x \in \mathcal{B}_{\mathcal{A}} \backslash \mathcal{A}$ such that, for the hybrid system $\mathcal{H}_\sigma$, the compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_{\mathcal{A}}$.

The idea in the next example is related to Lebesgue sampling presented in [3] and to the ideas in [80] used for control in the presence of communication or power constraints. The idea exploits the robustness described in Theorem 15, but for the special case of classical systems, to explain the asymptotic stability induced by a particular hybrid control strategy.

Example 16: Control Through Event-Based Hold Updates

Consider the continuous-time, nonlinear control system $\dot{\xi} = f(\xi, u)$, $(\xi, u) \in \mathbb{R}^n \times \mathbb{R}^m$, with the continuous state feedback $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ that stabilizes a compact set $\mathcal{A}$ with basin of attraction $\mathcal{B}_{\mathcal{A}} \subset \mathbb{R}^n$. Suppose the feedback is implemented by keeping the control value constant until an event triggers a change in the control value. The trigger comes from sensing that the state deviates from the state used to compute the control value by an amount that is significant enough to warrant a change in the input value. This amount is determined by a continuous function $\sigma_1 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ related to the function σ in Theorem 15.

The closed-loop system can be modeled as a hybrid system with the data

$$C := \{x \in \mathbb{R}^{2n} : |x_1 - x_2| \leq \sigma_1(x_1)\},$$

$$F(x) := \begin{bmatrix} f(x_1, \kappa(x_2)) \\ 0 \end{bmatrix} \quad \text{for all } x \in C,$$

$$D := \{x \in \mathbb{R}^{2n} : |x_1 - x_2| \geq \sigma_1(x_1)\},$$

$$G(x) := \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \quad \text{for all } x \in D,$$

where σ_1 is specified as follows. Let $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfy the conditions of Theorem 15 for the (classical) system

Converse Lyapunov Theorems

Despite the Lyapunov-based sufficient conditions for pre-asymptotic stability given by Theorem 32 and [6, 19, 90], it is a mistake to conclude that Lyapunov functions do not exist for asymptotically stable hybrid systems. Early results on the existence of Lyapunov functions for hybrid systems can be found in [90]. More recently, [14] establishes that a hybrid system with a pre-asymptotically stable compact set admits a smooth (C^∞) Lyapunov function as long as the hybrid system satisfies the Basic Assumptions. Converse theorems of this type are typically established theoretically, constructing smooth Lyapunov functions from the system's solutions, which are not usually available explicitly. Thus, converse theorems are of limited help in constructing Lyapunov functions. Nevertheless, converse theorems justify searching for Lyapunov functions that can be constructed without knowledge of the system's trajectories.

Converse theorems also play a role in establishing stabilization results for nonlinear control systems. For example, early results [S40, Theorem 4] and [S34, Theorem 2.1] on backstepping, that is, proving that smooth stabilizability is not destroyed by adding an integrator, construct a stabilizing feedback by using the gradient of a Lyapunov function whose existence is guaranteed by a converse theorem. Also, the result in [76] showing that smooth stabilization implies coprime factorization, that is, input-to-state stabilization with respect to disturbances that add to the control variable, exploits a converse Lyapunov theorem. Similar results apply for hybrid systems. For example, in [14] converse theorems are developed and used to show that smooth stabilization with logic-based feedback implies smooth, logic-based input-to-state stabilization with respect to matched disturbances. Converse Lyapunov theorems can also be used to establish various forms of robustness, including robustness to small, persistent perturbations.

The converse theorems for hybrid systems given in [14] draw heavily from the literature on converse theorems for continuous-time and discrete-time systems. We recount some of the major milestones in the development of converse Lyapunov theorems for time-invariant, finite-dimensional dynamical systems having compact, asymptotically stable sets.

In his 1892 Ph.D. dissertation, in particular [S38, Section 20, Theorem II], Lyapunov provides the first contribution to converse theorems, where he addresses asymptotically stable linear systems. Generalizations of this result to nonlinear systems did not appear until the 1940s and 1950s. For example, the 1949 paper [S39] provides a converse Lyapunov theorem for time-invariant, continuously differentiable systems having a locally asymptotically stable equilibrium.

Essentially all of the converse Lyapunov theorems until [42] pertain only to dynamical systems with unique solutions. In contrast, [42] establishes the first converse Lyapunov theorems for differential equations with continuous right-hand side without assuming uniqueness of solutions. This extension is significant in anticipation of hybrid systems where nonuniqueness of solutions is not uncommon. The contribution of [42] to the development of converse theorems is immense; it appears to be the first work that relies explicitly on robustness and advanced smoothing techniques to establish the existence of smooth Lyapunov functions.

$\dot{\xi} = \widetilde{F}(\xi) := f(\xi, \kappa(\xi))$. Take $\sigma_1 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ to be continuous, positive on $\mathcal{B}_{\mathcal{A}} \backslash \mathcal{A}$, zero on $\mathcal{A}$, such that $\sigma_1(x_1) \leq \sigma(x_1)$ for all $x_1 \in \mathcal{B}_{\mathcal{A}}$, and

$$|f(x_1, \kappa(x_2)) - f(x_2, \kappa(x_2))| \leq \sigma(x_1) \quad \text{for all } x_1 \in \mathcal{B}_{\mathcal{A}},\ x_2 \in x_1 + \sigma_1(x_1)\mathbb{B}.$$

For all $x \in C$, it follows that

$$\begin{aligned} f(x_1, \kappa(x_2)) &= f(x_2, \kappa(x_2)) + f(x_1, \kappa(x_2)) - f(x_2, \kappa(x_2)) \\ &\in \overline{\text{con}}\, \widetilde{F}(x_1 + \sigma(x_1)\mathbb{B}) + \sigma(x_1)\mathbb{B}. \end{aligned}$$

The jumps do not change x_1. Thus, according to Theorem 15, x_1 stays close to $\mathcal{A}$ and converges to $\mathcal{A}$ whenever the solution has a hybrid time domain that is unbounded in the t direction. For such solutions, x_2 also stays close to $\mathcal{A}$ and converges to $\mathcal{A}$. For solutions with a hybrid time domain bounded in the t direction, notice that when $x \in D$ but $x_1 \notin \mathcal{A}$, it follows that $G(x) \notin D$. Thus, the only way that a solution can have a domain bounded in the t direction is if x_1 converges to $\mathcal{A}$. Whenever x_1 converges to $\mathcal{A}$, so does x_2. It thus follows for the closed-loop system that the set $\mathcal{A} \times \mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}} \times \mathbb{R}^n$. ■

For problems where the size of the perturbations does not become arbitrarily small as the state approaches the set $\mathcal{A}$ or the boundary of the basin of pre-attraction, the following theorem is relevant.

Theorem 17 [26, Theorem 6.6]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, suppose that a compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_{\mathcal{A}}$. In particular, suppose that there exist $\beta \in \mathcal{KL}$ and a proper indicator function ω for $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$ such that, for all solutions starting in $\mathcal{B}_{\mathcal{A}}$,

$$\omega(x(t,j)) \leq \beta(\omega(x(0,0)), t + j) \quad \text{for all } (t,j) \in \operatorname{dom} x.$$

Then, for each $\varepsilon > 0$ and compact set $K \subset \mathcal{B}_{\mathcal{A}}$, there exists $\delta > 0$ such that, with the function σ defined as $\sigma(x) = \delta$ for all $x \in \mathbb{R}^n$, each solution to the hybrid system $\mathcal{H}_\sigma$ starting in K satisfies

The converse theorem of [S41] is noteworthy for its smoothing technique, which is adopted in more recent converse results. We emphasize that continuously differentiable Lyapunov functions are needed to establish robustness in a straightforward manner, yet establishing the existence of continuously differentiable Lyapunov functions is challenging for systems with nonunique solutions.

Continuing down the path of systems with nonunique solutions, in the context of the input-to-state stability property, the work in [S37] establishes a converse Lyapunov theorem for locally Lipschitz differential inclusions, establishing smoothness of the Lyapunov function. This work is extended in [S36] to differential inclusions satisfying assumption (A2) of the Basic Assumptions with $C = \mathbb{R}^n$. The approach of [S36] again emphasizes the fundamental link between robustness and smoothness of the Lyapunov function.

The results in [14] draw heavily on earlier smoothing techniques, especially the smoothing techniques of [S36] adapted to discrete-time inclusions, as in [S27]. Here we state a simple version of a converse Lyapunov theorem in [14].

Theorem S11 [14, Thm. 3.14]

For the hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions, if the compact set $\mathcal{A}$ is globally pre-asymptotically stable, then there exist a C^∞ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$\langle \nabla V(x), f \rangle \leq -V(x) \quad \text{for all } x \in C,\ f \in F(x),$$

$$V(g) \leq \frac{V(x)}{2} \quad \text{for all } x \in D,\ g \in G(x).$$

Extensions of this result to more general asymptotic stability notions are given in [S35]. A version for local pre-asymptotic stability is given in [14], which also specializes Theorem S11 to the case of hybrid systems with discrete states and gives a Lyapunov-based proof of robustness to various sources of perturbations, including slowly varying parameters.

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$$\omega(x(t,j)) \leq \beta(\omega(x(0,0)), t+j) + \varepsilon \quad \text{for all } (t,j) \in \operatorname{dom} x. \tag{22}$$

The property concluded in Theorem 17 is referred to as either semiglobal, practical robustness to persistent perturbations or semiglobal, practical pre-asymptotic stability. "Semiglobal" refers to the fact that the bound (22) can be achieved from an arbitrary, compact subset of the basin of pre-attraction. "Practical" refers to the fact that the value of ε in (22) can be made arbitrarily small.

We now discuss two applications of Theorem 17. An additional application can be found in "Zeno Solutions," where temporal regularization is discussed.

Example 18: Hybrid Control with Slowly Varying Parameters

Consider the parameterized nonlinear control system $\dot{\xi} = F_1(\xi, p, u)$, $\xi \in \mathbb{R}^{n_1}$, where, for now, p represents a constant parameter taking values in a compact set $\mathcal{P} \subset \mathbb{R}^{n_2}$. Consider also a dynamic, hybrid controller with state $\eta \in \mathbb{R}^{n_3}$ perhaps consisting of timers, discrete states, and continuous states, designed under the assumption that p is constant. The resulting hybrid closed-loop system with state $x = (\xi, p, \eta)$ has the data (C, F, D, G), where $\dot{x} \in F(x)$ implies $\dot{p} = 0$, $x^+ \in G(x)$ implies $p^+ = p$, and $C \cup D \subset \mathbb{R}^{n_1} \times \mathcal{P} \times \mathbb{R}^{n_3}$. Moreover, suppose that the compact set $\mathcal{A} \subset \mathbb{R}^{n_1} \times \mathcal{P} \times \mathbb{R}^{n_3}$ is asymptotically stable. We now consider what happens when p can vary, slowly during flows and with small changes during jumps. Such variations are captured within the modeling of perturbations in (16)–(19) with $\sigma(x) \equiv \bar{\sigma} > 0$ since $\dot{x} \in F_\sigma(x)$ implies $\dot{p} \in \bar{\sigma}\mathbb{B}$ and $x^+ \in G_\sigma(x)$ implies $p^+ \in p + \bar{\sigma}\mathbb{B}$. Hence, we conclude with the help of Theorem 17 that the pre-asymptotic stability of the compact set $\mathcal{A}$ is semiglobally, practically robust to small parameter variations. Similar results for differential equations are pointed out in [40] using results from [37]. ■

The following corollary of Theorem 17 is a reduction result that is related to results for continuous-and discrete-time nonlinear systems in [38] and [28], respectively.

Corollary 19

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions. If the compact set $\mathcal{A}_1$ is globally pre-asymptotically stable for $\mathcal{H}$ and the compact set $\mathcal{A}_2 \subset \mathcal{A}_1$ is globally pre-asymptotically stable for $\mathcal{H}|_{\mathcal{A}_1} := (C \cap \mathcal{A}_1, F, D \cap \mathcal{A}_1, G \cap \mathcal{A}_1)$, then $\mathcal{A}_2$ is globally pre-asymptotically stable for $\mathcal{H}$.

Corollary 19 follows from Theorem 17 since solutions that start close to $\mathcal{A}_2$ start close to $\mathcal{A}_1$ and thus stay close to $\mathcal{A}_1$. Thus the solutions of $\mathcal{H}$ are contained in a small perturbation of $\mathcal{H}|_{\mathcal{A}_1}$. Moreover, these perturbations vanish with time since $\mathcal{A}_1$ is assumed to be globally pre-asymptotically stable.

Corollary 19 can be applied to analyze cascaded systems [75]. For example, consider the classical system with state $x = (x_1, x_2)$ and dynamics

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2) \end{aligned}\right\} = F(x), \tag{23}$$

where the functions f_1 and f_2 are continuous, the origin of $\dot{x}_2 = f_2(x_2)$ is globally asymptotically stable, and the origin of $\dot{x}_1 = f_1(x_1, 0)$ is globally asymptotically stable. For the system

$$\dot{x} = F(x), \quad x \in C = M\mathbb{B} \times M\mathbb{B}, \quad M > 0, \tag{24}$$

it follows that the compact set $\mathcal{A}_1 = M\mathbb{B} \times \{0\}$ is globally pre-asymptotically stable and that the system $\dot{x} = F(x)$, $x \in C \cap \mathcal{A}_1$ has the compact set $\mathcal{A}_2 = \{(0,0)\}$ globally pre-asymptotically stable. We conclude from Corollary 19 that the origin of the system (24) is globally pre-asymptotically stable. This conclusion means that, for the original system (23), the origin is asymptotically stable. Moreover, since M is arbitrary in (24), the basin of attraction for the origin for the system (23) contains the set of initial conditions from which each solution of (23) is bounded.

STABILITY ANALYSIS USING LYAPUNOV FUNCTIONS

Lyapunov functions are familiar to most control engineers. These functions provide a way of establishing asymptotic stability without having to construct the system's solutions explicitly, a daunting task for almost anything other than a linear system. Sufficient Lyapunov conditions for asymptotic stability are well known for both continuous- and discrete-time systems. These conditions amount to finding a nonnegative-valued function that is strictly decreasing along solutions. In the case of a smooth Lyapunov function for a continuous-time system, the decrease condition can be verified from negative definiteness of the inner product of the Lyapunov function's gradient and the vector field that generates solutions. The standard Lyapunov-based sufficient conditions for asymptotic stability are covered in the version below for hybrid dynamical systems; see also [72, Cor. 7.7]. First, we give a definition.

Given the hybrid system $\mathcal{H}$ with data (C, F, D, G) and the compact set $\mathcal{A} \subset \mathbb{R}^n$, the function $V : \operatorname{dom} V \to \mathbb{R}$ is a *Lyapunov-function candidate* for $(\mathcal{H}, \mathcal{A})$ if i) V is continuous and nonnegative on $(C \cup D) \setminus \mathcal{A} \subset \operatorname{dom} V$, ii) V is continuously differentiable on an open set $\mathcal{O}$ satisfying $C \setminus \mathcal{A} \subset \mathcal{O} \subset \operatorname{dom} V$, and iii)

$$\lim_{\{x \to \mathcal{A},\ x \in \operatorname{dom} V \cap (C \cup D)\}} V(x) = 0.$$

Conditions i) and iii) hold when dom V contains $\mathcal{A} \cup C \cup D$, V is continuous and nonnegative on its domain, and $V(z) = 0$ for all $x \in \mathcal{A}$. These conditions are typical of Lyapunov-function candidates for discrete-time systems. Condition ii) holds when V is continuously differentiable on an open set containing $C \setminus \mathcal{A}$, which is typical of Lyapunov-function candidates for continuous-time systems. We impose continuous differentiability for simplicity, but it is possible to work with less regular Lyapunov functions and their generalized derivatives. When $x = (\xi, q) \in \mathbb{R}^n \times Q$, where Q is a discrete set, it is natural to define V only on $\mathbb{R}^n \times Q$. To satisfy condition ii), the definition of V can be extended to a neighborhood of $\mathbb{R}^n \times Q$, with $V(\xi, q) = V(\xi, q_0)$ for all q near $q_0 \in Q$.

We now state a hybrid Lyapunov theorem.

Theorem 20

Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions and the compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfying $G(\mathcal{A} \cap D) \subset \mathcal{A}$. If there exists a Lyapunov-function candidate V for $(\mathcal{H}, \mathcal{A})$ such that

$$\begin{aligned} \langle \nabla V(x), f \rangle < 0 &\quad \text{for all } x \in C \setminus \mathcal{A},\ f \in F(x), \\ V(g) - V(x) < 0 &\quad \text{for all } x \in D \setminus \mathcal{A}, g \in G(x) \setminus \mathcal{A}, \end{aligned}$$

then the set $\mathcal{A}$ is pre-asymptotically stable and the basin of pre-attraction contains every forward invariant, compact set.

A consequence of Theorem 20 is that the compact set $\mathcal{A}$ is globally pre-asymptotically stable if $C \cup D$ is compact or the sublevel sets of $V|_{\operatorname{dom} V \cap (C \cup D)}$ are compact. A sublevel set of $V|_{\operatorname{dom} V \cap (C \cup D)}$ is the set $\{x \in \operatorname{dom} V \cap (C \cup D) : V(x) \leq c\}$, where $c \geq 0$.

Theorem 20 encompasses classical Lyapunov theorems, both for continuous- and discrete-time systems. For example, consider the case where F is a continuous function, and suppose there exist a continuously differentiable, positive semidefinite function V and a compact neighborhood C of the origin such that $\langle \nabla V(x), F(x) \rangle < 0$ for all $x \in C \setminus \{0\}$. According to Theorem 20, the origin of $\dot{x} = F(x)$, $x \in C$, is globally pre-asymptotically stable. Then, since C contains a neighborhood of the origin, it follows that the origin is asymptotically stable for $\dot{x} = F(x)$. This conclusion parallels the conclusion of [39, Thm. 4.1]. Similarly, for the case where G is a continuous, single-valued mapping, if $G(0) = 0$ and there exist a continuous, positive semidefinite function V and a compact neighborhood D of the origin such that $V(G(x)) < V(x)$ for all $x \in D \setminus \{0\}$, then the origin of $x^+ = G(x)$ is asymptotically stable.

The following examples illustrate the use of Theorem 20.

Example 2 Revisited: Lyapunov Analysis

Consider the hybrid model in Example 2. Note that $g(0) = 0$. Now consider the piecewise quadratic Lyapunov-function candidate

$$V(x) = \begin{cases} \max\{2x_1^2, x_2^2\}, & \text{for all } x_1 \geq 0,\ x_2 \geq 0, \\ \max\{x_1^2, 2x_2^2\}, & \text{for all } x_1 \leq 0,\ x_2 \leq 0, \\ 2x_1^2 + 2x_2^2, & \text{for all } x_1 \geq 0, x_2 \leq 0, \\ x_1^2 + x_2^2, & \text{for all } x_1 \leq 0, x_2 \geq 0, \end{cases}$$

where $\operatorname{dom} V = \mathbb{R}^2$. The function V is continuous on its domain, and the sublevel sets of V are compact. Define $C_\odot = \{x \in C : x_2 \leq c\}$ where $c \in (0, 3)$, and note that $x \in C_\odot \backslash \{0\}$ implies that $x_1 \geq x_2$. It follows that V is continuously differentiable on an open set containing $C_\odot \backslash \{0\}$. Then, through routine calculations, we obtain $\langle \nabla V(x), f(x) \rangle < 0$ for all $C_\odot \backslash \mathcal{A}$. Next define $D_\odot = \{x \in D : |x_1| \leq d\}$ where $d \in (0, \sqrt{1/6})$. Then, for $x \in D_\odot \backslash \{0\}$ such that $g(x) \neq 0$, we obtain $V(g(x)) \leq 2(2x_1^2)^2 \leq 6d^2 x_1^2 < x_1^2 \leq V(x)$. These calculations and Theorem 20 establish that the origin of the system $(C_\odot, f, D_\odot, g)$ is globally pre-asymptotically stable. Since there exists a neighborhood K of the origin such that $C \cap K \subset C_\odot$ and $D \cap K \subset D_\odot$, the origin is pre-asymptotically stable for the system (C, f, D, g). Since $C \cup D = \mathbb{R}^2$, nontrivial solutions exist from each initial point in $C \cup D$, and consequently, bounded maximal solutions are complete. Thus, the origin is asymptotically stable for the system (C, f, D, g). ■

Example 3 Revisited: Lyapunov Analysis of the Bouncing Ball System

For the hybrid bouncing ball model of Example 3, we establish global asymptotic stability of the origin using Theorem 20. First note that $g(0) = 0$. Now consider the Lyapunov-function candidate

$$V(x) = x_2 + k\sqrt{\frac{1}{2}x_2^2 + \gamma x_1}$$

where $k > \sqrt{2}(1+\rho)/(1-\rho)$ and $\operatorname{dom} V = \{x \in \mathbb{R}^2 : (1/2) x_2^2 + \gamma x_1 \geq 0\}$. This choice for V is motivated by the ideas in [45]. We obtain $\langle \nabla V(x), f(x) \rangle = -\gamma < 0$ for all $x \in C \backslash \{0\}$ and, since $x \in D \backslash \{0\}$ implies $x_1 = 0$ and $x_2 \neq 0$, it follows that

$$\begin{aligned} V(g(x)) &= -\rho x_2 + k\sqrt{\frac{1}{2}\rho^2 x_2^2} \\ &\leq \rho\left(1 + \frac{k}{\sqrt{2}}\right)|x_2| \\ &< \left(-1 + \frac{k}{\sqrt{2}}\right)|x_2| \quad \text{for all } x \in D\backslash\{0\} \\ &\leq V(x). \end{aligned}$$

It now follows from Theorem 20 that the origin is globally pre-asymptotically stable. Since the sublevel sets of $V|_{\operatorname{dom} V \cap (C \cup D)}$ are compact, the origin is globally pre-asymptotically stable. It also can be shown that nontrivial solutions exist from each point in $C \cup D$. See "Existence, Uniqueness, and Other Well-Posedness Issues." Therefore, the origin is globally asymptotically stable. ■

Example 9 Revisited: Lyapunov Analysis of a Planar System

For the system in Example 9, assume $\exp(\alpha\pi/\omega)\gamma < 1$. Let α^* satisfy $\varepsilon_c := \alpha^* - \alpha > 0$ and $\varepsilon_d := 1 - \exp(2\alpha^*\pi/\omega)\gamma^2 > 0$. Then consider the Lyapunov-function candidate $V(x) := \exp(2\alpha^* T(x))|x|^2$, where T is a continuously differentiable function on an open set containing $C\backslash(0,0)\}$ and, for all $x \in C$, is equal to the time required to go from x to D. Equivalently, for $x \in C$, $T(x)$ is equal to ω^{-1} times the angle of x in the counterclockwise direction with the angle equal to zero on the negative x_2 -axis and equal to π on the positive x_2 -axis. The function V is continuous and continuously differentiable on an open set containing $C\backslash\{(0,0)\}$. Moreover, the sublevel sets of $V|_{\operatorname{dom} V \cap (C \cup D)}$ are compact. The function T satisfies $\langle \nabla T(x), f(x) \rangle = -1$ for all $x \in C\backslash\{(0,0)\}$. Also note that $\langle \nabla |x|^2, f(x) \rangle = 2\alpha|x|^2$ for all $x \in C$. It follows that

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &= -2\alpha^* V(x) + 2\alpha V(x) \\ &= -2\varepsilon_c V 1 x 2 \text{ for all } x \in C\backslash\{(0,0)\}. \end{aligned}$$

In addition,

$$\begin{aligned} V(g(x)) &= \exp(2\alpha^*\pi/\omega)\gamma^2|x|^2 \leq (1-\varepsilon_d)|x|^2 \\ &= (1-\varepsilon_d)V(x) \text{ for all } x \in D. \end{aligned}$$

It follows from Theorem 20 that the origin is globally asymptotically stable. In fact, the origin is globally exponentially stable since $V(x(t,j)) \leq \exp(-\lambda(t+j))V(x(0,0))$, where $\lambda = \min\{2\varepsilon_c, -\ln(1-\varepsilon_d)\}$, for all solutions x and all $(t,j) \in \operatorname{dom} x$, and $|x|^2 \leq V(x) \leq \exp(2\alpha^*\pi/\omega)|x|^2$ for all $x \in C \cup D$. ■

Example 21: A Bounded-Rate Hybrid System

This example is based on [36, Ex. 1]. Consider the hybrid system with data (C, F, D, G), where $D = \{0\}$, $G(0) = 0$, $C = \mathbb{R}^3_{\geq 0}$, and

$$F = \overline{\operatorname{con}}\left\{ f_1 := \begin{bmatrix} 100 \\ -90 \\ 1 \end{bmatrix}, \ f_2 := \begin{bmatrix} -90 \\ 100 \\ -90 \end{bmatrix}, \ f_3 := \begin{bmatrix} 1 \\ -90 \\ 100 \end{bmatrix}, \ f_4 := \begin{bmatrix} 1 \\ 50 \\ -90 \end{bmatrix}, \ f_5 := \begin{bmatrix} -90 \\ 50 \\ 1 \end{bmatrix} \right\}.$$

Consider the Lyapunov-function candidate $V(x) = x_1 + 1.5x_2 + x_3$, for which the sublevel sets of $V|_{\mathrm{dom}\, V \cap (C \cup D)}$ are compact. It is easy to verify, for each of the vectors f_i, $i = 1, \ldots, 5$, that $\langle \nabla V(x), f_i \rangle < 0$. Then it is a simple calculation to verify that $\langle \nabla V(x), f \rangle < 0$ for all $f \in F$. It follows from Theorem 20 that the origin is globally pre-asymptotically stable. In fact, the "pre" can be dropped since, for each initial point in $C \cup D$, there exists a nontrivial solution, and consequently, all maximal solutions to the hybrid system are complete. Indeed, for each $x \neq 0$ on the boundary of C there exists $f \in F$ along which flow in C is possible. In fact, for each $x \neq 0$ there exists $i \in \{1, \ldots, 5\}$ such that $f_i \in T_C(z)$ for all $z \in C$ near x. See "Existence, Uniqueness, and Other Well-Posedness Issues" for details. Existence of nontrivial solutions also holds when defining F to be the convex hull of the vectors f_i, $i = 1, 2, 3$, which corresponds to the system considered in [36, Ex. 1], but it cannot be verified by using only the three generating vectors at the nonzero points $(0, 0, x_3)$ and $(x_1, 0, 0)$. ■

The next example demonstrates that a Lyapunov function can be zero at points outside of $\mathcal{A}$.

Example 22: Lyapunov Function Can Be Zero at Points Outside of $\mathcal{A}$

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \mathbb{R}^2_{\geq 0}, \quad f(x) := \begin{bmatrix} x_1^2 + x_2^2 \\ -x_1^2 - x_2^2 \end{bmatrix} \quad \text{for all } x \in C,$$
$$D := \{x \in \mathbb{R}^2_{\geq 0} : x_2 = 0\}, \quad G(x) := 0 \quad \text{for all } x \in D.$$

We establish pre-asymptotic stability of $\mathcal{A} = \{0\}$. Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined as $V(x) := x_2$, which is a Lyapunov-function candidate for $(\mathcal{H}, \mathcal{A})$. Note that V is zero on D and satisfies

$$\langle \nabla V(x), f(x) \rangle = -x_1^2 - x_2^2 < 0 \quad \text{for all } x \in C \setminus \mathcal{A}.$$

In addition, the set $G(x) \setminus \mathcal{A}$ is empty. Thus, the origin is pre-asymptotically stable according to Theorem 20. In fact, global pre-asymptotic stability follows from the fact that the compact sets $\{x \in C \cup D : x_1 + x_2 \leq c\}$, where $c > 0$, cover $C \cup D$, and each one is forward invariant. ■

STABILITY ANALYSIS USING LYAPUNOV-LIKE FUNCTIONS AND AN INVARIANCE PRINCIPLE

Pre-asymptotically stable compact sets always admit Lyapunov functions. See "Converse Lyapunov Theorems." Nevertheless, it can be challenging to construct a Lyapunov function even for systems that are not hybrid. This fact inspires the development of relaxed, Lyapunov-based conditions for asymptotic stability. Perhaps the most well-known result in this direction is the Barbasin-Krasovskii theorem [5]. This result guarantees asymptotic stability when the Lyapunov function is not increasing along solutions and the only value of the Lyapunov function that can be constant along solutions is a value of the Lyapunov function taken only on the set $\mathcal{A}$. It is a specialization to asymptotic stability questions of the invariance principle [46], [47]. The next result, which can be called a hybrid Barbasin-Krasovskii-LaSalle theorem, is essentially contained in [72, Thm. 7.6] and combines the idea of the Barbasin-Krasovskii theorem with the invariance principle for hybrid systems. For more information on the invariance principle, see "Invariance."

For $\mu \in \mathbb{R}$ and a function $V : \mathrm{dom}\, V \to \mathbb{R}$ let $L_V(\mu) := \{x \in \mathrm{dom}\, V : V(x) = \mu\}$.

Theorem 23

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions and a compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfying $G(D \cap \mathcal{A}) \subset \mathcal{A}$. If there exists a Lyapunov-function candidate V for $(\mathcal{H}, \mathcal{A})$ that is positive on $(C \cup D) \setminus \mathcal{A}$ and satisfies

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq 0 \quad \text{for all } x \in C \setminus \mathcal{A},\ f \in F(x), \\ V(g) - V(x) &\leq 0 \quad \text{for all } x \in D \setminus \mathcal{A},\ g \in G(x) \setminus \mathcal{A} \end{aligned}$$

then the set $\mathcal{A}$ is stable. If furthermore there exists a compact neighborhood K of $\mathcal{A}$ such that, for each $\mu > 0$, no complete solution to $\mathcal{H}$ remains in $L_V(\mu) \cap K$, then the set $\mathcal{A}$ is pre-asymptotically stable. In this case, the basin of pre-attraction contains every compact set contained in K that is forward invariant.

A consequence of Theorem 23 is that $\mathcal{A}$ is globally pre-asymptotically stable if K can be taken to be arbitrarily large and $C \cup D$ is compact or the sublevel sets of $V|_{\mathrm{dom}\, V \cap (C \cup D)}$ are compact.

Realizing the full potential of Theorem 23 hinges on detecting whether $\mathcal{H}$ has complete solutions in $L_V(\mu) \cap K$. This property can be assessed by focusing on a hybrid system that is contained in the original hybrid system. For a vector $v \in \mathbb{R}^n$, let $v^\perp := \{w \in \mathbb{R}^n : \langle v, w \rangle = 0\}$. For each $\mu > 0$, define

$$\begin{aligned} F_\mu(x) &:= F(x) \cap \nabla V(x)^\perp \quad \text{for all } x \in C \cap L_V(\mu), && (25) \\ G_\mu(x) &:= G(x) \cap L_V(\mu) \quad \text{for all } x \in D \cap L_V(\mu), && (26) \\ C_\mu &:= \mathrm{dom}\, F_\mu, && (27) \\ D_\mu &:= \mathrm{dom}\, G_\mu \cap (\mathrm{dom}\, F_\mu \cup G_\mu(\mathrm{dom}\, G_\mu)). && (28) \end{aligned}$$

Ruling out complete solutions to $\mathcal{H}$ that remain in $L_V(\mu) \cap K$ is equivalent to ruling out complete solutions to the hybrid system $\mathcal{H}_{\mu, K} := (C_\mu \cap K, F_\mu, D_\mu \cap K, G_\mu \cap K)$.

Sometimes the absence of complete solutions to the system $\mathcal{H}_{\mu, K}$ can be verified by inspection. Otherwise, we may try to use an auxiliary function W, which does not need to be sign definite, to rule out complete solutions to

Hybrid dynamical systems can model a variety of closed-loop feedback control systems.

$\mathcal{H}_{\mu,K}$. When this auxiliary function is always decreasing along solutions to $\mathcal{H}_{\mu,K}$, the system $\mathcal{H}_{\mu,K}$ cannot have complete solutions since $(C_\mu \cup D_\mu) \cap K$ is compact for each $\mu > 0$ and each compact set K. More generally, Proposition 24 below can be applied iteratively to reduce the size of the flow and jump sets in an attempt to make them empty eventually, thereby ruling out complete solutions. This result is another consequence of the general invariance principle discussed in "Invariance."

Proposition 24

Consider the hybrid system $\mathcal{H} := (C, F, D, G)$ satisfying the Basic Assumptions, and assume that $C \cup D$ is compact. Let W be continuously differentiable on an open set containing C and continuous on D. Suppose

$$\begin{aligned} \langle \nabla W(x), f\rangle &\le 0 \quad \text{for all } x \in C, \quad f \in F(x),\\ W(g) - W(x) &\le 0 \quad \text{for all } x \in D, \ g \in G(x) \cap (C \cup D). \end{aligned}$$

For each $\nu \in W(\operatorname{dom} W \cap (C \cup D))$, define

$$\begin{aligned} F_\nu(x) &:= F(x) \cap \nabla W(x)^\perp \quad \text{for all } x \in C \cap L_W(\nu),\\ G_\nu(x) &:= G(x) \cap L_W(\nu) \quad \text{for all } x \in D \cap L_W(\nu),\\ C_\nu &:= \operatorname{dom} F_\nu,\\ D_\nu &:= \operatorname{dom} G_\nu \cap (\operatorname{dom} F_\nu \cup G_\nu(\operatorname{dom} G_\nu)). \end{aligned}$$

Then the hybrid system $\mathcal{H}$ has complete solutions if and only if there exists $\nu \in W(\operatorname{dom} W \cap (C \cup D))$ such that the hybrid system $\mathcal{H}_\nu := (C_\nu, F_\nu, D_\nu, G_\nu)$ has complete solutions.

Whenever $\langle \nabla W(x), f\rangle < 0$ for all $x \in C$, $f \in F(x)$, and $W(g) - W(x) < 0$ for all $x \in D$, $g \in G(x) \cap (C \cup D)$, it follows that C_ν and D_ν are empty and thus $\mathcal{H}_\nu$ has no complete solutions. Typically, C_ν and D_ν have fewer points than C and D, and so it may be easier to establish that $\mathcal{H}_\nu$ has no complete solutions than to establish that $\mathcal{H}$ has no complete solutions. The combination of Theorem 23 with a repeated application of Proposition 24 is the idea behind Matrosov's theorem for time-invariant hybrid systems, as presented in [74]. The original Matrosov theorem, conceived for time-varying systems, appears in [53]. Matrosov theorems reach their full potency for time-varying systems, where invariance principles typically do not hold. Refinements of Matrosov's original idea have appeared over the years. See [60] and [50] and the references therein. A version for time-varying hybrid systems appears in [52].

We illustrate Theorem 23 with several examples.

Example 3 Revisited: Lyapunov Analysis with the Invariance Principle.

Consider the hybrid model of a bouncing ball in Example 3. We establish global asymptotic stability of the origin. Let $\mathcal{A}$ be the origin in $\mathbb{R}^2$. The condition $g(D \cap \mathcal{A}) \subset \mathcal{A}$ holds. Consider a Lyapunov-function candidate $V : \mathbb{R}^2 \to \mathbb{R}$ given by $V(x) = \gamma x_1 + (1/2)x_2^2$. The sublevel sets of $V|_{C \cup D}$ are compact, and V satisfies

$$\langle \nabla V(x), f(x)\rangle = \gamma x_2 - \gamma x_2 = 0 \quad \text{for all } x \in C$$

and

$$V(g(x)) = \frac{1}{2}\rho^2 x_2^2 < \frac{1}{2}x_2^2 = V(x) \quad \text{for all } x \in D \backslash \mathcal{A}.$$

The set D_μ defined in (25)–(28) is empty for each $\mu > 0$ since $\operatorname{dom} G_\mu$ is empty for each $\mu > 0$. Thus, $\mathcal{A}$ is globally asymptotically stable as long as there are no complete solutions that flow only and keep V equal to a positive constant. It may be possible to see by inspection that no solutions of this type exist. Otherwise, consider using the idea in Proposition 24 to rule out complete solutions to $\mathcal{H}_\mu$, $\mu > 0$, that only flow.

Next, applying Proposition 24 with $C := \{x \in \mathbb{R}^2 : x_1 \ge 0, V(x) = \mu\}$, D given by the empty set, and $W_1(x) = x_2$, we obtain

$$\langle \nabla W_1(x), f(x)\rangle = -\gamma < 0 \quad \text{for all } x \in C.$$

The set C_ν in Proposition 24 is empty for all ν, and thus complete solutions that only flow are ruled out. It follows from Theorem 23 that the set $\mathcal{A}$ is globally asymptotically stable. ■

Example 25: Interacting Fireflies.

Consider the firefly model given earlier with two fireflies, and suppose each flow map is equal to a constant $f > 0$. This choice results in a hybrid system $\mathcal{H}$ with $x \in \mathbb{R}^2$ and the data

$$C := [0,1] \times [0,1], \quad F(x) := \begin{bmatrix} f \\ f \end{bmatrix} \quad \text{for all } x \in C,$$

$$D := \{x \in C : \max\{x_1, x_2\} = 1\},$$

$$G(x) := \begin{bmatrix} g((1+\varepsilon)x_1) \\ g((1+\varepsilon)x_2) \end{bmatrix} \quad \text{for all } x \in D,$$

Invariance

The invariance principle has played a fundamental role over the years as a tool for establishing asymptotic stability in nonlinear control algorithms. Prime examples include the Jurdjevic-Quinn approach to nonlinear control design [S45], [S47] as well as general passivity-based control [S42], [S48].

The invariance principle transcends stability analysis by characterizing the nature of the sets to which a bounded solution to a dynamical system converges. The basic invariance principle for dynamical systems with unique solutions is due to LaSalle [47], [S46]. It has been extended to systems with possibly nonunique solutions, in particular, differential inclusions, as well as integral and differential versions with nonsmooth functions [S49]. A version of the invariance principle for hybrid systems with unique solutions and continuous dependence on initial conditions appears in [51]. An invariance principle for left-continuous dynamical systems having unique solutions and quasi-continuous dependence on initial conditions is given in [S43]. An invariance principle for switched systems is given in [S3] and [S44]. The results quoted below are from [72], which contains results analogous to those in [S49] but for hybrid dynamical systems.

Lemma S12

Suppose that the hybrid system $\mathcal{H}$ satisfies the Basic Assumptions, and let $x : \operatorname{dom} x \to \mathbb{R}^n$ be a complete and bounded solution to $\mathcal{H}$. Then the *omega-limit* of x, that is, the set

$$\omega(x) = \{z \in \mathbb{R}^n : \text{ there exists } (t_i, j_i) \in \operatorname{dom} x \text{ there exists } t_i + j_i \to \infty,\ x(t_i, j_i) \to z\},$$

is nonempty, compact, and *weakly invariant* in the sense that the following conditions are satisfied:

i) $\omega(x)$ is *weakly forward invariant*, that is, for each $z \in \omega(x)$ there exists a complete solution y to $\mathcal{H}$ such that $y(0,0) = z$ and $y(t,j) \in \omega(x)$ for all $(t,j) \in \operatorname{dom} y$.

ii) $\omega(x)$ is *weakly backward invariant*, that is, for each $z \in \omega(x)$ and each $m > 0$, there exists a solution y to $\mathcal{H}$ such that $y(t,j) \in \omega(x)$ for all $(t,j) \in \operatorname{dom} y$ and such that $y(t_z, j_z) = z$ for some $(t_z, j_z) \in \operatorname{dom} y$ with $t_z + j_z > m$.

Furthermore, the distance from $x(t,j)$ to $\omega(x)$ decreases to zero as $t + j \to \infty$, and, in fact, $\omega(x)$ is the smallest closed set with this property.

Theorem S13 [72, Thm. 4.3]

Suppose that the hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the Basic Assumptions. Let $V : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable on an open set containing C, continuous on $C \cup D$, and satisfy

$$u_C(x) \le 0 \text{ for all } x \in C, \text{ where } u_C(x) := \max_{f \in F(x)} \langle \nabla V(x), f \rangle,$$
$$u_D(x) \le 0 \text{ for all } x \in D, \text{ where } u_D(x) := \max_{g \in G(x)} V(g) - V(x).$$

Then, there exists $r \in \mathbb{R}$ such that each complete and bounded solution x to $\mathcal{H}$ converges to the largest weakly invariant subset of the set

$$\{z : V(z) = r\} \cap (u_C^{-1}(0) \cup (u_D^{-1}(0) \cap G(u_D^{-1}(0)))), \quad \text{(S8)}$$

where $u_C^{-1}(0) := \{z \in C : u_C(z) = 0\}$ and $u_D^{-1}(0) := \{z \in D : u_D(z) = 0\}$.

Further information on a particular complete and bounded solution x may lead to more precise descriptions of the set to which x converges. For example, if x is Zeno (see "Zeno Solutions") then both forward and backward weak invariance of $\omega(x)$ can be verified by a complete solution to $\mathcal{H}$ that never flows, and, moreover, we can limit our attention to weakly invariant subsets of

$$\{z \in D : u_D(z) = 0\} \cap G(\{z \in D : u_D(z) = 0\}).$$

On the other hand, if there exists $\tau > 0$ such that all jumps of x are separated by at least an amount of time τ, then we can limit our attention to weakly invariant subsets of

$$\{z \in C : u_C(z) = 0\}.$$

Example 14: Illustration of the Invariance Principle.

Consider the hybrid system in the plane with data

$$C := \{x \in \mathbb{R}^2 : x_2 \ge 0\}, \quad F(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \text{ for all } x \in C,$$
$$D := \{x \in \mathbb{R}^2 : x_2 \le 0\}, \quad G(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \text{ for all } x \in D.$$

Solutions flow in the counterclockwise direction when in the closed upper-half plane. From the closed lower-half plane, which is where solutions are allowed to jump, solutions undergo an instantaneous rotation of $\pi/2$, also in

where $\varepsilon > 0$, $g(s) = s$ when $s < 1$, $g(s) = 0$ when $s > 1$ and $g(s) = \{0, 1\}$ when $s = 1$.

Note that the compact set $\mathcal{A} := \{x \in C : x_1 = x_2\}$ represents synchronized flashing. We establish asymptotic stability $\mathcal{A}$ and characterize the basin of attraction.

Define $k := \varepsilon/(2+\varepsilon)$ and consider the Lyapunov-function candidate

$$V(x) := \min\{|x_1 - x_2|, 1 + k - |x_1 - x_2|\}.$$

This function is continuously differentiable on the open set $\mathcal{X} \backslash \mathcal{A}$, where

$$\mathcal{X} := \{x \in \mathbb{R}^2 : V(x) < (1+k)/2 = (1+\varepsilon)/(2+\varepsilon)\}$$
$$= \{x \in \mathbb{R}^2 : |x_1 - x_2| \ne (1+\varepsilon)/(2+\varepsilon)\}.$$

Let $\nu^* = (1+\varepsilon)/(2+\varepsilon)$, let $\nu \in (0, \nu^*)$, $K_\nu = \{x \in C \cup D : V(x) \le \nu\}$, and define $C_\odot = C \cap K_\nu$ and $D_\odot = D \cap K_\nu$. Since V is a function of only $x_1 - x_2$ and $F_1(x) = F_2(x)$, it follows that the counterclockwise direction. We analyze the asymptotic behavior of the complete solution x with $x(0,0)=(1,0)$. A natural function V to consider is $V(x)=|x|^2$. Then, $u_C(x)=0$ for all $x\in C$ and $u_D(x)=0$ for all $x\in D$. Since F and G are single-valued maps, this property is enough to conclude that $V(x(t,j))=V(x(0,0))=1$ for all $(t,j)\in \operatorname{dom} x$, and, in particular, that x is bounded. Now note that the set of all $z\in C$ such that $u_C(z)=0$ is the whole set C, similarly, the set of all $z\in D$ such that $u_D(z)=0$ is the whole set D. Furthermore, the set $G(u_D^{-1}(0))=G(\{z\in D: u_D(z)=0\})$ is the closed right-half plane. Hence, Theorem S13 implies that x converges to the largest weakly invariant subset of

$$S=\{z\in\mathbb{R}^2: |z|=1,\ z_1\geq 0 \text{ or } z_2\geq 0\}.$$

This subset turns out to be

$$S^{\text{inv}}=\{z\in\mathbb{R}^2: |z|=1,\ z_2\geq 0\}\cup(0,-1).$$

As Figure S10 depicts, S^{inv} is also exactly the range of the periodic solution x and thus its omega-limit. Indeed, we have $x(t,j)=(\cos t,\sin t)$ for $t\in[0,\pi]$, $x(\pi,1)=(0,-1)$, and $x(t,j)=x(t-\pi,j-2)$ for $t\geq\pi$, $j\geq 2$. In particular, S^{inv} is the smallest closed set to which x converges.

Note that asserting that the periodic solution x converges to S^{inv} is possible only by considering both weak forward and backward invariance. Indeed, the largest weakly forward invariant subset of S is S itself. ■

In the conclusion of Theorem S13 we can equivalently consider the largest weakly invariant subset of

$$\{z: V(z)=r\}\cap(u_C^{-1}(0)\cup u_D^{-1}(0)).$$

This set has a simpler description but is usually larger than the set used in Theorem S13. More extensive investigation using invariance properties is then needed to obtain the same set to which solutions converge. If the term $G(u_D^{-1}(0))$ in (S8) were to be omitted when applying Theorem S13 to the hybrid system in Example 14, then the set S would be larger. In particular, S would be the whole unit circle. Nevertheless, considering forward and backward weak invariance leads to the same invariant subset S^{inv}.

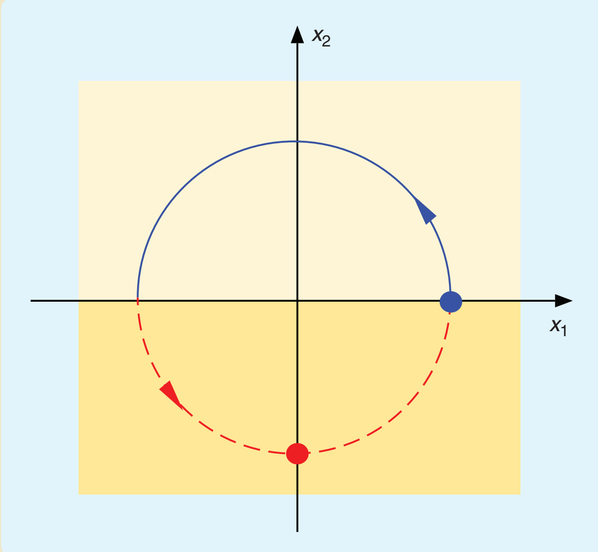

FIGURE S10 A solution to the hybrid system in Example S14. All solutions to this system flow in the counterclockwise direction when in the closed upper-half plane. From the closed lower-half plane, all solutions jump and undergo a rotation by $\pi/2$, also in the counterclockwise direction. An application of Theorem S13 indicates that all solutions converge to the set $S^{\text{inv}}=\{z\in\mathbb{R}^2: |z|=1,\ z_2\geq 0\}\cup(0,\ -1)$. This set coincides with the range of the solution that starts flowing from $x(0,0)=(1,0)$.

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$$\langle\nabla V(x), F(x)\rangle = 0 \quad \text{for all } x\in C_\odot\backslash\mathcal{A}.$$

Now consider $x\in D_\odot$. By symmetry, without loss of generality we can consider $x=(1,x_2)$, where $x_2\in[0,1]\backslash\{1/(2+\varepsilon)\}$. We obtain

$$V(x)=\min\{1-x_2, k+x_2\},$$
$$V(G(x))=\min\{g((1+\varepsilon)x_2), 1+k-g((1+\varepsilon)x_2)\}.$$

When $g((1+\varepsilon)x_2)=0$, it follows that $V(G(x))=0$. Now consider the case where $g((1+\varepsilon)x_2)=(1+\varepsilon)x_2$. There are two possibilities, namely, $x_2<1/(2+\varepsilon)$ and $x_2>1/(2+\varepsilon)$. In the first situation,

$$V(x)=k+x_2>(1+\varepsilon)x_2\geq V(G(x)).$$

In the second situation, $V(x)=1-x_2>1+k-(1+\varepsilon)x_2\geq V(G(x))$. It now follows from Theorem 23 that the set

$\mathcal{A}$ is globally pre-asymptotically stable for the system $(C_{\odot}, F, D_{\odot}, G)$. To see this fact, note that the calculations above imply that the set $\operatorname{dom} G_{\mu}$, and thus also D_{μ}, used in (25)–(28) are empty for each $\mu > 0$. Furthermore, complete solutions that only flow are impossible. This fact can be seen by inspection or by applying Proposition 24 with $W(x) = -x_1 - x_2$.

Since $\nu \in (0, \nu^*)$ used in the definitions of $C_{\odot}$ and $D_{\odot}$ is arbitrary, and the sets K_{ν} are compact and forward invariant, the basin of attraction of $\mathcal{A}$ for the system (C, F, D, G) contains $\mathcal{X}$. In fact, solutions starting from the condition $|x_1 - x_2| = (1+\varepsilon)/(2+\varepsilon)$ do not converge to $\mathcal{A}$. This behavior can be seen by noting that $V(G(x)) = V(x)$ when $x_1 = 1$ and $|x_1 - x_2| = (1+\varepsilon)/(2+\varepsilon)$. It then follows that $\mathcal{X}$ is the basin of attraction for $\mathcal{A}$ for the system (C, F, D, G). ∎

The final two illustrations of Theorem 23 address stability analysis for general classes of systems where sampling is involved. The first application pertains to classical sampled-data systems. The second application covers nonlinear networked control systems, which are becoming more prevalent due to the ubiquity of computers and communication networks.

Example 26: Absolute Stability for Sampled-Data Systems

In this example we study the absolute stability of periodic jump linear systems, including sampled-data systems. Absolute stability refers to asymptotic stability in the presence of arbitrary time-varying, sector-bounded nonlinearities [39, Sec. 7.1]. It is possible to model sector-bounded nonlinearities through a differential inclusion.

Consider the class of hybrid systems with state $x \in \mathbb{R}^{n+1}$, decomposed as $x = (\xi, \tau)$, where $\xi \in \mathbb{R}^n$, and data

$$\begin{aligned} C &:= \{x : \tau \in [0, T]\}, \\ F(x) &:= \left\{ \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, w\colon \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} M_1 & M_3 \\ M_3^\top & M_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \le 0 \right\} \\ &\quad \text{for all } x \in C, \\ D &:= \{x : \tau = T\}, \\ G(x) &:= \left\{ \begin{bmatrix} J\xi + Lw \\ 0 \end{bmatrix}, w\colon \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} N_1 & N_3 \\ N_3^\top & N_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \le 0 \right\} \\ &\quad \text{for all } x \in D, \end{aligned}$$

where M_1, M_2, N_1, and N_2 are symmetric. We assume that the eigenvalues of M_1 and N_1, which are real, are nonnegative; in other words, M_1 and N_1 are positive-semidefinite matrices. The matrices M_2 and N_2 are positive definite, meaning that their eigenvalues are positive. These conditions guarantee that the Basic Assumptions hold. The state τ corresponds to a timer state that forces jumps every T seconds. With w constrained to zero, the system is a simple periodic jump linear system with flow equation $\dot{\xi} = A\xi$ and jump equation $\xi^+ = J\xi$. More generally, w is constrained to satisfy a quadratic constraint that is a function of the state ξ, thereby modeling a sampled-data linear control system with sector-bounded nonlinearities.

To establish asymptotic stability of the compact set $\mathcal{A} := \{x : \xi = 0,\ \tau \in [0, T]\}$, consider a Lyapunov-function candidate of the form $V(x) := \xi^\top P(\tau)\xi$, where $P : [0, T] \to \mathcal{P}_n$ and $\mathcal{P}_n$ denotes the set of symmetric, positive definite matrices. The function P is chosen to satisfy

$$\begin{aligned} \nabla P(\tau) = &-A^\top P(\tau) - P(\tau)A + M_1 \\ &-(P(\tau)B - M_3)M_2^{-1}(B^\top P(\tau) - M_3^\top), \end{aligned} \tag{29}$$

where we assume that $P(T) = P^\top(T) > 0$ can be chosen to guarantee that $P(\tau)$ exists and is positive definite for all $\tau \in [0, T]$. For more on this condition, see below. With this choice for V, for all $x \in C$ and $f \in F(x)$, we obtain

$$\begin{aligned} \langle \nabla V(x), f \rangle &= 2\xi^\top P(\tau)Bw + \xi^\top M_1 \xi - \xi^\top (P(\tau)B - M_3) \\ &\quad \times M_2^{-1}(B^\top P(\tau) - M_3^\top)\xi \\ &\le 2\xi^\top P(\tau)Bw + \xi^\top M_1 \xi - \xi^\top (P(\tau)B - M_3) \\ &\quad \times M_2^{-1}(B^\top P(\tau) - M_3^\top)\xi \\ &\quad - \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} M_1 & M_3 \\ M_3^\top & M_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix}. \end{aligned}$$

Completing squares leads to the conclusion that

$$\langle \nabla V(x), f \rangle \le 0 \quad \text{for all } x \in C,\ f \in F(x)\,.$$

We now turn to the change in V during jumps. Asymptotic stability of the set $\mathcal{A}$ follows from Theorem 23 when there exists $\varepsilon > 0$ such that

$$\begin{aligned} &(\xi^\top J^\top + w^\top L^\top)P(0)(J\xi + Lw) - \xi^\top P(T)\xi \\ &\qquad - \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} N_1 & N_3 \\ N_3^\top & N_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \le -\varepsilon \xi^\top \xi. \end{aligned} \tag{30}$$

To check this condition, we need to relate $P(0)$ and $P(T)$. The solution to the matrix differential equation (29) can be written explicitly by forming the Hamiltonian matrix

$$H = \begin{bmatrix} A - BM_2^{-1}M_3^\top & BM_2^{-1}B^\top \\ M_1 - M_3 M_2^{-1} M_3^\top & -(A - BM_2^{-1}M_3^\top)^\top \end{bmatrix},$$

forming the matrix exponential $\Phi(\tau) := \exp(-H\tau)$, partitioning Φ in the same way that H is partitioned, and verifying that, where P is defined,

$$P(T - \tau) = (\Phi_{21}(\tau) + \Phi_{22}(\tau)P(T))(\Phi_{11}(\tau) + \Phi_{12}(\tau)P(T))^{-1}.$$

When $\Phi_{11}(t)$ is invertible for all $t \in [0, T]$, the quantities $-\Phi_{11}(t)^{-1}\Phi_{12}(t)$ and $\Phi_{21}(t)\Phi_{11}(t)^{-1}$ are positive semidefinite for all $t \in [0, T]$. Also, defining $\Psi := \Phi(T)$, $X := P(T)$, and letting S be a matrix satisfying $SS^\top = -\Psi_{11}^{-1}\Psi_{12}$,

$P(t)$ is defined and positive definite for all $t \in [0, T]$ when $X = X^\top > 0$ and $S^\top XS < I$. These conditions on X and S are guaranteed by what follows.

Using the fact that Ψ is symplectic, that is, $\Psi^\top \Omega \Psi = \Omega$, where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, the value $P(0)$ is related to X through the formula

$$P(0) = \Psi_{11}^{-1}[X + XS(I - S^\top XS)^{-1} S^\top X]\Psi_{11}^{-T} + \Psi_{21}\Psi_{11}^{-1}.$$

Substituting for $P(0)$ in (30) and using Schur complements, it follows that asymptotic stability of the set $\mathcal{A}$ is guaranteed when $\Phi_{11}(t)$ is invertible for all $t \in [0, T]$ and the matrix inequality

$$\begin{bmatrix} J^\top \\ L^\top \\ 0 \end{bmatrix} (\Psi_{11}^{-1} X \Psi_{11}^{-\top} + \Psi_{21}\Psi_{11}^{-1}) [J \quad L \quad 0] - \begin{bmatrix} X + N_1 & N_3 & J^\top \Psi_{11}^{-1} XS \\ N_3^\top & N_2 & L^\top \Psi_{11}^{-1} XS \\ S^\top X \Psi_{11}^{-\top} J & S^\top X \Psi_{11}^{-\top} L & (I - S^\top XS) \end{bmatrix} < 0 \quad (31)$$

is satisfied for some $X = X^\top > 0$. ■

Example 27: Networked Nonlinear Control Systems.

A wide variety of interesting control problems are associated with networked control systems. Stability analysis for some of these problems is similar to stability analysis for sampled-data systems. The primary differences are the following:

- » The length of time between updates can be unpredictable, due to network variability.
- » The update rules are often time varying, typically periodic as when using a round-robin protocol, or nonlinear, as when using the "try-once-discard" protocol presented in [89]. These attributes arise from communication constraints that limit how much of the state can be updated at a given time. Protocols are developed to make a choice about which component of the state to update at the current time.

Time-varying update rules can be addressed in the framework of sampled-data systems [20]. Uncertain and variable update times also can be addressed in the framework of sampled-data systems when we are satisfied with a common quadratic Lyapunov function for the various discrete-time systems that emerge from the variable update times. When we move to nonlinear dynamics and nonlinear updates, it becomes more challenging to exploit exact knowledge of the functions that describe the evolution of networked control system. In this situation, a reasonable approach is to analyze the closed-loop behavior using coarse information about the functions involved.

Consider the networked control system with state $x = (x_1, x_2, \tau) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}$ and data

$$C := \{x = (x_1, x_2, \tau) : \tau \in [0, T]\},$$

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ 1 \end{bmatrix} \text{ for all } x \in C,$$

$$D := \{x = (x_1, x_2, \tau) : \tau = T\},$$

$$g(x) = \begin{bmatrix} x_1 \\ g_2(x_1, x_2) \\ [0, \varepsilon T] \end{bmatrix} \text{ for all } x \in D,$$

where $\varepsilon \in [0, 1)$. The quantity x_1 denotes physical states in the plants that are being controlled over the communication network. The quantity x_2 denotes states associated with communication. For example, the state x_2 may denote mismatch between ideal control actions and control actions achieved by the network. Uncertainty and variability in the transmission times are captured by the set-valued nature of the update rule for the timer state τ, which can be updated to any value in the interval $[0, \varepsilon T]$. For simplicity, we consider the case where the desired steady-state behavior corresponds to $(x_1, x_2) = (0, 0)$. In a more general analysis, both x_1 and x_2 may contain states that don't tend to zero.

The function g_2 addresses the communication protocol used to update x_2 at transmission times. We are considering time-invariant protocols here, since g_2 does not depend on time. However, all of the ideas below extend to the time-varying, periodic case.

We assume there exist continuous functions ϕ_1 and ϕ_2 that are zero at zero and positive otherwise, positive numbers ε, k_{11}, k_{12}, k_{21}, k_{22}, and k_3, and continuously differentiable functions V_1, V_2 that are zero at zero and positive otherwise, with compact sublevel sets, such that

$$\langle \nabla V_1(x_1), f_1(x_1, x_2)\rangle \le -(\varepsilon + k_{11})\phi_1(x_1)^2 + k_{12}\phi_1(x_1)\phi(x_2), \quad (32)$$

$$\langle \nabla V_2(x_2), f_2(x_1, x_2)\rangle \le k_{21}\phi(x_1)\phi(x_2) + k_{22}\phi_2(x_2)^2, \quad (33)$$

$$\max\{k_3\phi(x_2)^2, \lambda^{-2} V_2(g_2(x_1, x_2))\} \le V_2(x_2). \quad (34)$$

Possible choices for the functions ϕ_1 and ϕ_2 include the Euclidean norms of the arguments. Condition (32) includes the assumption that the origin is globally asymptotically stable in the case of perfect communication, that is, $x_2 \equiv 0$. Condition (34) includes the assumption that the communication protocol, which is determined by the function g_2, makes $x_2 = 0$ asymptotically stable when the protocol dynamics are disconnected from the continuous-time dynamics. Condition (33) is a coupling condition between (32) and (34) that bounds the growth of x_2 during flows.

Now consider the Lyapunov-function candidate $V(x) := V_1(x_1) + p(\tau)V_2(x_2)$, where $p : [0, T] \to \mathbb{R}_{>0}$ satisfies

$$\nabla p(\tau) = -\frac{1}{k_3 k_{11}}\left[k_{22}k_{11}p(\tau) + \frac{1}{4}(k_{12} + p(\tau)k_{21})^2\right], \quad (35)$$

and we assume that $p(T) > 0$ can be chosen so that p is defined on $[0, T]$. This choice is similar to the Lyapunov function considered in [15]. We obtain

$$\begin{aligned}\langle \nabla V(x), f(x)\rangle \leq & -(k_{11} + \varepsilon)\phi_1(x_1)^2 + k_{12}\phi_1(x_1)\phi_2(x_2) \\ & + p(\tau)[k_{21}\phi_1(x_1)\phi_2(x_2) + k_{22}\phi_2(x_2)^2] \\ & - \frac{1}{k_3 k_{11}}\left[k_{22}k_{11}p(\tau) + \frac{1}{4}(k_{12} + p(\tau)k_{21})^2\right] \\ & \times k_3\phi_2(x_2)^2 \\ = & \begin{bmatrix}\phi_1(x_1) \\ \phi_2(x_2)\end{bmatrix}^\top \end{aligned}$$

$$\begin{bmatrix} -(k_{11}+\varepsilon) & \frac{1}{2}(k_{12} + p(\tau)k_{21}) \\ \frac{1}{2}(k_{12} + p(\tau)k_{21}) & -\frac{1}{4k_{11}}(k_{12} + p(\tau)k_{21})^2 \end{bmatrix} \begin{bmatrix}\phi_1(x_1) \\ \phi_2(x_2)\end{bmatrix}$$

$$\leq -\varepsilon\phi_1(x_1)^2 \leq 0.$$

We now turn to the change in V due to jumps. Since $p(0) \geq p(t)$ for all $t \in [0, \varepsilon T]$, we also have

$$\begin{aligned} V(g(x)) &\leq V_1(x_1) + p(0)V_2(g(x_1, x_2)) \\ &\leq V_1(x_1) + p(0)\lambda^2 V_2(x_2). \end{aligned}$$

When $p(T)$ can be chosen so that $p(0)\lambda^2 < p(T)$, global asymptotic stability of the set

$$\mathcal{A} := \{x: x_1 = 0, x_2 = 0, \tau \in [0, T]\}$$

follows from Theorem 23. Indeed, points in the set $\operatorname{dom} F_\mu$ used in (25) must have $x_1 = 0$, while points in the set $\operatorname{dom} G_\mu$ used in (26) must have $x_2 = 0$, and thus points in the set $G_\mu(\operatorname{dom} G_\mu)$ must have $x_2 = 0$. In turn, points in the set D_μ defined in (28) must have $x_1 = 0$, $x_2 = 0$, and $V(x) = \mu$. Since this is impossible for $\mu > 0$, D_μ is empty for $\mu > 0$. Therefore, to rule out complete solutions for $\mathcal{H}_\mu$ we just need to rule out solutions that only flow. These solutions are ruled out by $\dot{\tau} = 1$ and the fact that C_μ bounded in the τ direction.

Now we must relate $p(0)$ to $p(T)$. As in the sampled-data problem, the solution to the differential equation (35) can be written explicitly by forming the Hamiltonian matrix

$$H := \begin{bmatrix} -\frac{k_{22}}{2k_3} - \frac{k_{21}k_{22}}{k_3 k_{11}} & \frac{k_{21}^2}{4k_3 k_{11}} \\ -\frac{k_{12}^2}{4k_3 k_{11}} & \frac{k_{22}}{2k_3} + \frac{k_{21}k_{22}}{k_3 k_{11}} \end{bmatrix}$$

and the corresponding matrix exponential $\Phi(\tau) = \exp(-H\tau)$, and verifying

$$p(T - t) = (\phi_{21}(t) + \phi_{22}(t)p(T))(\phi_{11}(t) + \phi_{12}(t)p(T))^{-1} \quad \text{for all } t \in [0, T].$$

When $\phi_{11}(t) > 0$ for all $t \in [0, T]$, the quantities $\phi_{21}(t)$ and $-\phi_{12}(t)$ are nonnegative for all $t \in [0, T]$. When, in addition, $p(T) > 0$ and $-p(T)\phi_{12}(T)/\phi_{11}(T) < 1$, the quantity $p(t)$ is defined for all $t \in [0, T]$. Then, with the definitions $\Psi := \Phi(T)$, $s := \sqrt{-\psi_{12}/\psi_{11}}$, $r := \sqrt{\psi_{21}/\psi_{11}}$, and $q := p(T)$, it follows that

$$p(0) = r^2 + [q + (1 - s^2 q)^{-1} s^2 q^2]/\psi_{11}^2.$$

In turn, the stability condition $p(0)\lambda^2 < p(T)$ is guaranteed by the conditions that $\phi_{11}(t) > 0$ for all $t \in [0, T]$ and

$$\begin{bmatrix} \lambda^2\left(r^2 + \frac{q}{\psi_{11}^2}\right) - q & \lambda sq/\psi_{11} \\ \lambda sq/\psi_{11} & -1 + s^2 q \end{bmatrix} < 0, \quad 0 < q.$$

These conditions on q are feasible for T sufficiently small since $\lambda < 1$, and r and s tend to zero and ψ_{11} tends to one as T tends to zero. ■

LYAPUNOV-BASED HYBRID FEEDBACK CONTROL

In a control system, part of the data of the system is free to be designed. Sensors are used to measure state variables, and actuators are used to affect the system's behavior, resulting in a closed-loop dynamical system. When the design specifies regions in the state space where flowing and jumping, respectively, are allowed, it yields hybrid feedback control. In this case, the closed-loop system is a hybrid dynamical system. Since a typical goal of feedback control is asymptotic stability, construction of the hybrid control algorithm is guided by stability analysis tools for hybrid dynamical systems. In this section, we use several examples to illustrate how Lyapunov-based analysis tools, especially the Barbasin-Krasovskii-LaSalle theorem, are used to derive hybrid feedback control laws. In the examples below, the asymptotic stability induced by feedback is robust in the sense of theorems 15 and 17. This robustness is achieved by insisting on regularity of the data of the control laws so that the resulting closed loop hybrid systems satisfy the Basic Assumptions.

Example 25 Revisited: Impulsive Clock Synchronization Based on the Firefly Model

The synchronicity analysis in Example 25 for a network of fireflies can be thought of as a synchronization control problem where the impulsive control law $u_i = \varepsilon x_i$ in the jump equation $x_i^+ = g(x_i + u_i)$ is chosen to make Theorem 23 applicable, establishing almost global synchronization. Global synchronization can be achieved for two fireflies by redesigning u_i. Indeed, letting $k \in (0, 1)$ and picking

Some asymptotically controllable nonlinear control systems cannot be robustly stabilized to a point using classical, time-invariant state feedback ... hybrid feedback, and supervisory control in particular, makes robust stabilization possible.

$u_i = 0$ when $x_i < (1-k)/2$ and $u_i = 2k$ when $x_i > (1-k)/2$ results in global synchronization. This property is established using the same Lyapunov-function candidate used in Example 25.

Example 26 Revisited: Sampled-Data Feedback Control Design

The design of a linear sampled-data feedback controller can be carried out based on the analysis in Example 26. The matrix M in Example 26 is used to indicate a desired dissipation inequality for the closed-loop system or a desired stability robustness margin, while the matrices L and N are typically zero. In this case, using the definitions of S and Ψ given in Example 26, the matrix inequality (31) reduces to

$$\begin{bmatrix} J^\top(\Psi_{11}^{-1}X\Psi_{11}^{-\top} + \Psi_{21}\Psi_{11}^{-1})J - X & J^\top\Psi_{11}^{-1}XS \\ S^\top X\Psi_{11}^{-\top}J & -I + S^\top XS \end{bmatrix} < 0. \quad (36)$$

In turn, the matrix J decomposes as $J = G + H\Theta K$, where G, H, and K are fixed and Θ is a design parameter corresponding to feedback gains. Then feasibility of the matrix inequality (36) in the parameters Θ and $X = X^\top > 0$ can be identified with a synthesis problem for the discrete-time system

$$\begin{aligned} \xi^+ &= \Psi_{11}^{-\top}(G + H\Theta K)\xi + Sw, \\ y &= Y(G + H\Theta K)\xi, \end{aligned}$$

where $Y^\top Y = \Psi_{21}\Psi_{11}^{-1}$. Recall that the matrix $\Psi_{21}\Psi_{11}^{-1}$ is guaranteed to be positive semidefinite as long as $\Phi_{11}(t)$ is invertible for all $t \in [0, T]$. The synthesis problem corresponds to picking the parameter Θ to ensure that the ℓ_2-gain from the disturbance w to the output y is less than one, as certified by the energy function $V(\xi) = \xi^\top X\xi$ through the condition $V(x^+) - V(x) \le -\varepsilon|\xi|^2 - |y|^2 + |w|^2$ for some $\varepsilon > 0$.

When the parameters G, H, and K correspond to state feedback or full-order output feedback, the above synthesis problem can be cast as a convex optimization problem in the form of a linear matrix inequality [22]. Generalizations of this sampled-data control approach are developed for multirate sampled-data systems in [44] based on time-varying lifting. The approach discussed above, which is expressed directly in terms of a Lyapunov analysis and thus avoids lifting, is reminiscent of the approach to sampled-data control taken in [79] and [85].

Example 28: Resetting Nonlinear Control

This example is based on [11], [29], and [30]. Consider the nonlinear control system $\dot{\xi} = f(\xi, u)$, where $\xi \in \mathbb{R}^n$ and f is continuous, together with the dynamic controller $\dot{\eta} = \phi(\eta, \xi)$, $u = \kappa(\eta, \xi)$, where $\eta \in \mathbb{R}^m$ and ϕ and κ are continuous. Let $x = (\xi, \eta)$ and, for each $x \in \mathbb{R}^{n+m}$, define $F(x) := (f(\xi, \kappa(\eta, \xi)), \phi(\eta, \xi))$. We refer to the system $\dot{x} = F(x)$ as $\mathcal{H}_c$. Let V be a Lyapunov-function candidate for $(\mathcal{H}_c, \{0\})$ that is positive on $\mathbb{R}^{n+m}\setminus\{0\}$ and for which the sublevel sets of V are compact. Moreover, suppose

R1) The condition $\langle \nabla V(x), F(x)\rangle \le 0$ holds for all $x \in \mathbb{R}^{n+m}$.

We do not assume that the function V satisfies the conditions of Theorem 23 for the system $\mathcal{H}_c$. Instead, global asymptotic stability of the origin is achieved by resetting the controller state η so that V decreases at jumps. To that end, suppose that

R2) For each $\xi \in \mathbb{R}^n$ there exists $\eta_* \in \mathbb{R}^m$ such that $V(\xi, \eta_*) < V(\xi, \eta)$ for all $\eta \ne \eta_*$.

Since V is continuous and its sublevel sets are compact, the function $\xi \mapsto \eta_*(\xi)$ is continuous. Since $V(x)$ is positive when $x \ne 0$, $\eta_*(0) = 0$. As in [29] and [30], the hybrid controller resets η to $\eta_*(\xi)$ when (ξ, η) belong to the jump set D. We suppose that

R3) The set $D \subset \mathbb{R}^{n+m}$ is closed, intersects the set $\{(\xi, \eta) : \eta = \eta_*(\xi)\}$ only at the origin, and, for each $c > 0$, there does not exist a complete solution to

$$\dot{x} = F(x) \quad x \in (\overline{\mathbb{R}^{n+m}\setminus D}) \cap L_V(c),$$

where $L_V(c)$ denotes the c-level set of V.

Under conditions R1), R2), and R3), it follows from Theorem 23 with the Lyapunov-function candidate V that the hybrid resetting controller

$$\left.\begin{aligned} u &= \kappa(\eta, \xi) \\ \dot{\eta} &= \phi(\eta, \xi) \end{aligned}\right\} \quad (\xi, \eta) \in \overline{\mathbb{R}^{n+m}\setminus D},$$
$$\eta^+ = \eta_*(\xi) \qquad (\xi, \eta) \in D$$

Zeno Solutions

Zeno of Elea would have liked hybrid systems. According to Aristotle [S59] with embellishment from Simplicius [S57] and the authors of this article, Zeno was the ancient Greek philosopher whose Tortoise bested the swift Achilles in a battle of wits by appealing to hybrid systems theory. Here is our version of the encounter, a modern paraphrase of Zeno's paradox.

Tortoise: Hey, Achilles, how about a race?

Achilles: Are you kidding me? I'll trounce you!

Tortoise: Well, for sure, I need a head start...

Achilles: And how much of a head start would you like?

Tortoise: Doesn't matter. Any will do.

Achilles: And why is that?

Tortoise: Because this will be a hybrid race.

Achilles: What do you mean by that?

Tortoise: Well, Achilles, as you try to catch me, I want you to keep track of where I have been. This will help you see that you are catching up to me. Note where I start. When you reach that point, note where I am again. Keep doing this until you pass me! Surely, your brain can handle this, can't it Achilles?

Achilles: So, the race is hybrid because, while my feet are running a race I can win, my mind is racing without end? You know, I just felt a twinge in a tendon near my ankle. Do you think we can postpone this race for a couple of millennia?

Achilles would overtake the Tortoise and win if the race depended only on Achilles' feet. However, the Tortoise's ground rules require Achilles' mind to be involved as well. The mental task that the Tortoise gives to Achilles prevents Achilles from catching the Tortoise. Indeed, no matter how many steps of the mental assignment that Achilles completes, he will still trail the Tortoise physically.

In modern terms, here is the hybrid model that the Tortoise suggests. Let $\tau \in \mathbb{R}$ denote the position of the Tortoise, which is initialized to zero as a reference point. Let $\alpha \in \mathbb{R}$ denote the position of Achilles. Due to the Tortoise's head start, α is initialized to a negative value. The speed of the Tortoise is normalized to one, whereas the speed of Achilles, for sake of convenience, is taken to be two. If this description were the complete model, Achilles would reach the Tortoise at a time proportional to the initial separation between the Tortoise and Achilles, and then Achilles would continue past the Tortoise to victory. But this point is where the Tortoise gets clever. He gives Achilles the following instructions: "Note where I start. When you reach that point, note where I am again. Keep doing this until you pass me (... if you ever do)!"

To adhere to the Tortoise's ground rules, Achilles needs a state variable ρ that keeps track of the Tortoise's most recently observed position. Specifically, each time Achilles reaches ρ, he must update the value of ρ to be the Tortoise's current position. Thus, we have a hybrid system with the state $x = (\alpha, \tau, \rho)$ and data

$$C := \{x \in \mathbb{R}^3 : \alpha \le \rho \le \tau\}, \quad f(x) := \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ for all } x \in C,$$

$$D := \{x \in \mathbb{R}^3 : \alpha = \rho \le \tau\}, \quad g(x) := \begin{bmatrix} \alpha \\ \tau \\ \tau \end{bmatrix} \text{ for all } x \in D.$$

We call this system the Achilles. Achilles' initial goal is to reach the set $\mathcal{A} := \{x \in \mathbb{R}^3 : \alpha = \rho = \tau\}$, but he is observant enough to see that he cannot reach this goal. Indeed, Achilles cannot reach $\mathcal{A}$ by flowing, since then it would be possible to follow his flowing trajectory backward from a point in $\mathcal{A}$ to a point outside of $\mathcal{A}$ while remaining in C. However, no solution of the equation $\dot{x} = -f(x)$, $x \in C$ can start in $\mathcal{A}$ and leave $\mathcal{A}$. In fact, satisfying $\dot{x} = -f(x)$ from any initial condition in $\mathcal{A}$ immediately would force $\tau < \rho$, that is, the Tortoise is behind Achilles' most recent observation of the Tortoise's location, which is outside of C. Also, Achilles cannot reach $\mathcal{A}$ by jumping to it. Indeed, being in D but not in $\mathcal{A}$ means that $\alpha = \rho < \tau$. The value of the state after a jump would satisfy $\alpha < \rho = \tau$, which is neither in $\mathcal{A}$ nor D.

It is no consolation for Achilles that the set $\mathcal{A}$ is globally asymptotically stable for the Achilles system. This fact can be established using the Lyapunov-function candidate $V(x) := 2(x_2 - x_1) + x_2 - x_3$, which is zero on $\mathcal{A}$, positive otherwise, and such that the sublevel sets of $V|_{\operatorname{dom} V \cap (C \cup D)}$ are compact. The function V satisfies $\langle \nabla V(x), f(x) \rangle = -1$ for all $x \in C$, and $V(g(x)) \le (2/3)V(x)$ for all $x \in D$.

Neither is it a consolation for Achilles that the ordinary time t used to approach $\mathcal{A}$ is bounded, a property that can be established by integrating the derivative of V along flows of the solution to the Achilles system. Figure S11 shows typical hybrid time domains for solutions to the Achilles system. The hybrid time domain in Figure S11(a) occurs for solutions starting in $(C \cup D) \setminus \mathcal{A}$. The hybrid time domain in Figure S11(b) occurs for solutions starting in $\mathcal{A}$.

globally asymptotically stabilizes the origin of the closed-loop system. Indeed, by construction, the Lyapunov-function candidate V decreases at jumps that occur at points other than the origin, does not increase during flows, and no complete, flowing solution keeps V equal to a nonzero constant.

Regarding condition R3), consider the case in which $V(x) = V_1(\xi) + V_2(x)$, where $V_1(\xi) > 0$ for $\xi \ne 0$, $V_2(x) \ge 0$ for all $x \in \mathbb{R}^{n+m}$, and $V_2(x) = 0$ if and only if $x = (\xi, \eta)$ satisfies $\eta = \eta_*(\xi)$. This case is presented in [29] and [30]. In this situation, let

$$D := \{x = (\xi, \eta) : V_2(x) \ge \rho(V_1(\xi)), \langle \nabla V_2(x), F(x) \rangle \le 0\} \tag{37}$$

for some continuous, nondecreasing function $\rho : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ that is zero at zero and positive otherwise. The set D defined by (37) enables resets when η has moved from $\eta_*(\xi)$ and $V_2(x)$ is not increasing along a solution. Since the set D is closed and intersects the set $\{(\xi, \eta) : \eta = \eta_*(\xi)\}$, it follows that R3) reduces to the condition

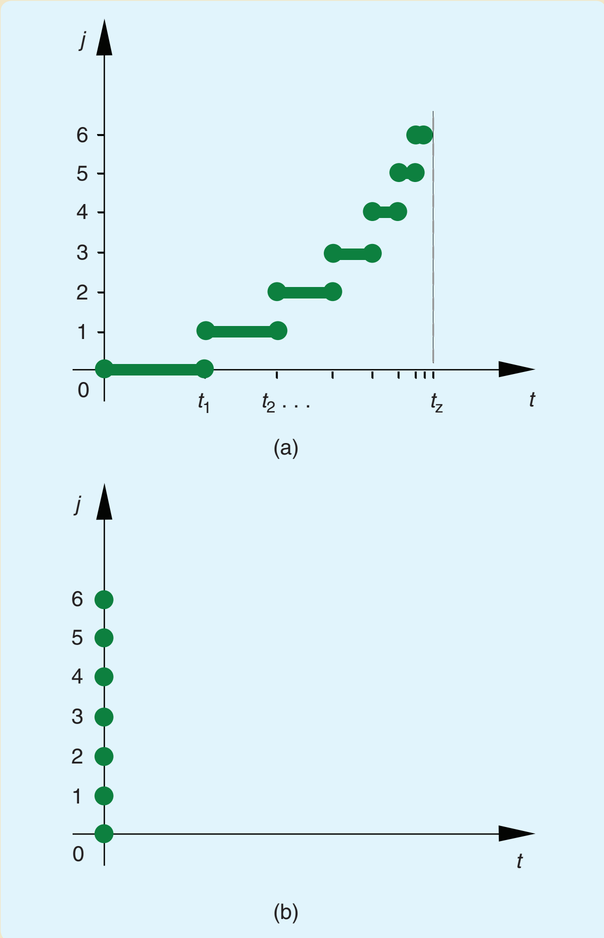

FIGURE S11 Hybrid time domains associated to two solutions of the Achilles system. (a) The hybrid time domain for a solution starting from the condition $\alpha < \tau$ and $\rho \in [\alpha, \tau]$. The hybrid time domain is unbounded in the j direction and bounded in the t direction by the Zeno time, which is denoted by t_Z. Solutions with domains of this type are called Zeno solutions. (b) The hybrid time domain for a solution starting from the condition $\alpha = \rho = \tau$. Its domain dom x is a subset of $\{0\} \times \mathbb{N}$. Solutions with domains of this type are called discrete solutions.

The bouncing ball model discussed in the main text is a physically based model that exhibits similar behavior, as illustrated in Figure S16.

In honor of Zeno and his Tortoise, sets $\mathcal{A}$ as in the Achilles system or the bouncing ball are often called Zeno attractors. Researchers have investigated sufficient conditions [36], [S50], [S51], and necessary conditions [45] for the existence of Zeno equilibria and have also grappled with the question of how to continue solutions past their "Zeno times." See, for example, [S52] and [18]. Some recent contributions to the characterization of Zeno equilibria can be found in [S54] and [S55].

Stabilizing hybrid feedback can induce Zeno equilibria or, more generally, solutions having a hybrid time domain bounded in the ordinary time direction. Such solutions sometimes exist in reset control systems [S53], [S58]. When asymptotically stable Zeno equilibria are induced by hybrid control, the equilibria can be made non-Zeno by introducing temporal regularization into the control algorithm. This operation can be carried out while preserving asymptotic stability in a practical sense. Results of this type, which follow from the robustness results discussed in the main text, are presented in [26]. Temporal regularization also appears in [S56].

The behavior of hybrid systems is rich and sometimes unexpected. Solutions with domains bounded in the ordinary time direction, including Zeno solutions, are a fascinating example of this behavior.

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R3′) For each $c > 0$ and $c_2 > 0$, there are no complete solutions to the systems

$$\dot{x} = F(x), \quad x \in \{x = (\xi, \eta) \in L_V(c) : V_2(x) \le \rho(V_1(\xi))\}$$

and

$$\dot{x} = F(x),$$
$$x \in \{x = (\xi, \eta) \in L_V(c) \cap L_{V_2}(c_2) : V_2(x) \ge \rho(V_1(\xi))\}.$$

The fact that R3′) implies R3) follows from Proposition 24 with $W := -V_2$ and the fact that the set $\{x = (\xi, \eta) : V_2(x) \ge \rho(V_1(\xi))\}$ is forward invariant for the system

$$\dot{x} = F(x), \quad x \in \{x \in L_V(c) : \langle \nabla V_2(x), F(x) \rangle \ge 0\}.$$

Finally, note that when the set D satisfies R3), condition R3) is also satisfied for any closed set that contains D and does not intersect $\{(\xi, \eta) : \eta = \eta_*(\xi)\}$ except at the origin. So, for example, jumps might be enabled when x belongs to the set D defined in (37) and also when x belongs to the set $\{x = (\xi, \eta) : V_2(x) \geq V_1(\xi)\}$. This observation is related to the thermodynamic stabilization technique described in [29] and [30]. ■

STABILITY ANALYSIS THROUGH LIMITED EVENTS

Background and Definitions

The concepts of average dwell-time switching [34] and multiple Lyapunov functions [6], [19], which are applicable to switched systems, extend to hybrid systems. These results for switching systems are based on how frequently switches occur or on the value of a Lyapunov function when switches occur. A switch between systems can be generalized to the occurrence of an event in a hybrid system, where an event is a particular type of jump. For example, in a networked control system, an event might correspond to a packet dropout, which can be modeled as a jump that is different from a jump that corresponds to successful communication. For more details, see "Example 27 Revisited." As another example, in a multimodal sampled-data controller, a timer state jump, which captures the sampled-data nature of the system, would typically not count as an event, whereas a jump corresponding to a change in the mode may correspond to an event. See Example 29.

We discuss below when pre-asymptotic stability of a compact set for a hybrid system can be deduced from pre-asymptotic stability of the set for the system with its events removed.

Events are defined by an event indicator, which is an outer semicontinuous set-valued mapping $\mathcal{E}: \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \{0\}$. An event is a pair (g, x) such that $\mathcal{E}(g, x)$ is empty. The number of events experienced by a hybrid arc x is given by the cardinality of the set

$$\{j : \text{for some } t \geq 0,\ (t, j), (t, j+1) \in \operatorname{dom} x,\quad \mathcal{E}(x(t, j+1), x(t, j)) = \emptyset\}.$$

Example 27 Revisited: Networked Control Systems with Transmission Dropouts

Consider a networked control system with the possibility of communication dropouts, modeled by the hybrid system with data $C := \mathbb{R}^{n_1+n_2} \times [0, T] \times \{0, 1\}$, $D := \mathbb{R}^{n_1+n_2} \times \{T\} \times \{0, 1\}$,

$$F(x) := \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \\ 1 \\ 0 \end{bmatrix} \quad \text{for all } x \in C,$$

$$G(x) := \left\{ \begin{bmatrix} x_1 \\ g_4 G_2(x_1, x_2) + (1 - g_4) x_2 \\ [0, \varepsilon T] \\ g_4 \end{bmatrix} : g_4 \in \{0, 1\} \right\}$$

for all $x \in D$,

where $\varepsilon \in [0, 1)$, $g_4 = 1$ corresponds to successful transmission and $g_4 = 0$ corresponds to unsuccessful transmission. If we associate events with unsuccessful transmission then the event indicator is given by $\mathcal{E}(g, x) = 0$ when $g_4 = 1$ and $\mathcal{E}(g, x) = \emptyset$ otherwise. Solutions that do not experience events behave like the networked control systems analyzed in Example 27. ■

Example 29: Mode Transitions as Events

Consider a hybrid system with state $x = (\xi, q) \in \mathbb{R}^n \times \mathbb{R}$, and data (C, F, D, G), where $C \cup D \subset \mathbb{R}^n \times Q$ and Q is a finite set. Suppose we want to identify jumps that change the value of q as events. For $x = (\xi, q) \in D$ and $(g_\xi, g_q) \in G(x)$, we let $\mathcal{E}(g, x) = 0$ if $g_q = q$, and we let $\mathcal{E}(g, x) = \emptyset$ otherwise.

A refinement is to associate events with jumps that change the mode q only when the state is outside of a target compact set $\mathcal{A} \times Q$. In this case, we let $\mathcal{E}(g, x) = 0$ if $g_q = q$ or $x = (\xi, q) \in \mathcal{A} \times Q$. Otherwise, we let $\mathcal{E}(g, x) = \emptyset$. ■

For a hybrid system $\mathcal{H} := (C, F, D, G)$, the eventless hybrid system $\mathcal{H}^0 := (C, F, D^0, G^0)$, with data contained in the data of $\mathcal{H}$, is constructed by removing events from (G, D) through the definitions

$$G^0(x) := G(x) \cap \{v : \mathcal{E}(v, x) = \{0\}\} \text{ for all } x \in D, \quad (38)$$

$$D^0 := D \cap \{x : G^0(x) \neq \emptyset\}. \quad (39)$$

The data of $\mathcal{H}^0$ satisfies the Basic Assumptions and the solutions to $\mathcal{H}^0$ experience no events.

Example 30: Jumps Out of $C \cup D$

For a hybrid system $\mathcal{H} = (C, F, D, G)$, it is possible that $g \notin C \cup D$ for some $x \in D$ and $g \in G(x)$. These jumps can be inhibited with an event indicator $\mathcal{E}$ satisfying $\mathcal{E}(g, x) = 0$ if $g \in C \cup D$ and $\mathcal{E}(g, x) = \emptyset$ otherwise. In this case, the definition (38) gives $G^0(x) = G(x) \cap (C \cup D)$ for each $x \in D$. Each solution of $\mathcal{H}$ experiences no more than one event, since a solution cannot be extended from a point $g \notin C \cup D$.

According to Theorem 31, the compact set $\mathcal{A}$ is globally pre-asymptotically stable for the hybrid system (C, F, D, G) whenever $G(D \cap \mathcal{A}) \subset \mathcal{A}$, and $\mathcal{A}$ is globally pre-asymptotically stable for the hybrid system (C, F, D^0, G^0), where $G^0(x) = G(x) \cap (C \cup D)$ for each $x \in D$ and D^0 is the set of points $x \in D$ for which $G^0(x) \neq \emptyset$. ■

The stability properties of $\mathcal{H}^0 = (C, F, D^0, G^0)$, with D^0 and G^0 defined in (38)–(39), can be used to derive stability properties for $\mathcal{H} = (C, F, D, G)$ under a variety of conditions. Below, we consider the cases where 1) the solutions of $\mathcal{H}$ experience a finite number of events or 2) there is a relationship between the values of a Lyapunov function when different events occur. Other results can be formulated that involve an average dwell-time requirement on events, like for switched systems discussed in "Switching Systems" or

Pre-asymptotic stability for compact sets is equivalent to uniform pre-asymptotic stability, which is sometimes called KL-stability.

that combine dwell-time conditions and Lyapunov conditions, as in [33].

Finite Number of Events

An approach used in adaptive supervisory control to establish convergence properties is to establish that the number of switches experienced by a solution is finite [58]. A related result for asymptotic stability in hybrid systems is given in the next theorem.

Theorem 31

Suppose that the hybrid system $\mathcal{H} = (C, F, D, G)$ with state $x \in \mathbb{R}^n$ satisfies the Basic Assumptions. Let the compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfy $G(D \cap \mathcal{A}) \subset \mathcal{A}$, and assume that $\mathcal{A}$ is globally pre-asymptotically stable for the eventless hybrid system $\mathcal{H}^0 = (C, F, D^0, G^0)$, where D^0 and G^0 are defined in (38)–(39). Also suppose that, for the hybrid system $\mathcal{H}$ and each compact set $K \subset \mathbb{R}^n$, there exists $N > 0$ such that each solution starting in K experiences no more than N events. Then the set $\mathcal{A}$ is globally pre-asymptotically stable for the system $\mathcal{H}$.

Theorem 31 is behind the proof of asymptotic stability for the mode-switching controller for the disk drive system in Example 1. As shown for "Example 1 Revisited," the origin of the closed-loop system is pre-asymptotically stable when jumps are eliminated, and when jumps are allowed the maximum number of jumps is two.

An outline of the proof of Theorem 31 is as follows. Boundedness of maximal solutions and global pre-attractivity of $\mathcal{A}$ for $\mathcal{H}$ follow from the fact that the number of events is finite and that the maximal solutions of $\mathcal{H}^0$ are bounded and the complete ones converge to $\mathcal{A}$. Stability can be established by concatenating trajectories of the system $\mathcal{H}^0$, which can include jumps but not events, with jumps from G that correspond to events, and repeating this process up to N times. In particular, let $\varepsilon =: \varepsilon_0 > 0$ be given. Since $G(D \cap \mathcal{A}) \subset \mathcal{A}$ and G is outer semicontinuous and locally bounded, there exists $\delta_0 > 0$ such that $G(D \cap (\mathcal{A} + \delta_0\mathbb{B})) \subset \mathcal{A} + \varepsilon_0\mathbb{B}$. Since $\mathcal{A}$ is stable for $\mathcal{H}^0$, there exists $\varepsilon_1 > 0$ such that, for each solution x_1 to $\mathcal{H}^0$, $|x_1(0,0)|_{\mathcal{A}} \le \varepsilon_1$ implies $|x_1(t,j)|_{\mathcal{A}} \le \delta_0$ for all $(t,j) \in \operatorname{dom} x_1$. Repeating this construction N times and letting ε_j generate ε_{j+1} for $j \in \{0, \ldots, N-1\}$, it follows that $|x_1(0,0)|_{\mathcal{A}} \le \varepsilon_N$ implies $|x_1(t,j)|_{\mathcal{A}} \le \varepsilon_0 = \varepsilon$ for all solutions x of $\mathcal{H}_N$ and all $(t,j) \in \operatorname{dom} x$.

Multiple Lyapunov Functions

Stability results involving multiple Lyapunov functions [6], [19] for switched systems, as well as related results for discontinuous dynamical systems in [90] and for differential equations in [1], extend to hybrid systems. The stability conditions are expressed in terms of an event indicator $\mathcal{E}$, the eventless system with events inhibited, as in (38)–(39), and a decomposition of the event indicator into a finite number of event types.

Theorem 32

For a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions with an event indicator $\mathcal{E}$, the compact set $\mathcal{A}$ is globally pre-asymptotically stable if and only if the following conditions are satisfied:

1) $G(D \cap \mathcal{A}) \subset \mathcal{A}$,
2) $\mathcal{A}$ is globally pre-asymptotically stable for the system without events $\mathcal{H}^0 := (C, F, D^0, G^0)$.
3) There exist a function $W: C \cup D \to \mathbb{R}_{\ge 0}$, and class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, such that

$$\alpha_1(|x|_{\mathcal{A}}) \le W(x) \le \alpha_2(|x|_{\mathcal{A}}) \text{ for all } x \in C \cup D,$$
$$W(g) \le \alpha_3(W(x)) \text{ for all } x \in D,\ g \in G(x).$$

4) There exists a positive integer ℓ and outer semicontinuous mappings $\mathcal{E}_i: \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \{0\}$, $i = 1, \ldots, \ell$, such that, for each (v, x) satisfying $\mathcal{E}(v,x) = \emptyset$, there exists $i \in \{1, \ldots, \ell\}$ satisfying $\mathcal{E}_i(v,x) = \emptyset$, and there exists a continuous function α_4 satisfying $0 < \alpha_4(s) < s$ for all $s > 0$ and such that, for each $i \in \{1, \ldots, \ell\}$ and each solution x,

$$W(x(t_2, j_2)) \le \alpha_4(W(x(t_1, j_1))),$$

where $(t_k, j_k) \in \operatorname{dom} x$, $k \in \{1, 2\}$, $t_2 + j_2 > t_1 + j_1$, are the two smallest elements of $\operatorname{dom} x$ satisfying $\mathcal{E}_i(x(t_k, j_k), x(t_k, j_k - 1)) = \emptyset$.

Figure 17 indicates a possible evolution of $W(x(t,j))$. Some jumps, such as the one from $(t_2, 1)$ to $(t_2, 2)$, do not correspond to events. When a particular type of event occurs twice, the value of W at the second occurrence is required to be strictly less than the value of W at the first occurrence of the event. In the figure, one type of event is indicated in blue while the other is indicated in red. Blue events occur at $(t_1, 1)$ and $(t_4, 4)$. Thus, the value of $W(x(t_4, 4))$ is required to be less than the value of $W(x(t_1, 1))$.

Consider Theorem 32 in the context of multiple Lyapunov functions V_q, where $q \in \{1, \ldots, \ell\}$, for a switching system with state z. We take $W(x) := V_q(z)$, where $x = (z, q)$. Sometimes it is assumed that V_q does not increase during flows [6], although this property is not necessary [90], and is not assumed in Theorem 32. For $i \in \{1, \ldots, \ell\}$, we take $\mathcal{E}_i(v, x) = 0$ for switches that turn the state q into a value $j \ne i$, otherwise we take $\mathcal{E}_i(v, x) = \emptyset$. Then the third assumption

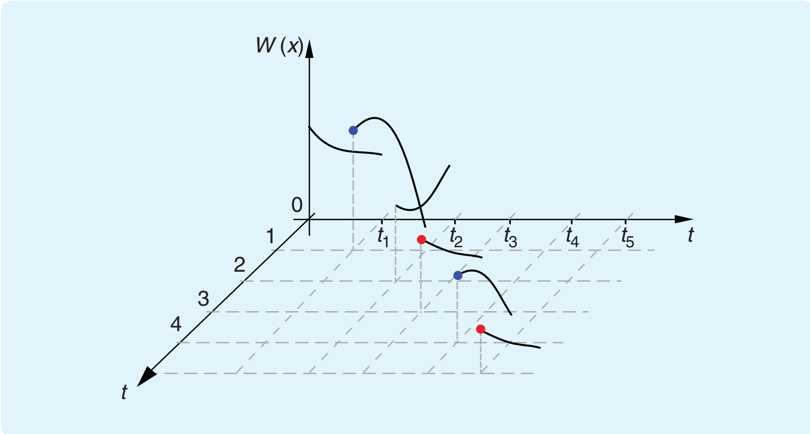

FIGURE 17 The evolution of a candidate function W for Theorem 32 along a trajectory of a hybrid system. The blue and red dots indicate different types of events. The blue dots, at $(t_1, 1)$ and $(t_4, 4)$, denote points $v \in G(x)$ at which $\mathcal{E}_1(v, x) = \emptyset$, whereas the red dots, at $(t_3, 3)$ and $(t_5, 5)$, denote points $v \in G(x)$ at which $\mathcal{E}_2(v, x) = \emptyset$. Some jumps, such as the one from $(t_2, 1)$ to $(t_2, 2)$, do not correspond to events. When a particular type of event occurs twice, the value of W at the second occurrence must be less than the value of W at the first occurrence of the event. For example, the value of $W(x(t_4, 4))$ must be less than the value of $W(x(t_1, 1))$.

of Theorem 32 asks that, when comparing W at an event where q changes to i to the value of W at the previous event where q changes to i, we see a decrease in W. The case of q belonging to a compact set can be handled in a similar way by generating a finite covering of the compact set, to generate a finite number of event indicators $\mathcal{E}_i$, and using extra continuity properties for the function $(z, q) \mapsto V_q(z)$ to ensure a decrease when an event repeats.

The necessity in Theorem 32 follows from the results discussed in "Converse Lyapunov Theorems," which show that W can always be taken to be smooth. See also [90]. The sufficiency, which doesn't use continuity properties for W, follows using the same argument used for multiple Lyapunov functions in [6]. We sketch the idea here.

For stability, note that due to the first two assumptions and Theorem 31, there exists $\alpha \in \mathcal{K}_\infty$ such that, for each solution x and each $(t, j) \in \operatorname{dom} x$ such that only one event has occurred

$$W(x(t, j)) \le \alpha(W(x(0, 0))).$$

Without loss of generality, we associate the first event with the event indicator $i = 1$, and we use condition 3) of Theorem 32 to conclude that, for all event times (t_{k_1}, j_{k_1}) where event $i = 1$ occurs, that is, $\mathcal{E}_1(x(t_{k_1}, j_{k_1}), x(t_{k_1}, j_{k_1} - 1)) = \emptyset$, we have $W(x(t_{k_1}, j_{k_1})) \le \alpha(W(x(0, 0)))$. We associate the first event that is different from events with $i = 1$ with event indicator $i = 2$ and conclude that, for all event times (t_{k_2}, j_{k_2}) at which event $i = 2$ occurs, that is, $\mathcal{E}_2(x(t_{k_2}, j_{k_2}), x(t_{k_2}, j_{k_2} - 1)) = \emptyset$, we have $W(x(t_{k_2}, j_{k_2})) \le \alpha^2(W(x(0, 0)))$. Continuing in this way, we conclude that, for each solution x and each $(t, j) \in \operatorname{dom} x$

$$W(x(t, j)) \le \alpha^{\ell}(W(x(0, 0))).$$

With the first assumption of the theorem, we conclude that the set $\mathcal{A}$ is stable.

Pre-attractivity is established as follows. Complete solutions that experience a finite number of events converge to $\mathcal{A}$ according to the first assumption of Theorem 32. For complete solutions that experience an infinite number of events, at least one type of event $\mathcal{E}_i$ does not cease happening and thus, using this i in the third assumption of Theorem 32, $W(x(t, j))$ converges to zero, that is, the solution converges to $\mathcal{A}$. Thus, the set $\mathcal{A}$ is globally pre-asymptotically stable.

We refer the reader to [6] and [19] for examples that illustrate Theorem 32 in the context of multiple Lyapunov functions. An illustration is also given at the end of the next section.

HYBRID FEEDBACK CONTROL BASED ON LIMITED EVENTS

In this section, we illustrate how stability analysis based on limited events motivates hybrid control algorithms for the nonlinear system (5). The control algorithms have a flow set C_c, flow map f_c, jump set D_c, jump map G_c, and feedback law κ_c, resulting in a closed-loop hybrid system with data given in (6) and (7).

Supervising a Family of Hybrid Controllers

The literature contains an extensive list of papers on supervisory control. For example, in the context of discrete-event systems see [68], in the setting of adaptive control see [57], and for hybrid systems see [41].

We discuss the design of a supervisor for a family of hybrid controllers $\widetilde{\mathcal{H}}_{c,q}$, where $q \in Q := \{1, \ldots, k\}$, each with state $\eta \in \mathbb{R}^m$ and data $(\widetilde{C}_q, \widetilde{f}_q, \widetilde{D}_q, \widetilde{G}_q, \widetilde{\kappa}_q)$. Classical controllers correspond to the special case where $\widetilde{C}_q = \mathbb{R}^{n+m}$, $\widetilde{D}_q = \emptyset$, and $\widetilde{G}_q(x) = \emptyset$ for all $x \in \mathbb{R}^{n+m}$.

Following [83], a supervisor $\mathcal{H}_c$ of these hybrid controllers is specified by indicating closed sets $C_q \subset \widetilde{C}_q$ and $D_q \subset \widetilde{D}_q$ in which flowing and jumping, respectively, using controller q is allowed, and a set $H_q \subset \mathbb{R}^{n+m}$, in which switching from controller q to another controller is allowed. In addition to the sets C_q, D_q, and H_q, the supervisor specifies a rule for how η and q change when the supervisor switches authority from controller q to a different controller. This rule is given through the set-valued mappings $J_q : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{m+1}$, $q \in Q$. Without loss of generality, we take $J_q(x_p, \eta) = \emptyset$ for all $(x_p, \eta) \notin H_q$. We also define f_q and κ_q to be the restrictions of $\widetilde{f}_q$ and $\widetilde{\kappa}_q$ to C_q, and we define G_q to be the set-valued mapping that is equal to $\widetilde{G}_q$ on $\overline{D_q \setminus H_q}$ and is empty otherwise.

The hybrid supervisor $\mathcal{H}_c$ has state $x_c = (\eta, q)$, input $x_p \in \mathbb{R}^n$, and is defined by the data

$$f_c(x_p, x_c) = \begin{bmatrix} f_q(x_p, \eta) \\ 0 \end{bmatrix}, \; C_c = \bigcup_{q \in Q}(C_q \times \{q\}),$$
$$G_c(x_p, x_c) = \begin{bmatrix} G_q(x_p, \eta) \\ q \end{bmatrix} \bigcup J_q(x_p, \eta),$$
$$D_c = \bigcup_{q \in Q}((D_q \cup H_q) \times \{q\}), \tag{40}$$

and $\kappa_c(x_p, x_c) = \kappa_q(x_p, \eta)$.

The goal of the supervisor is to achieve global asymptotic stability of the set $\mathcal{A} \times Q$, where $\mathcal{A} \subset \mathbb{R}^{n+m}$ is compact, while satisfying $C_c \cup D_c = \Theta \times Q$, where $\Theta \subset \mathbb{R}^{n+m}$ is closed, and to have that all maximal solutions starting in $C_c \cup D_c$ are complete. To guarantee that a supervisor can be constructed to achieve the goal, as in [83] we assume the following.

(Supervisor Assumption)

There exist closed sets $C_q \subset \widetilde{C}_q \cap \Theta$ and $D_q \subset \widetilde{D}_q \cap \Theta$ so that the following conditions hold:

1) There exist closed sets $\Psi_q \subset C_q \cup D_q$ such that $\bigcup_{q \in Q} \Psi_q = \Theta$.
2) For each $q \in Q$, the interconnection of the plant (5) and the controller $\mathcal{H}_{c,q} = (C_q, f_q, D_q, G_q, \kappa_q)$, having closed-loop data given in (6), (7), is such that, using the definition $\Phi_q := \bigcup_{\{i \in Q,\, i > q\}} \Psi_i$, the following conditions are satisfied:
   a) The set $\mathcal{A}$ is globally pre-asymptotically stable.
   b) No solution starting in Ψ_q reaches $\overline{\Theta \setminus (C_q \cup D_q \cup \Phi_q)} \setminus \mathcal{A}$.
   c) Each maximal solution is either complete or ends in $\Phi_q \cup \overline{\Theta \setminus (C_q \cup D_q \cup \Phi_q)}$.

Condition 2c) is a type of local existence of solutions condition. It is assumed to guarantee that a supervisor can be constructed to achieve not only global pre-asymptotic stability but also that all maximal solutions are complete.

The combination of conditions 2b) and 2c) imply that solutions starting in Ψ_q are either complete or end in $\mathcal{A}$ or Φ_q. This property enables building a hybrid supervisor to guarantee that each solution experiences no more than $k+1$ events, where events are defined to be jumps in the value of q when $(x_p, \eta) \notin \mathcal{A}$.

The sets involved in the "Supervisor Assumption" for the problem of stabilizing the inverted position of a pendulum on a cart are depicted in Figure 20.

Given the sets C_q, D_q, Ψ_q, and Φ_q from the "Supervisor Assumption," we define the sets H_q and mappings J_q in (40) to be

$$H_q = \Phi_q \cup \overline{\Theta \setminus (C_q \cup D_q \cup \Phi_q)} \tag{41}$$

and in (42), shown at the bottom of the page. By construction, the mapping J_q is outer semicontinuous.

For the closed-loop system, events are defined to be jumps from $D \setminus (\mathcal{A} \times Q)$ that change the value of q. In other words, events are jumps that are due to J_q. Due to the definition of the map J_q, if a solution experiences one event then there is a time where the state (x_p, η, q) satisfies $(x_p, \eta) \in \Psi_q$. Thereafter, due to the "Supervisor Assumption," until the state reaches the set $\mathcal{A} \times Q$, the index q jumps monotonically. Therefore, the number of events experienced by a solution will be bounded by $k+1$. Moreover, by the "Supervisor Assumption," the corresponding eventless system has the set $\mathcal{A} \times Q$ globally pre-asymptotically stable. These observations lead to the following corollary of Theorem 31.

Corollary 33

Under the "Supervisor Assumption" on the family of controllers $\mathcal{H}_{c,q}$, $q \in Q$, the closed-loop interconnection of the plant (5) and the hybrid supervisor $\mathcal{H}_c$, with data given in (40)–(42), satisfies the Basic Assumptions and has the compact set $\mathcal{A} \times Q$ globally asymptotically stable.

For the disk drive in Example 1 and for additional examples reported below, the "Supervisor Assumption" holds and the supervisory control algorithm indicated in Corollary 33 is used.

Some asymptotically controllable nonlinear control systems cannot be robustly stabilized to a point using classical, time-invariant state feedback. Examples include systems that fail Brockett's necessary condition for time-invariant, continuous stabilization [7], like the nonholonomic mobile robot, and other systems that satisfy Brockett's condition, like the system known as Artstein's circles [2]. The topological obstruction to robust stabilization in Artstein's circles is the same one encountered when trying to globally stabilize a point on a circle and is related to the issues motivating the development of topological complexity in the context of motion planning in [21]. We now illustrate how hybrid feedback, and supervisory control in particular, makes robust stabilization possible.

$$J_q(x_p, \eta) := \begin{cases} \begin{bmatrix} \{x_c\} \\ \{i \in Q : (x_p, \eta) \in \Psi_i\} \end{bmatrix}, & (x_p, \eta) \in \overline{\Theta \setminus (C_q \cup D_q \cup \Phi_q)}, \\ \begin{bmatrix} \{x_c\} \\ \{i \in Q : i > q,\ (x_p, \eta) \in \Psi_i\} \end{bmatrix}, & (x_p, \eta) \in H_q \setminus (\overline{\Theta \setminus (C_q \cup D_q \cup \Phi_q)}). \end{cases} \tag{42}$$

Simulation in Matlab/Simulink

Numerous software tools have been developed to simulate dynamical systems, including Matlab/Simulink, Modelica [S61], Ptolemy [S63], Charon [S60], HYSDEL [S66], and HyVisual [S62]. Hybrid systems $\mathcal{H} = (C, f, D, g)$ can be simulated in Matlab/Simulink with an implementation such as the one shown in Figure S12. This implementation uses the reset capabilities of integrator blocks in Matlab/Simulink.

Four basic blocks are used to input the data (C, f, D, g) of the hybrid system $\mathcal{H}$ for simulation by means of *Matlab function* blocks. The outputs of these blocks are connected to the input of the *Integrator System*, which integrates the flow map and executes the jumps. The *Integrator System* generates both the time variables t and j as well as the state trajectory x. Additionally, the *Integrator System* provides the value of the state before a jump occurs, which is denoted as x'. The four blocks are defined as follows:

- » The f block uses the user function *f.m* to compute the flow map. Its input is given by the state of the system x, and its output is the value of the flow map f.
- » The C block executes the user function *C.m*, which codes the flow set C. Its input is given by x'. The output of this block is set to one if x' belongs to the set C and is set to zero otherwise.
- » The g block executes the user function *g.m*, which contains the jump map information. Its input is x', while its output is the value of the jump map g. Its output is passed through an IC block to specify the initial condition of the simulation.
- » The D block evaluates the jump condition coded in the user function *D.m*. Its input is given by x', while its output is set to one if x' belongs to D, otherwise is set to zero.

For example, to simulate the bouncing ball system in Example 3 with gravity constant $\gamma = 9.8$, the flow map f and the flow set C can be coded into the functions *f.m* and *C.m*, respectively, as follows:

```
function out = f(u)
% state
x1 = u(1);
x2 = u(2);
% flow map
x1dot = x2;
x2dot = –9.8;
out = [x1dot; x2dot];                    (S9)

function out = C(u)
% state
x1 = u(1);
x2 = u(2);
% flow condition
if (x1 >= 0)
  out = 1;
else
  out = 0;
end                                      (S10)
```

The subsystems of the *Integrator System*, which are depicted in Figure S13, are used to compute the flows and jumps. These subsystems are the following.

The integrator block, labeled $1/s$, is the main block of the *Integrator System*. Its configuration uses the following settings: *external reset* set to "level hold" (setting for Matlab/Simulink R2007a and beyond; see [S65] for details about this setting for previous Matlab/Simulink versions), *initial condition source* set to "external" and *show state port* set to "checked." Simulink's zero-crossing detection algorithm for numerical computations is globally disabled.

The *CT dynamics* subsystem computes the flow map of the hybrid system as well as updates the parameters t and j during flows. Its output is integrated by the integrator block. This computation is accomplished by setting the dynamics of the integrator block to be $\dot{t} = 1$, $\dot{j} = 0$, $\dot{x} = f(x)$.

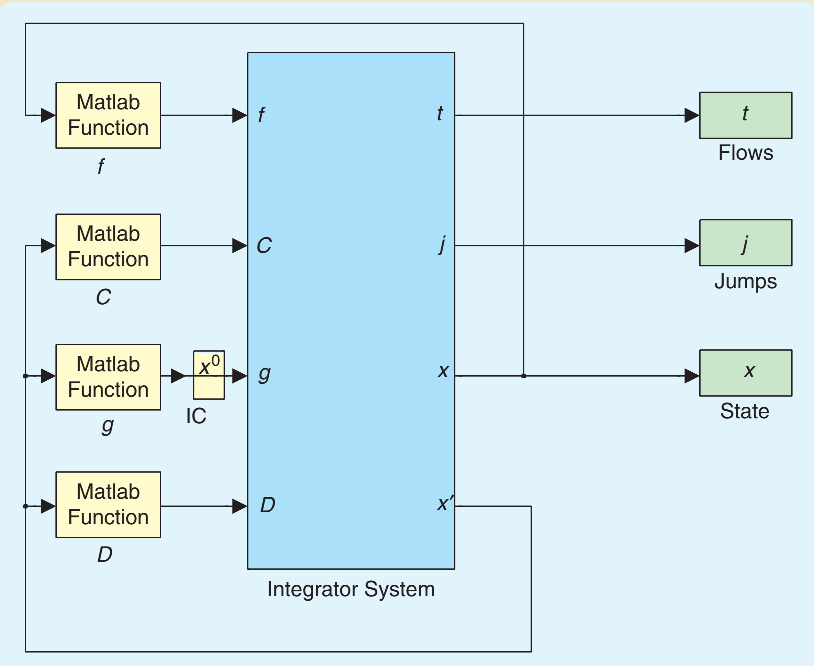

FIGURE S12 Matlab/Simulink implementation of a hybrid system $\mathcal{H} = (C, f, D, g)$. From the output of the Matlab functions, the *Integrator System* takes the values of the flow map f and jump map g as well as the indications of whether or not the state belongs to the flow set C and jump set D. The output of the *Integrator System* consists of t, j, and $x(t, j)$ at every simulation time step. The output x', which corresponds to the value of the state x before the jump instant, is used in the computations. The initial condition for the simulation is specified by x^0 in the *IC* block.

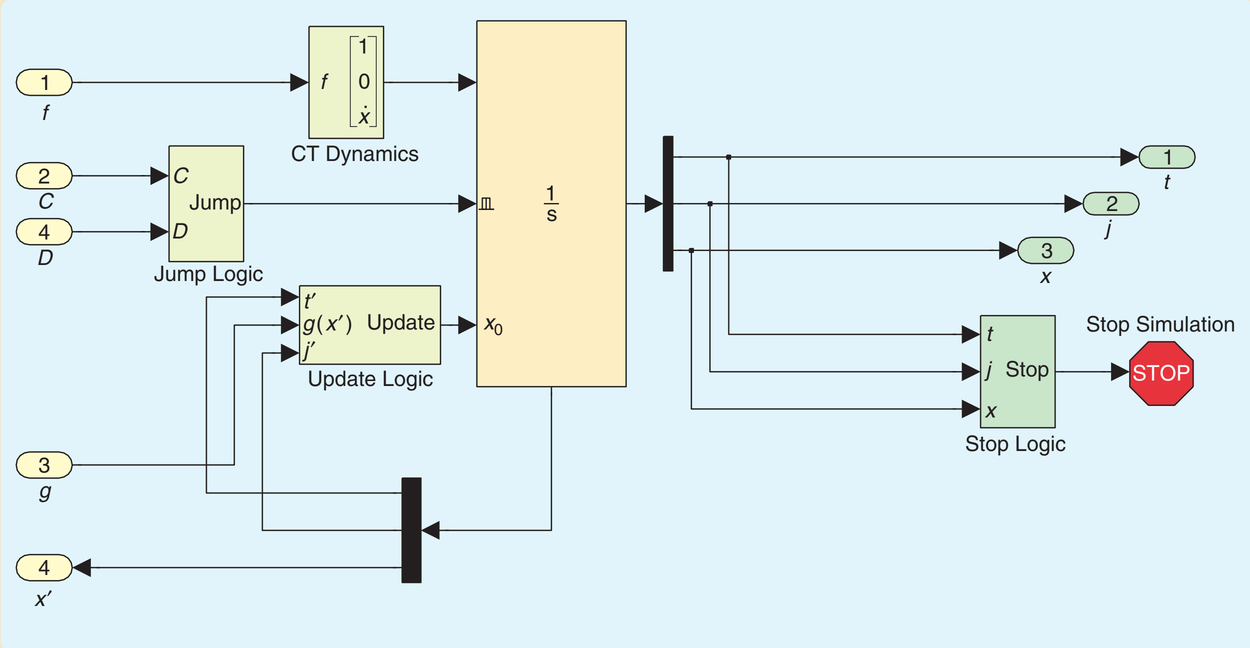

FIGURE S13 Internal diagram of the *Integrator System*. The CT *dynamics* subsystem computes the flow of the hybrid system and updates the parameter t. The *Jump Logic* subsystem triggers the jumps when the state x belongs to the jump set D. The *Update Logic* subsystem updates the value of the simulation as well as the pair (t, j) at jumps. The *Stop Simulation* subsystem stops the simulation when either the flow time is larger than or equal to the maximum flow time specified, the jump time is larger than or equal to the maximum number of jumps specified, or the simulation is in neither C nor D.

The *Jump Logic* subsystem triggers the jumps. Its inputs, which are the outputs of the blocks C and D, indicate whether the simulation is in either the flow set or the jump set. With this information, the *Jump Logic* subsystem generates the reset input of the integrator block as follows: when the simulation belongs only to the flow set, no reset is generated; when the simulation belongs only to the jump set, a reset is generated; and when the simulation belongs to both sets, a reset is generated depending on the value of a flag, which is denoted as *priority*.

The *Update Logic* subsystem updates the value of the simulation as well as the pair (t, j) at jumps. By using the state port of the integrator block, which reports the value of the state of the integrator at the exact instant that the reset condition becomes true, the value of the simulation is updated by $g(x')$. The flow time t is kept constant at jumps, and j is incremented by one. That is, at jumps the update law is $t^+ = t'$, $j^+ = j' + 1$, $x^+ = g(x')$.

The *Stop Logic* subsystem stops the simulation under any of the following events:

- The flow time is larger than or equal to the maximum flow time, which is specified by the parameter T.
- The jump time is larger than or equal to the maximum number of jumps, which is specified by the parameter J.
- The simulation is neither in C nor in D.

Under any of these events, the output of the logic operator connected to the *Stop logic* becomes one and the simulation is stopped.

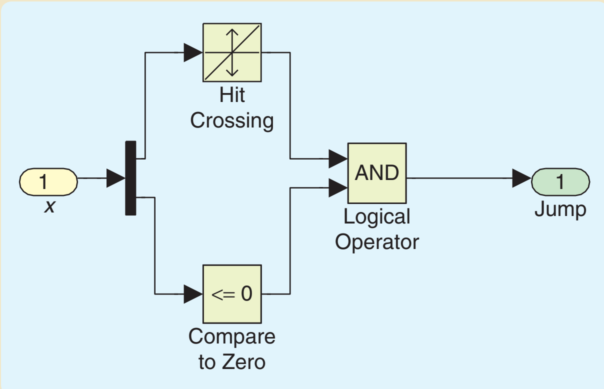

FIGURE S14 Alternative implementation in Matlab/Simulink of the jump set for the bouncing ball system in Example 3, which is given by $\{x : x_1 = 0,\ x_2 \le 0\}$. The "Hit Crossing" block detects when the state component x_1 crosses zero, while the "Compare to Zero" block is employed to report when the state component x_2 satisfies the condition $x_2 \le 0$. When the outputs of the "Hit Crossing" and "Compare to Zero" blocks are equal to one, a jump is triggered.

The following parameters are used in a simulation: initial condition for x, which is given by x^0; simulation horizon for the flows and jumps, which are given by T and J, respectively; and flag *priority* for flows/jumps, where $priority = 1$ indicates that jumps have the highest priority (forced jumps) and $priority = 2$ indicates

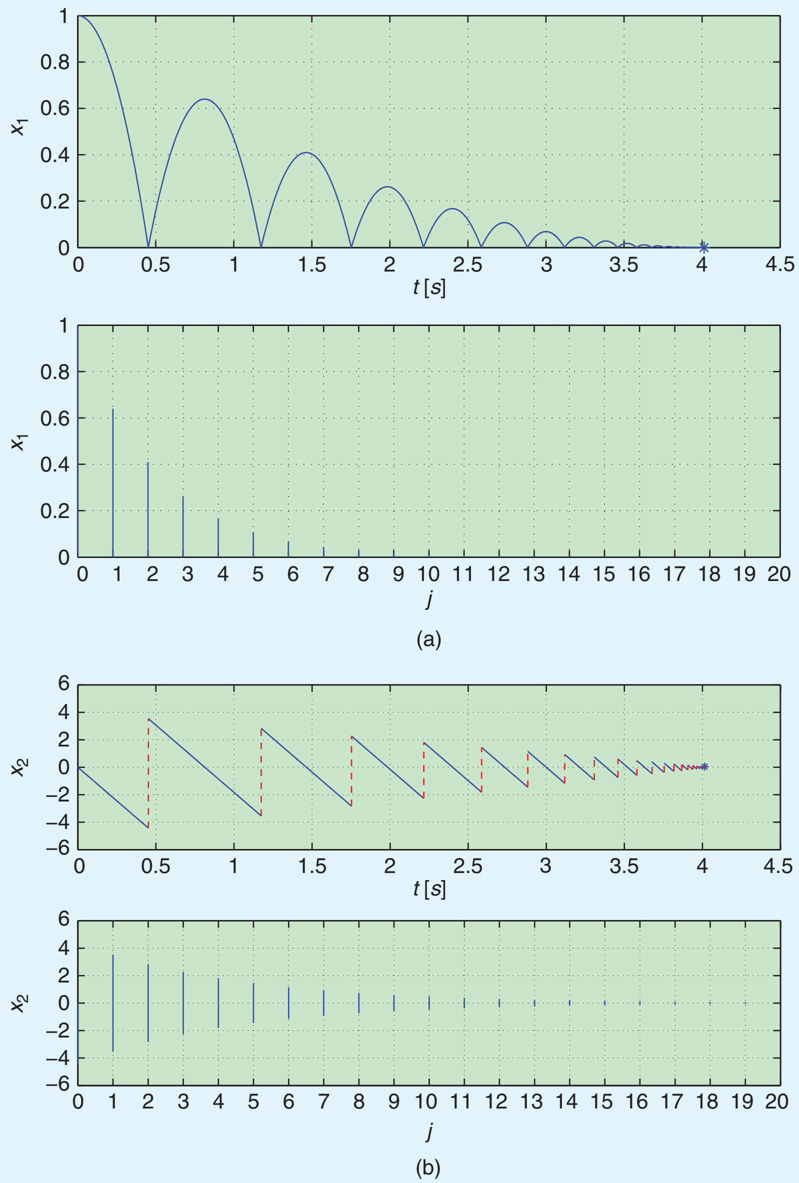

FIGURE S15 Simulation of the bouncing ball system in Matlab/Simulink for an initial condition with height $x_1 = 1$ and vertical velocity $x_2 = 0$. The maximum flow time is $T = 10$ s, the maximum number of jumps is $J = 20$, and the flag *priority* is set to one, which indicates that priority is given to the jump condition coded in *D.m*. (a) shows the height of the ball, which is denoted by x_1, while (b) shows the velocity of the ball, which is denoted by x_2, projected to the t axis and to the j axis of its hybrid time domain. The plots indicate that the solution jumps more frequently as the height of the ball approaches zero, which occurs when the flow time approaches the Zeno time of the bouncing ball system. For the given initial condition and parameters, the Zeno time is given by $t_Z = 4.066$ s.

that flows have the highest priority (forced flows). Matlab/Simulink documentation recommends using numerical solvers with variable step size for simulations of systems with discontinuities/jumps, such as ode45, which is a one-step solver based on an explicit Runge-Kutta (4, 5) formula. This setting, which is the default, is recommended as a first-try solver [S64].

Additionally, this implementation of a simulation of a hybrid system (C, f, D, g) permits replacing the Matlab functions that define the data of the hybrid system by a Simulink subsystem, and, hence, it enables using general purposes blocks in Simulink's library. For instance, this implementation permits the use of Simulink blocks to detect signal crossings of a given threshold with the "Hit Crossing" block, as illustrated for the bouncing ball example below, and quantizing a signal with the "Quantizer" block.

To simulate the bouncing ball system in Example 3 in Matlab/Simulink using the implementation in Figure S12, we employ the functions *f.m* and *C.m* defined in (S9) and (S10) for the flow map and flow set, respectively. The jump map and jump set are implemented with restitution coefficient $\rho = 0.8$ in the functions *g.m* and *D.m*, respectively, as follows:

```
function out = g(u)
% state
x1 = u(1);
x2 = u(2);
% jump map
x1plus = −0.8*x1;
x2plus = −0.8*x2;
out = [x1plus; x2plus];

function out = D(u)
% state
x1 = u(1);
x2 = u(2);
% jump condition
if (x1 <= 0 && x2 <= 0)
```

Example 34: Robust Global Asymptotic Stabilization of a Point on a Circle

Consider a plant with state ξ that is constrained to evolve on the unit circle with angular velocity u, where positive angular velocity corresponds to counterclockwise rotation. Moving on the unit circle constrains the velocity vector to be perpendicular to the position vector, as generated by rotating the position vector using the

```
    out = 1;
  else
    out = 0;
  end
```

Note that due to numerical integration error, the condition $x_1 = 0$ may never be satisfied. Then, solutions obtained from simulations can miss jumps as well as leave the set $C \cup D$, causing a premature stop of the simulation. To overcome these issues, instead of using the jump set in Example 4, which is given by $\{x : x_1 = 0, x_2 \le 0\}$, the jump set $D = \{x : x_1 \le 0, x_2 \le 0\}$ is used. This jump set has no effect on solutions starting from the set $\{x : x_1 \ge 0\}$. Another alternative is to replace the function *D.m* by a Simulink subsystem implementing the condition $x_1 = 0$ in the jump set $\{x : x_1 = 0, x_2 \le 0\}$ with a Simulink block that is capable of detecting zero crossings of x_1 as shown in Figure S14. This implementation uses three Simulink blocks, namely, a "Hit Crossing" block, a "Compare to Zero" block, and an "AND Logical Operator" block. The last block combines the outputs of the first two blocks. The "Hit Crossing" block is used to detect when the state component x_1 crosses zero. For this purpose, this block is required to have enabled Simulink's zero-crossing detection algorithm for numerical computations. Then, when x_1 crosses zero, this algorithm sets back the simulation step to a time instant close enough to the time at which x_1 is zero, recomputes the state, and sets its output to one. The "Compare to Zero" block is used to report when the state component x_2 satisfies the condition $x_2 \le 0$ by setting its output to one. Then, when the outputs of the "Hit Crossing" and "Compare to Zero" blocks are equal to one, which indicates that $x_1 = 0$ and $x_2 \le 0$ are satisfied, a jump is reported to the *Integrator System.*

Figure S15 depicts the height and velocity of a solution to the bouncing ball system projected to the t axis and j axis of its hybrid time domain. The solution is generated from the initial condition $(1, 0)$ with maximum flow time $T = 10$ s, maximum number of jumps $J = 20$, and highest priority for jumps, that is, $priority = 1$. It is easy to verify that the solutions to the bouncing ball system are Zeno and that the Zeno time, which is given by $t_Z = \sup\{t : (t, j) \in \operatorname{dom} x\}$, is given by

$$t_Z = \frac{x_2(0,0) + \sqrt{x_2(0,0)^2 + 2\gamma x_1(0,0)}}{\gamma} + \frac{2\rho\sqrt{x_2(0,0)^2 + 2\gamma x_1(0,0)}}{\gamma(1-\rho)}.$$

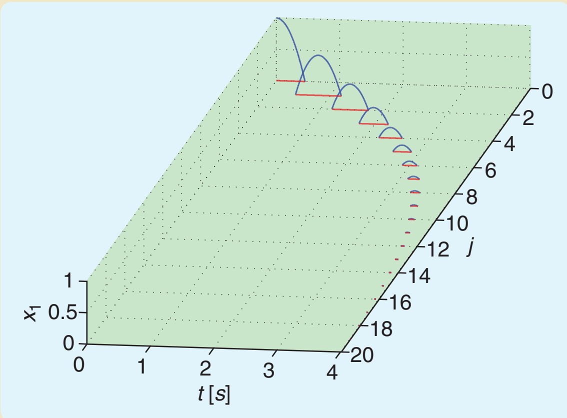

FIGURE S16 Hybrid arc and hybrid time domain corresponding to the height component of the simulation to the bouncing ball system in Matlab/Simulink depicted in Figure S15. The hybrid arc is shown in blue, while the hybrid time domain is shown in red.

For the initial conditions $x_1(0,0) = 1$, $x_2(0,0) = 0$ and the parameters $\gamma = 9.8\ \text{m/s}^2$ and $\rho = 0.8$, the simulation yields $t_Z = 4.066\ s$. The solution obtained from the numerical simulation indicates that as the flow time approaches t_Z, the height and velocity of the ball as well as the time elapsed between jumps approach zero. Finally, Figure S16 depicts the hybrid arc and hybrid time domain for the height component of the solution obtained from the numerical simulation.

The Matlab/Simulink files for this implementation and the functions used for plotting solutions as well as several examples are available online [S65].

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rotation matrix $R(-\pi/2)$. The constrained equations of motion are

$$\dot{\xi} = uR(-\pi/2)\xi \quad \xi \in \mathbb{S}^1.$$

For this system, global asymptotic stabilization of the point $(1, 0)$ with a continuous state-feedback law is impossible. Global asymptotic stabilization of this point by discontinuous feedback is possible, although the resulting

With the use of hybrid time domains and the notion of graphical convergence, sequential compactness of the space of solutions and semicontinuous dependence of solutions on initial conditions and perturbations can be established under mild conditions.

global closed-loop behavior is not robust to arbitrarily small measurement noise. For an explanation, see "Robustness and Generalized Solutions." We now describe a hybrid controller that achieves robust, global asymptotic stabilization of the point (1, 0).

Consider a hybrid supervisor for classical controllers $\widetilde{\mathcal{H}}_q$, $q \in Q = \{1, 2\}$, that are static and are given by $\widetilde{\kappa}_q(\xi) = \xi_q$. The "Supervisor Assumption" is satisfied for $\mathcal{A} = (1, 0)$ and $\Theta = \mathbb{S}^1$ by taking $D_1 = D_2 = \varnothing$ and

$$\begin{aligned}\Psi_1 = C_1 &:= \{\xi \in \mathbb{S}^1 : \xi_1 \leq -1/3\},\\ C_2 &:= \{\xi \in \mathbb{S}^1 : \xi_1 \geq -2/3\},\\ \Psi_2 &:= \{\xi \in \mathbb{S}^1 : \xi_1 \geq -1/3\}.\end{aligned}$$

Following the prescription of the components of a hybrid supervisor in (41) and (42), we get $H_1 = \Psi_2$, $H_2 = \overline{\mathbb{S}^1 \setminus C_2} = \{\xi \in \mathbb{S}^1 : \xi_1 \leq -2/3\}$ and $J_q = 3 - q$. Then, using the definition of a supervisory controller in (40), we get a hybrid controller with state $x_c = q$, data $C_c = (C_1 \times \{1\}) \cup (C_2 \times \{2\})$, $f_c = 0$, $D_c = (H_1 \times \{1\}) \cup (H_2 \times \{2\})$, $G_c = 3 - q$, and $\kappa_c(\xi, q) = \xi_q$. See Figure 18.

According to Corollary 33, this controller renders the set $\{(1, 0)\} \times Q$ globally asymptotically stable. In fact, it can be established that the point $(1, 0, 2)$ is globally asymptotically stable. ■

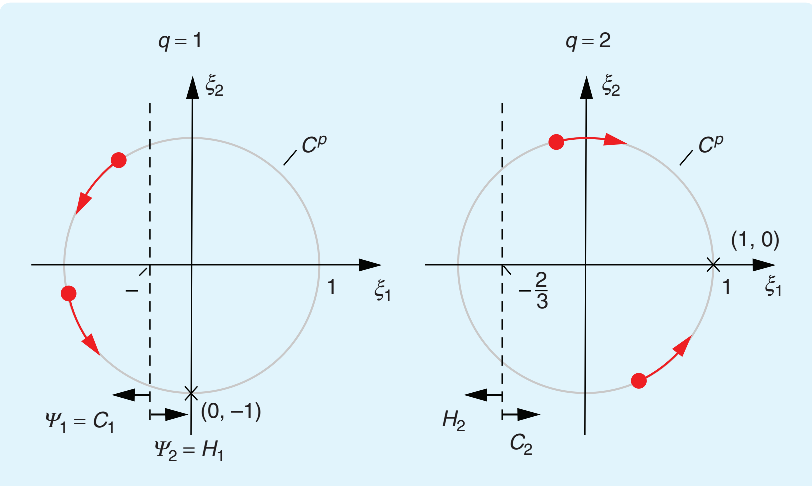

FIGURE 18 Illustration of the hybrid controller, for robust global stabilization of the point $(1, 0)$ on a circle $\mathbb{S}^1$, in Example 34. The sets C_1, Ψ_1, H_1 and C_2, Ψ_2, H_2 are subsets of the circle, contained in the half-planes as indicated by the arrows. These sets define the flow and jump sets in modes $q = 1$ and $q = 2$, respectively. In mode $q = 1$, the controller drives the state toward the point $(0, -1)$ from every point in C_1, as indicated by the arrows, and the switch to mode $q = 2$ happens when the state is in H_1. In mode $q = 2$, the controller drives the state toward the point $(1, 0)$ from every point in C_2, as indicated by the arrows, and the switch to mode $q = 1$ happens when the state is in H_2. This controller results in robust asymptotic stability of $(1, 0)$ on the circle. Note that the solutions starting from $\xi_1 = -1/3$, $\xi_2 > 0$, $q = 1$ are not unique. This initialization results in one solution that jumps immediately to $q = 2$ and then flows to $(1, 0)$ and another solution that flows to $\xi_1 = -1/3$, $\xi_2 < 0$, $q = 1$ before jumping. Similarly, the solutions starting from $\xi_1 = -2/3$, $\xi_2 > 0$, $q = 2$ are not unique.

Example 35: Mobile Robot Control

Consider a stabilization problem for the control system

$$\left.\begin{aligned}\dot{z} &= \xi v\\ \dot{\xi} &= uR(-\pi/2)\xi\end{aligned}\right\} \quad (z, \xi) \in \mathbb{R}^2 \times \mathbb{S}^1,$$

where $z \in \mathbb{R}^2$ corresponds to the position of a nonholonomic vehicle, $v \in \mathbb{R}$ denotes its velocity, which is treated as a control variable, $\xi \in \mathbb{S}^1$ denotes the orientation of the vehicle, and $u \in \mathbb{R}$ denotes angular velocity, which is the other control variable. The plant state is $x_p = (z, \xi)$. As in Example 34, positive angular velocity corresponds to rotation of ξ in the counterclockwise direction. The control objective is to asymptotically stabilize the point $x_p^* = (0, \xi^*)$, where $\xi^* \in \mathbb{S}^1$. We start by picking $v = -\rho(z^\top \xi)$, where ρ is a continuous function satisfying $s\rho(s) > 0$ for all $s \neq 0$. A discontinuous or set-valued choice for ρ is also possible. The condition $s\rho(s) > 0$ for all $s \neq 0$ ensures that, during flows, the size of z decreases except when $z^\top \xi = 0$ at which points $\dot{z} = 0$.

With the vehicle velocity specified, what remains is a control system with a single input u. Following the ideas in Example 34, a hybrid feedback algorithm for u exists that drives ξ to a given point on the unit circle. More generally, a comparable hybrid algorithm can be developed to track any continuously differentiable function, of the plant state z and perhaps of extra controller variables, that takes values on the unit circle. This task is accomplished by adding to the point stabilization algorithm an appropriate feedforward signal that renders the desired angular profile invariant.

Following ideas that are similar to those in [65], we consider the supervision of two hybrid controllers for the mobile robot system using a hybrid supervisor $\mathcal{H}_c$ with data given in (40)–(42).

The first controller is efficient at bringing the position of the robot close to the desired target while the second controller works to simultaneously regulate orientation and position. We give the two controllers a common state $\eta \in \mathbb{R}^m$ and suppose η takes values in a compact set Ω. The first hybrid controller drives ξ to $-z/|z|$. The data of the first controller satisfies

$$\widetilde{C}_1 \cup \widetilde{D}_1 = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^1 \times \Omega.$$

This specification avoids the singularity in $-z/|z|$ at $z = 0$.

The second hybrid controller drives ξ to $R(\sigma(z)\chi_1)\xi^*$, where σ is a continuously differentiable function that is zero at zero and positive otherwise, and χ_1 is a state of the controller satisfying

$$\dot{\chi} = \omega R(\pi/2)\chi, \quad \chi \in \mathbb{S}^1,$$

where $\omega > 0$. This controller satisfies $\widetilde{C}_2 \cup \widetilde{D}_2 = \mathbb{R}^2 \times \mathbb{S}^1 \times \Omega$. This second controller is closely related to time-varying stabilization algorithms that have appeared previously in the literature. See, for example, [64] and [71].

To satisfy the "Supervisor Assumption" for $\mathcal{A} = \{0\} \times \{\xi^*\} \times \Omega$ and $\Theta = \mathbb{R}^2 \times \mathbb{S}^1 \times \Omega$, we let $0 < \varepsilon_1 < \varepsilon_2$ and take

$$\begin{aligned} C_1 &:= \widetilde{C}_1 \cap (\overline{(\mathbb{R}^2 \setminus \varepsilon_1\mathbb{B})} \times \mathbb{S}^1 \times \Omega), \\ D_1 &:= \widetilde{D}_1 \cap (\overline{(\mathbb{R}^2 \setminus \varepsilon_1\mathbb{B})} \times \mathbb{S}^1 \times \Omega), \\ \Psi_1 &:= C_1 \cup D_1 = \overline{(\mathbb{R}^2 \setminus \varepsilon_1\mathbb{B})} \times \mathbb{S}^1 \times \Omega, \end{aligned}$$

and

$$\begin{aligned} C &:= \widetilde{C}_2 \cap (\varepsilon_2\mathbb{B} \times \mathbb{S}^1 \times \Omega), \\ D_2 &:= \widetilde{D}_2 \cap (\varepsilon_2\mathbb{B} \times \mathbb{S}^1 \times \Omega), \\ \Psi_2 &:= \varepsilon_1\mathbb{B} \times \mathbb{S}^1 \times \Omega. \end{aligned}$$

The components of the supervisor are then given by

$$\begin{aligned} H_1 &= \Psi_2 = \varepsilon_1\mathbb{B} \times \mathbb{S}^1 \times \Omega, \\ H_2 &= \overline{(\mathbb{R}^2 \setminus \varepsilon_2\mathbb{B})} \times \mathbb{S}^1 \times \Omega, \\ J_q &= 3 - q. \end{aligned}$$

With these specifications, according to Corollary 33 the hybrid controller (40)–(42) renders the set $\mathcal{A} \times Q$ globally asymptotically stable. ■

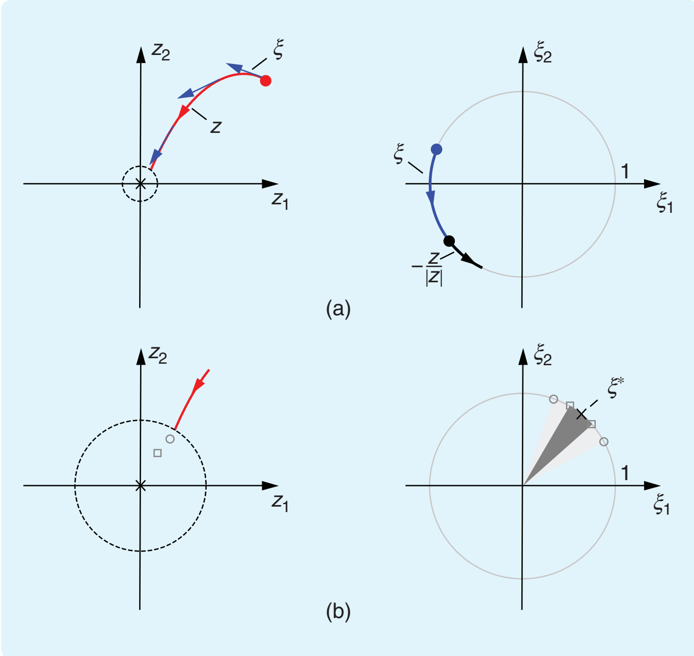

FIGURE 19 Position $z = (z_1, z_2)$ and orientation $\xi = (\xi_1, \xi_2)$ of the mobile robot with hybrid controller in Example 35. (a) The position, orientation, and ratio $-z/|z|$ with the first controller are shown on the z and ξ planes. The position z is in red, the orientation ξ is in blue, and $-z/|z|$ is in black. The first controller drives z to a neighborhood of $z = 0$ and ξ to $-z/|z|$. (b) The second controller drives the orientation to a cone centered at ξ^* with aperture depending on z. Cones associated to two points in a neighborhood of $z = 0$ are depicted. The two points, indicated by a square and a circle, along with the neighborhood around $z = 0$, indicated with dashed line, are depicted in the z plane, while their associated cones are denoted on the ξ plane. As z approaches the origin, the aperture of the cone approaches zero.

Example 36: Global Asymptotic Stabilization of the Inverted Position for a Pendulum on a Cart

Consider the task of robustly, globally asymptotically stabilizing the point $x_p^* := (0, 1, 0)$ for the system with state $x_p = (\xi, z) \in \mathbb{R}^3$ and dynamics

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} zR(-\pi/2)\xi \\ \xi_1 + \xi_2 u \end{bmatrix} =: f_p(x_p, u)$$
$$x_p \in \mathbb{S}^1 \times \mathbb{R}.$$

This model corresponds to a pendulum on a cart. The model includes an input transformation from force to acceleration of the cart, which is the control input u in the system above. Moreover, the ratio of the gravitational constant to the pendulum length has been normalized to one. The cart dynamics are ignored to simplify the presentation; however, global asymptotic stabilization of the full cart-pendulum system can be addressed with the same tools used below. The state ξ denotes the angle of the pendulum. The point $\xi = (0, 1)$ corresponds to the inverted position while $\xi = (0, -1)$ corresponds to the down position. The state z corresponds to the angular velocity, with positive velocity in the clockwise direction.

A feature of hybrid systems satisfying the Basic Assumptions is that pre-asymptotic stability is robust.

We construct a hybrid feedback stabilizer that supervises three classical controllers. The first controller steers the state x_p out of a neighborhood of the point $-x_p^*$. The second controller moves the system to a neighborhood of the point x_p^*. The third controller locally asymptotically stabilizes the point x_p^*. For an illustration of the data of these controllers, see Figure 20.

The third controller, a local asymptotic stabilizer for x_p^*, is simple to synthesize. For example, the idea of partial feedback linearization with "output" ξ_1 can be used since ξ_2 is positive and bounded away from zero in a neighborhood of x_p^*. Let $\kappa_3 : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ denote this local asymptotic stabilizer, let C_3 be a compact neighborhood of the point x_p^* that is also a subset of the basin of attraction for x_p^* for the system $\dot{x}_p = f_p(x_p, \kappa_3(x_p))$, $x_p \in \mathbb{S}^1 \times \mathbb{R}$, and let Ψ_3 be a compact neighborhood of the point x_p^* with the property that solutions of $\dot{x}_p = f_p(x_p, \kappa_3(x_p))$ starting in Ψ_3 do not reach the boundary of C_3. Then, redefine C_3 and Ψ_3 by intersecting the original choices with $\mathbb{S}^1 \times \mathbb{R}$. The set Ψ_3 is indicated in green in Figure 20(d) while the set C_3 is the union of the green and yellow regions in the same figure.

For the second controller, let $0 < \delta < \varepsilon < 1$ and define

$$
\begin{aligned}
W(x_p) &:= \frac{1}{2}z^2 + 1 + \xi_2, \\
\Psi_2 &:= \overline{\{(\xi, z) \in \mathbb{S}^1 \times \mathbb{R} : W(x_p) \geq \varepsilon\} \setminus \Psi_3}, \\
C_2 &:= \overline{\{(\xi, z) \in \mathbb{S}^1 \times \mathbb{R} : W(x_p) \geq \delta\} \setminus \Psi_3}, \\
\kappa_2(x_p) &:= -z\xi_2(W(x_p) - 2) \quad \text{for all } x_p \in C_2.
\end{aligned}
$$

The set Ψ_2 is indicated in green in Figure 20(b) or, alternatively, in Figure 20(c). The set C_2 is the union of the green and yellow regions in the same figures.

For the first controller, define $C_1 := \overline{(\mathbb{S}^1 \times \mathbb{R}) \setminus (\Psi_2 \cup \Psi_3)}$, $\Psi_1 := C_1$, and $\kappa_1(x_p) := k$ for all $x_p \in C_1$, where $k > \sqrt{\delta(2-\delta)}/(1-\delta)$. The set Ψ_1 is indicated in green in Figure 20(a).

We now establish that the indicated C_q and Ψ_q satisfy the "Supervisor Assumption" for $\mathcal{A} := \{x_p^*\}, Q := \{1, 2, 3\}$, and $\Theta := \mathbb{S}^1 \times \mathbb{R}$. We take $\widetilde{C}_q = \mathbb{S}^1 \times \mathbb{R}$ and $\widetilde{D}_q = \varnothing$ for each $q \in Q$. First, by construction, $\Psi_q \subset C_q$ for each $q \in Q$. Also,

$$
\begin{aligned}
\Psi_1 \cup \Psi_2 \cup \Psi_3 &= \overline{(\mathbb{S}^1 \times \mathbb{R}) \setminus (\Psi_2 \cup \Psi_3)} \cup \Psi_2 \cup \Psi_3 \\
&= \mathbb{S}^1 \times \mathbb{R} = \Theta.
\end{aligned}
$$

Next, we check that $\mathcal{A}$ is globally pre-asymptotically stable for $\dot{x}_p = f_p(x_p, \kappa_q(x_p))$, $x_p \in C_q$ for each $q \in Q$. This property holds for $q = 3$ since C_3 is a subset of the basin of attraction for x_p^* for the system $\dot{x}_p = f_p(x_p, \kappa_3(x_p))$.

For $q = 1$, we note that $x_p \in C_1$ implies that $W(x_p) \leq \delta$. In particular, $|z| \leq \sqrt{2\delta} < \sqrt{2}$ and $\xi_2 \leq \delta - 1 < 0$. We use the Lyapunov function $V_1(x_p) := 2 + z$, which is positive on C_1, and we obtain

$$
\begin{aligned}
\langle \nabla V_1(x_p), f_p(x_p, \kappa_1(x_p)) \rangle &= \xi_1 + \xi_2 k \\
&\leq \sqrt{1 - \xi_2^2} - |\xi_2| k \\
&\leq \sqrt{\delta(2-\delta)} - (1-\delta)k \\
&< 0
\end{aligned}
$$

for all $x_p \in C_q$. It follows from Theorem 20 that $\mathcal{A}$ is globally pre-asymptotically stable for $\dot{x}_p = f_p(x_p, \kappa_1(x_p))$, $x_p \in C_1$. This conclusion is equivalent to saying that there are no complete solutions that remain in the green region in Figure 20(a).

For $q = 2$, we use the Lyapunov function $V_2(x_p) := 1 + (W(x_p) - 2)^2$, which is positive on C_2. For all $x_p \in C_2$,

$$
\langle \nabla V_2(x_p), f_p(x_p, \kappa_2(x_p)) \rangle = -(z\xi_2)^2 (W(x_p) - 2)^2 \leq 0.
$$

It follows from Theorem 23 that $\mathcal{A}$ is globally pre-asymptotically stable for $\dot{x}_p = f_p(x_p, \kappa_2(x_p))$, $x_p \in C_2$. This is seen by noting that $V_2(x_p) = \mu \geq 1$ implies

$$
\langle \nabla V_2(x_p), f_p(x_p, \kappa_2(x_p)) \rangle = -(z\xi_2)^2(\mu - 1).
$$

For $\mu > 1$, nontrivial solutions evolving in the set where $\langle \nabla V_2(x_p), f_p(x_p, \kappa_2(x_p)) \rangle = 0$ must either be the constant solution $x_p(t, j) = x_p^*$ or the constant solution $x_p(t, j) = -x_p^*$. Since neither x_p^* nor $-x_p^*$ belong to C_2, the constrained system admits no nontrivial solutions. When $\mu = 1$, solutions evolve in the set where $W(x_p) = 2$. Belonging to this set and also to the set C_2 implies that z is bounded away from zero. Otherwise, the state would be near x_p^*, which is not a part of C_2. When z is bounded away from zero, ξ must eventually approach the point $(0, 1)$. Then, because we are considering solutions satisfying the condition $W(x_p) = 2$, z must eventually approach zero as ξ approaches $(0, 1)$. In other words, x_p must eventually approach x_p^*. Since x_p^* does not belong to C_2, there are no complete solutions for the constrained system.

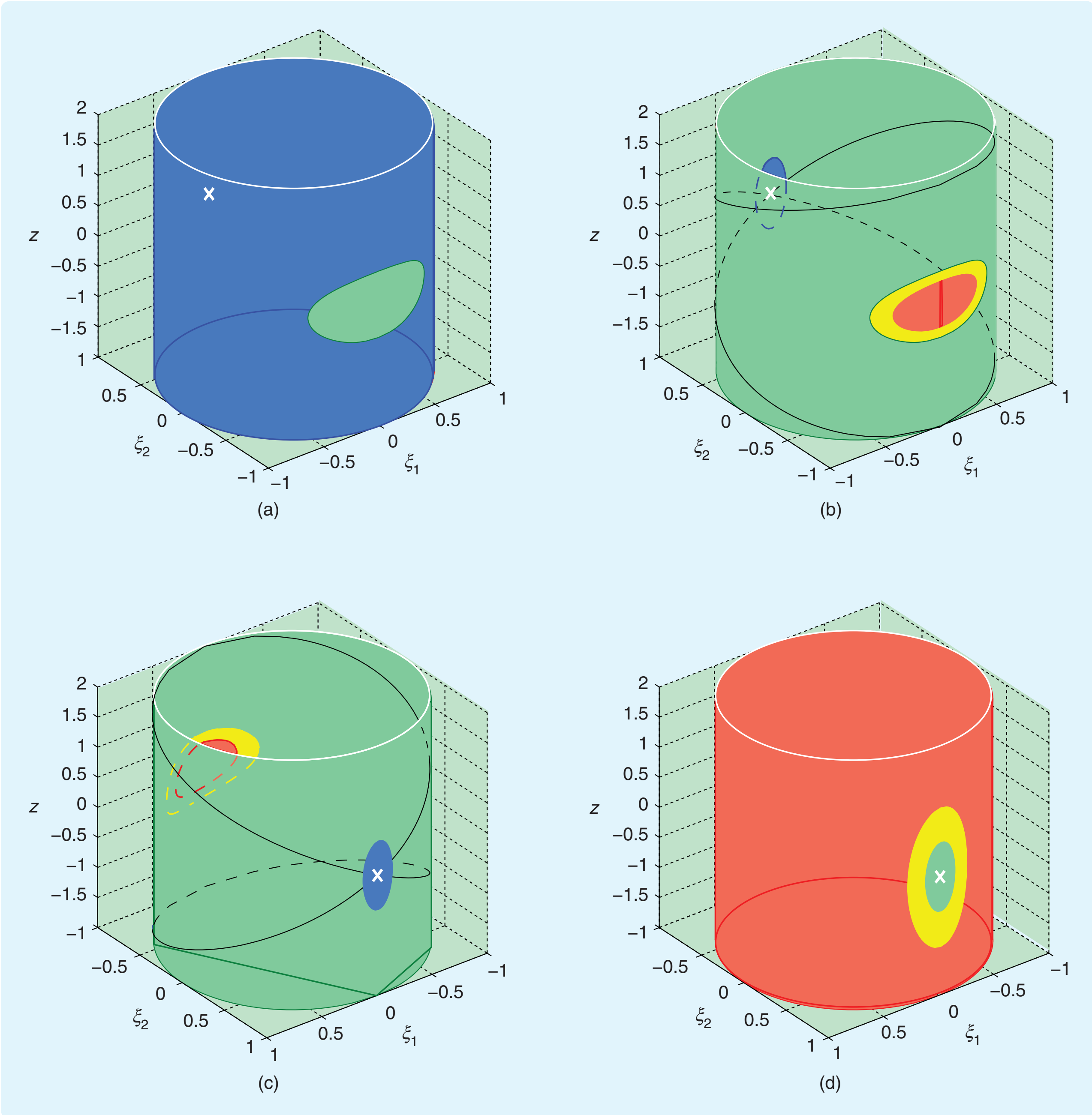

FIGURE 20 The sets used in the construction of the hybrid controller in Example 36 for $q=1$, $q=2$, and $q=3$ are shown in (a), (b), and (d), respectively. In (c), the sets for $q=2$ are depicted with perspective rotated by 180°. The state $x=(\xi, z)$ evolves on the cylinder $\mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^3$, while $q \in \{1, 2, 3\}$. The vertical position $(\xi, z)=((0, 1), 0)$ is indicated by a black $\mathbf{x}$. The black curve represents the set of points where $W(x) := 0.5z^2 + 1 + \xi_2 = 2$. To meet the "Supervisor Assumption," the sets D_q can be taken to be empty. The sets $\Phi_q := \bigcup_{i>q} \Psi_q$ are indicated in blue, the sets $\Psi_q \subset C_q$ are indicated in green, the sets $C_q \backslash \Psi_q$ are indicated in yellow, and the sets $\overline{(\mathbb{S}^1 \times \mathbb{R}) \backslash (C_q \cup (\bigcup_{i>q} \Psi_i))}$ are indicated in red. The sets H_q constructed in the supervisory control algorithm are the unions of the blue and red regions. The key properties are that, for each $q \in \{1, 2, 3\}$, the vertical position is globally pre-asymptotically stable for the system $\dot{x}_p = f_p(x_p, \kappa_q(x_p))$, $x_p \in C_q$, and each solution starting in a green region does not reach a red region. Taken together, these assumptions guarantee that solutions starting in a green region reach a blue region, for $q \in \{1, 2\}$.

This deduction is equivalent to saying that there are no complete solutions that remain in the union of the green and yellow regions in Figure 20(b) or, alternatively, Figure 20(c). This behavior is because trajectories in those regions converge to the set where $W(x_p) = 2$, which is indicated by the black curve, and moving along this set leads to points near x_p^* where the flow $\dot{x}_p = f_p(x_p, \kappa_2(x_p))$ cannot be continued in the union of the green and yellow regions.

Pre-asymptotically stable compact sets always admit Lyapunov functions.

Next we verify that, for each $q \in Q$, each solution of $\dot{x}_p = f_p(x_p, \kappa_q(x_p))$, $x_p \in C_q$ is either complete or ends in $\Phi_q \cup \overline{(\mathbb{S}^1 \times \mathbb{R}) \setminus C_q}$, where $\Phi_q = \cup_{i \in Q, i>q} \Psi_i$. This condition follows immediately from the fact that the second condition in Proposition 2 holds for each $x_p \in C_q \setminus \overline{(\mathbb{S}^1 \times \mathbb{R}) \setminus C_q}$. Thus, maximal solutions that are not complete must end in $\overline{(\mathbb{S}^1 \times \mathbb{R}) \setminus C_q}$. With respect to Figure 20, this conclusion amounts to the statement that the only places in green or yellow regions where flowing solutions cannot be continued are on the boundaries that touch red or blue regions.

Finally, we verify that, for each $q \in Q$, no solution starting in Ψ_q reaches $\overline{(\mathbb{S}^1 \times \mathbb{R}) \setminus (C_q \cup \Phi_q)} \setminus \mathcal{A}$. In other words, in terms of Figure 20, no solution starting in a green region reaches a red region. For $q = 1$, the red set is empty and so there is nothing to check. For $q = 3$, the property follows from the definition of Ψ_3, which is taken sufficiently small to satisfy the condition. For $q = 2$, this property follows from the fact that $\langle \nabla W(x_p), f_p(x_p, \kappa_2(x_p)) \rangle \geq 0$ for x_p such that $\delta \leq W(x_p) \leq 2$ and the fact that $x_p \in \Psi_2$ implies $W(x_p) \geq \varepsilon > \delta$.

This establishes that the "Supervisor Assumption" holds. In turn, the hybrid supervisory control algorithm given in (40)–(42), which is expressed explicitly in terms of C_q, Ψ_q, and κ_q, robustly, globally asymptotically stabilizes the point x_p^* for the pendulum on a cart. ■

PATCHY CONTROL LYAPUNOV FUNCTIONS

Patchy control-Lyapunov functions (PCLFs) are defined in [24] as an extension of the now classical concept of a control-Lyapunov function (CLF) [77], [2]. A CLF for a given control system is a smooth function whose value decreases along the solutions to the system under an appropriate choice of controls. The existence of such a function guarantees, in most cases, the existence of a robust stabilizing feedback [48]. Unfortunately, some nonlinear control systems do not admit a CLF.

A PCLF for system (5) is, broadly speaking, an object consisting of several local CLFs, the domains of which cover $\mathbb{R}^n$ and have certain weak invariance properties. PCLFs exist for far broader classes of nonlinear systems than CLFs, especially when an infinite number of patches (that is, of local CLFs) is allowed. Moreover, PCLFs lead to a robustly stabilizing hybrid feedback.

A smooth PCLF for the nonlinear system (5) with respect to the compact set $\mathcal{A} \subset \mathbb{R}^n$ consists of a finite set $Q \subset \mathbb{Z}$ and a collection of functions V_q and sets Ω_q, Ω'_q for each $q \in Q$, such that:

i) $\{\Omega_q\}_{q\in Q}$ and $\{\Omega'_q\}_{q \in Q}$ are families of nonempty open subsets of $\mathbb{R}^n$ such that

$$\mathbb{R}^n = \bigcup_{q\in Q} \Omega_q = \bigcup_{q \in Q} \Omega'_q,$$

and, for each $q \in Q$, the unit outward normal vector to Ω_q is continuous on $\partial\Omega_q \setminus (\cup_{i>q} \Omega'_i \cup \mathcal{A})$, and

$$\overline{\Omega'_q} \setminus \mathcal{A} \subset \Omega_q.$$

ii) For each q, V_q is a smooth function defined on an open set containing $\overline{\Omega_q \setminus \cup_{i>q} \Omega'_i} \setminus \mathcal{A}$.

iii) For each $q \in Q$ there exist a continuous, positive definite function $\alpha_q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions $\underline{\gamma}_q$, $\overline{\gamma}_q$, and a positive and continuous function $\mu_q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that

a) For all $\xi \in \Omega_q \setminus \cup_{i>q} \Omega'_i$,

$$\underline{\gamma}_q(|\xi|_{\mathcal{A}}) \leq V_q(\xi) \leq \overline{\gamma}_q(|\xi|_{\mathcal{A}}).$$

b) For all $\xi \in \Omega_q \setminus (\cup_{i>q} \Omega'_i \cup \mathcal{A})$, there exists $u_{q,\xi} \in \mathcal{U} \cap \mu_q(|\xi|_{\mathcal{A}})\mathbb{B}$, $\mathcal{U} \subset \mathbb{R}^r$, such that

$$\langle \nabla V_q(\xi), f_p(\xi, u_{q,\xi}) \rangle \leq -\alpha_q(|\xi|_{\mathcal{A}}).$$

c) For all $\xi \in \partial\Omega_q \setminus (\cup_{i>q} \Omega'_i \cup \mathcal{A})$, the $u_{q,\xi}$ of b) can be chosen such that

$$\langle n_q(\xi), f_p(\xi, u_{q,\xi}) \rangle \leq -\alpha_q(|\xi|_{\mathcal{A}}),$$

where $n_q(\xi)$ is the unit outward normal vector to Ω_q at ξ.

A stabilizing hybrid feedback can be constructed from a PCLF under an additional assumption. Suppose that, for each $\xi, v \in \mathbb{R}^n$, and $c \in \mathbb{R}$, the set $\{u \in \mathcal{U} : \langle v, f_p(\xi, u) \rangle \leq c\}$ is convex, which always holds if $\mathcal{U}$ is a convex set and $f_p(\xi, u)$ is affine in u. For each $q \in Q$, let

$$C_q = \overline{\Omega_q \setminus \bigcup_{i>q} \Omega'_i}, \qquad \Psi_q = \overline{\Omega'_q \setminus \bigcup_{i>q} \Omega'_i}, \qquad D_q = \varnothing.$$

It can be shown, in part through arguments similar to those used when constructing a feedback from a CLF, that for each

$q \in Q$ there exists a continuous mapping $k_q : C_q \setminus \mathcal{A} \to U$ such that, for all $\xi \in C_q \setminus \mathcal{A}$,

$$\langle \nabla V_q(\xi), f_p(\xi, k_q(\xi)) \rangle \leq -\alpha_q(|\xi|_{\mathcal{A}})/2,$$

which implies that the set $\mathcal{A}$ is globally pre-asymptotically stable for each controller $q \in Q$; moreover, for all $\xi \in \partial C_q \setminus \left(\mathcal{A} \cup \bigcup_{i>q} \Psi_i\right)$,

$$\langle n_q(\xi), f_p(\xi, u_{q,\xi}) \rangle \leq -\alpha_q(|\xi|_{\mathcal{A}})/2,$$

which implies that no maximal solution starting in Ψ_q reaches

$$\overline{\mathbb{R}^n \setminus \left(C_q \cup \bigcup_{i>q} \Psi_i\right)} \setminus \mathcal{A},$$

independently of how k_q is extended from $C_q \setminus \mathcal{A}$ to C_q. To preserve outer semicontinuity and local boundedness, set $k_q(\xi) = \mu_q(0)\mathbb{B}$ for $\xi \in C_q \cap \mathcal{A}$. The "Supervisor Assumption" holds. In particular, conditions 2a) and 2b) are addressed by the inequalities displayed above, while 2c) holds since $\Theta = \mathbb{R}^n$. By taking $\kappa_q(x_p, x_c) = k_q(x_p)$ for each $q \in Q$, the feedbacks k_q can be combined in a hybrid supervisor $\mathcal{H}_c$ with data given in (40). Although κ_q are set-valued on $\mathcal{A}$, the arguments used still apply. Furthermore, robustness results like Theorem 15 hold.

Example 37: Global Asymptotic Stabilization of a Point on the Three-Sphere

Consider the problem of globally asymptotically stabilizing the point $\xi^* := (0, 0, 0, 1) \in \mathbb{S}^3 := \{x \in \mathbb{R}^4 : x^T x = 1\}$ for the system

$$\dot{\xi} = \Lambda(u)\xi \quad \xi \in \mathbb{S}^3,$$

where

$$\Lambda(u) = \begin{bmatrix} 0 & u_3 & -u_2 & u_1 \\ -u_3 & 0 & u_1 & u_2 \\ u_2 & -u_1 & 0 & u_3 \\ -u_1 & -u_2 & -u_3 & 0 \end{bmatrix}.$$

These equations can model the orientation kinematics of a rigid body expressed in terms of unit quaternions. In this case, ξ^* corresponds to a desired orientation and the inputs u_i correspond to angular velocities.

First we study the effect of the feedback controls $u_i := -\xi_i$, $i \in \{1, 2, 3\}$. Denote this feedback κ_2. Using the Lyapunov-function candidate $V(\xi) = 1 - \xi_4$, and noting that

$$\langle \nabla V(\xi), \Lambda(\kappa_2(\xi))\xi \rangle = 1 - \xi_4^2 \quad \text{for all } \xi \in \mathbb{S}^3,$$

it follows that the point ξ^* is rendered locally asymptotically stable with basin of attraction containing every point except $-\xi^*$.

Next we study the effect of the feedback controls $u_1 = -\xi_2$, $u_2 = \xi_1$, $u_3 = \xi_4$. Denote this feedback κ_1. Using the Lyapunov-function candidate $V(\xi) = 1 - \xi_3$, it follows that the point $(0, 0, 1, 0)$ is rendered locally asymptotically stable with basin of attraction containing every point except $(0, 0, -1, 0)$.

Consider a hybrid supervisor of the classical controllers $\widetilde{\mathcal{H}}_q$, $q \in Q = \{1, 2\}$, that are static and are given by $\widetilde{\kappa}_q(\xi) = \kappa_q(\xi)$. The "Supervisor Assumption" is satisfied for $\mathcal{A} = \{\xi^*\}$ and $\Theta = \mathbb{S}^3$ by taking $D_1 = D_2 = \varnothing$ and

$$\begin{aligned} \Psi_1 = C_1 &:= \{\xi \in \mathbb{S}^3 : \xi_4 \leq -1/3\}, \\ C_2 &:= \{\xi \in \mathbb{S}^3 : \xi_4 \geq -2/3\}, \\ \Psi_2 &:= \{\xi \in \mathbb{S}^3 : \xi_4 \geq -1/3\}. \end{aligned}$$

Following the prescription of the components of a hybrid supervisor in (41) and (42), we get $H_1 = \Psi_2$, $H_2 = \overline{\mathbb{S}^3 \setminus C_2} = \{\xi \in \mathbb{S}^3 : \xi_4 \leq -2/3\}$, and $J_q = 3 - q$. Then, using the definition of a supervisory controller in (40), we get a hybrid controller with state $x_c = q$, data $C_c = (C_1 \times \{1\}) \cup (C_2 \times \{2\})$, $f_c = 0$, $D_c = (H_1 \times \{1\}) \cup (H_2 \times \{2\})$, $G_c = 3 - q$, and $\kappa_c(\xi, q) = \kappa_q(\xi)$.

According to Corollary 33, this controller renders the set $\{\xi^*\} \times Q$ globally asymptotically stable. In fact, it can be established that the point $(\xi^*, 2)$ is globally asymptotically stable.

This example can also be interpreted in terms of the concept of patchy control-Lyapunov functions. Indeed, it can be verified that, with $Q = \{1, 2\}$,

$$\begin{aligned} V_1(\xi) &= 1 - \xi_3, \quad \Omega_1' = \Omega_1 := \mathbb{S}^3, \\ V_2(\xi) &= 1 - \xi_4, \quad \Omega_2' := \{\xi \in \mathbb{S}^3 : \xi_4 \geq -1/3\}, \\ & \qquad\qquad\quad \Omega_2 := \{\xi \in \mathbb{S}^3 : \xi_4 \geq -2/3\}, \end{aligned}$$

constitutes a patchy control-Lyapunov function. ■

Example 38: Nonholonomic Integrator

Consider the nonlinear control system

$$\left. \begin{aligned} \xi_1 &= u \\ \xi_2 &= u_2 \\ \xi_3 &= \xi_1 u_2 - \xi_2 u_1 \end{aligned} \right\} =: f_p(\xi, u)$$

known as the nonholonomic integrator. According to Brockett's necessary condition, the origin is not stabilizable by continuous, static-state feedback, nor it is robustly stabilizable by discontinuous, static state feedback. It is robustly stabilizable by time-varying feedback and by hybrid feedback. See, for example, [35]. We construct a robustly stabilizing hybrid feedback using the idea of PCLFs.

Define $r(\xi) := \sqrt{\xi_1^2 + \xi_2^2}$ and let the constants $\rho > 1$ and $\varepsilon > 0$ satisfy $\rho(1+\varepsilon) < 2$. Let $V_1, V_2 : \mathbb{R}^3 \to \mathbb{R}$ and $\Omega_1', \Omega_1, \Omega_2', \Omega_2 \subset \mathbb{R}^3$ be given by

$$V_1(\xi) := (1+\varepsilon)\sqrt{\rho|\xi_3|} - \xi_1, \quad V_2(\xi) := \frac{1}{2}\xi^\top \xi,$$
$$\Omega_1' := \mathbb{R}^3, \quad \Omega_2' := \{r(\xi)^2 > \rho|\xi_3|\},$$
$$\Omega_1 := \mathbb{R}^3, \quad \Omega_2 := \{r(\xi)^2 > |\xi_3|\}.$$

We show that these choices correspond to a PCLF with respect to $\mathcal{A} = \{0\}$ for the nonholonomic integrator control system.

Observe that $\overline{\Omega_2'} \setminus \mathcal{A} \subset \Omega_2$. Also, the unit outward normal vector to Ω_2, given by

$$n_2(\xi) := \frac{(-2\xi_1, -2\xi_2, \operatorname{sgn}(\xi_3))}{\sqrt{4r(\xi)^2 + 1}},$$

is continuous on $\partial\Omega_2 \setminus \mathcal{A}$ since $\xi_3 \neq 0$ in that set. For the same reason, V_1 is smooth on an open set containing $\overline{\mathbb{R}^3 \setminus \Omega_2'} \setminus \mathcal{A}$.

Next we bound V_1 and V_2, examine their directional derivatives, and determine whether it is possible to pick the controls to make these functions decrease while also satisfying the appropriate condition with respect to the unit outward normal vectors.

Since, for each $\xi \in \mathbb{R}^3 \setminus \Omega_2'$, we have $\sqrt{\rho|\xi_3|} \geq r(\xi) \geq \xi_1$, it follows that

$$0.5\varepsilon(\sqrt{\rho|\xi_3|} + r(\xi)) \leq V_1(\xi) \leq (1+\varepsilon)(\sqrt{\rho|\xi_3|} + r(\xi)), \quad \xi \in \mathbb{R}^3 \setminus \Omega_2'.$$

This condition implies the existence of class-$\mathcal{K}_\infty$ functions $\underline{\gamma}_1$ and $\overline{\gamma}_1$ such that $\underline{\gamma}_1(|\xi|) \leq V_1(\xi) \leq \overline{\gamma}_1(|\xi|)$ for all $\xi \in \Omega_1 \setminus \Omega_2'$. Also, $0.5|\xi|^2 \leq V_2(\xi) \leq 0.5|\xi|^2$ for all $\xi \in \Omega_2$.

Now we consider possible control choices. Take $u_{1,\xi} = (1, 0)$ for all $\xi \in \mathbb{R}^3$. Then, for all $\xi \in \mathbb{R}^3 \setminus (\Omega_2' \cup \mathcal{A})$,

$$\begin{aligned}\langle \nabla V_1(\xi), f_p(\xi, u_{1,\xi})\rangle &= -1 + 0.5(1+\varepsilon)\rho\frac{-\xi_2}{\sqrt{\rho|\xi_3|}} \\ &\leq -1 + 0.5\,(1+\varepsilon)\,\rho < 0.\end{aligned}$$

Since there is no unit outward normal for Ω_1, this condition is all that needs to be checked for $q = 1$.

For $q = 2$, consider the control choice

$$u_{2,\xi} = \begin{bmatrix} -\xi_1 + 4\dfrac{\xi_2\xi_3}{r(\xi)^2} \\ -\xi_2 - 4\dfrac{\xi_1\xi_3}{r(\xi)^2} \end{bmatrix} \quad \text{for all } \xi \in \Omega_2 \setminus \mathcal{A}$$

which is continuous on $\Omega_2 \setminus \mathcal{A}$ and approaches zero as ξ approaches $\mathcal{A}$. We then have

$$\langle \nabla V_2(\xi), f_p(\xi, u_{2,\xi})\rangle = -\xi_1^2 - \xi_2^2 - 4\xi_3^2 \quad \text{for all } \xi \in \Omega_2 \setminus \mathcal{A}$$

and, for all $\xi \in \partial\Omega_2 \setminus \mathcal{A}$,

$$\langle n_2(\xi), f_p(\xi, u_{2,\xi})\rangle = \frac{2\xi_1^2 + 2\xi_2^2 - 4|\xi_3|}{\sqrt{4r(\xi)^2 + 1}} = -\frac{r(\xi)^2 + |\xi_3|}{\sqrt{4r(\xi)^2 + 1}}.$$

These calculations verify that the given patches and functions constitute a PCLF for the nonholonomic integrator system. Then, to construct a hybrid feedback control algorithm for this system following the presentation on PCLFs, we take $\Psi_1 = C_1 := \overline{\mathbb{R}^3 \setminus \Omega_2'}$, $C_2 := \Omega_2$, $\Psi_2 = \Omega_2'$, $\kappa_1(\xi) = u_{1,\xi}$ for all $\xi \in C_1$, $\kappa_2(\xi) = u_{2,\xi}$ for all $\xi \in C_2 \setminus \mathcal{A}$ and $\kappa_2(0) = 0$. Finally, the hybrid feedback stabilizer is defined using Ψ_q, C_q, and κ_q, $q \in \{1, 2\}$. ■

AN EXAMPLE BASED ON MULTIPLE LYAPUNOV FUNCTIONS

The final example uses many of the analysis tools that have been presented in this article. In particular, it uses Lyapunov functions, the invariance principle, results on stability with a finite number of events, and Theorem 32 based on the idea of multiple Lyapunov functions.

Example 37 Revisited: Stabilization on the Three-Sphere with Restricted Controls

We again consider the problem of stabilizing the point $\xi^* := (0, 0, 0, 1) \in \mathbb{S}^3$ for the kinematic equations in Example 37. This time, we restrict our attention to controls in the set $\mathcal{U} := \{u \in \mathbb{R}^3 : u_3 = u_1u_2 = 0, |u| \leq 1\}$. This problem can be associated with stabilizing a desired orientation for an underactuated rigid body. For example, see [16]. We focus only on the local asymptotic stabilization problem. Following the ideas outlined in this article, this solution can be combined with other hybrid controllers to achieve global asymptotic stabilization. The ideas used here are taken from [84], where global asymptotic stabilization is achieved.

Controller Specification

We use a hybrid controller with state $q \in Q = \{1, 2, 3, 4, 5, 6\}$, a timer state $\tau \in [0, 1]$, a state $\varsigma \in \{-1, 1\}$, and a state $\varphi \in [0, \pi/4]$. The closed-loop system state is $x := (\xi, q, \tau, \varsigma, \varphi) \in \mathbb{R}^8$, constrained to the set $K := \{\xi \in \mathbb{S}^3 : \xi_4 \geq \varepsilon\} \times Q \times [0,1] \times \{-1, 1\} \times [0, \pi/4]$, where $\varepsilon \in (0, 1)$. Our goal is to globally pre-asymptotically stabilize the set $\mathcal{A} := \{\xi^*\} \times Q \times [0, 1] \times \{-1, 1\} \times [0, \pi/4]$, resulting in local asymptotic stability of $\mathcal{A}$ when the constraint $\xi_4 \geq \varepsilon$ is removed.

Let $\sigma_i : \mathbb{N} \to \{0, 1\}$ satisfy $\sigma_i(i) = 1$ and $\sigma_i(j) = 0$ for $j \neq i$. For $q \in \{1, 2\}$, we use $u_i = -\sigma_i(q)\xi_i$. For $q \in \{3, 4, 6\}$, we

Synchronization in groups of biological oscillators occurs in swarms of fireflies, groups of crickets, ensembles of neuronal oscillators, and groups of heart muscle cells.

use $u_i = \varsigma\sigma_i(1)$. For $q = 5$, we use $u_i = \varsigma\sigma_i(2)$. For each $q \in Q$, the resulting feedback law is denoted $u = \kappa_q(x)$.

For $q \in \{1, 2\}$, we define $C_q := \{x \in K : \xi_q^2 \geq \mu(\xi_{3-q}^2 + \xi_3^2)\}$, where $\mu \in (0, 1/2)$. For $q \in \{3, 6\}$, we define $C_q := \{x \in K : -\varsigma\xi_1 \geq 0\}$. We also define $C_5 := \{x \in K : -\varsigma\xi_2 \geq 0\}$ and $C_4 := \{x \in K : \tau \in [0, \min\{\sqrt{1-\xi_4}, \varphi\}]\}$. For $q \in Q$, we define $D_q = \overline{K \setminus C_q}$.

We pick the flow map so that $\dot{q} = \dot{\varsigma} = \dot{\varphi} = 0$. In addition, $\dot{\tau} = 1$ for $q = 4$ while $\dot{\tau} = 0$ for $q \neq 4$.

We pick the jump map G so that $x^+ \in G(x)$ gives the following relationships: $\xi^+ = \xi$; $\tau^+ = 0$; for $q \in \{3, 4, 5\}$, $q^+ = q + 1$; for $q \in \{1, 2, 6\}$, $q^+ = 3$ when $\xi_1^2 + \xi_2^2 < \hat{\mu}\xi_3^2$, where $\hat{\mu} > 2\mu/(1-\mu)$, $q^+ \in \mathcal{I}(\xi)$ when $\xi_1^2 + \xi_2^2 > \hat{\mu}\xi_3^2$, where $\mathcal{I}(\xi) := \{i \in \{1, 2\} : \xi_i^2 = \max\{\xi_1^2, \xi_2^2\}\}$, and $q^+ = \{3\} \cup \mathcal{I}(\xi)$ when $\xi_1^2 + \xi_2^2 = \hat{\mu}\xi_3^2$. The possible mode transitions are indicated in Figure 21.

When $q \in \{1, 2, 5, 6\}$, $\varsigma^+ \in \{s \in \{-1, 1\} : -s\xi_1 \geq 0\}$, when $q = 3$, $\varsigma^+ = \operatorname{sgn}(\xi_2\xi_3)$ for $\xi_2\xi_3 \neq 0$ and $\varsigma^+ = \{-1, 1\}$ for $\xi_2\xi_3 = 0$; when $q = 4$, $\varsigma^+ \in \{s \in \{-1, 1\} : s\xi_2 \geq 0\}$, When $q \neq 3$, $\varphi^+ = \varphi$; when $q = 3$, $\varphi^+ = 0.5\cot^{-1}(|\xi_2/\xi_3|)$ for $\xi_3 \neq 0$, $\varphi^+ = 0$ for $\xi_3 = 0$ and $\xi_2 \neq 0$, and $\varphi^+ \in [0, \pi/4]$ for $\xi_2 = \xi_3 = 0$.

Verifying the Conditions of Theorem 32 for the Closed-Loop System

Condition 1

Since $\xi^+ = \xi$, it follows from the definitions above that $G(D \cap \mathcal{A}) \subset \mathcal{A}$.

Condition 2

We associate events with transitions to modes $q = 3$. Events are indicated by red arrows in Figure 21. To assess stability with these primary events inhibited, we define secondary events, corresponding to transitions to modes $q \in \{4, 5, 6\}$. With primary events inhibited, there are no more than three secondary events. Therefore, according to Theorem 31, the system with primary events inhibited has $\mathcal{A}$ pre-asymptotically stable if the flow dynamics on C_q, $q \in \{3, 4, 5, 6\}$ has $\mathcal{A}$ pre-asymptotically stable, and switching back and forth between between $q = 1$ and $q = 2$ results in $\mathcal{A}$ being pre-asymptotically stable.

To assess pre-asymptotic stability of the flow dynamics on C_4, we use the Lyapunov function $W(x) = \rho\sqrt{V(\xi)} - \tau$, where $V(\xi) = 1 - \xi_4$ and $\rho \in (1, \sqrt{2})$. For all $x \in C_4 \setminus \mathcal{A}$, it follows that $(\rho - 1)\sqrt{V(\xi)} \leq W(x) \leq \rho\sqrt{V(\xi)}$. Also, since

$$\langle \nabla V(\xi), \Lambda(\kappa_4(x))\xi \rangle \leq |\xi_1| \leq \sqrt{2}\sqrt{V(\xi)},$$

for all $x \in C_4 \setminus \mathcal{A}$ it follows that

$$\langle \nabla W(x), F_4(x) \rangle \leq \frac{\rho}{\sqrt{2}} - 1 < 0,$$

where F_4 is the closed-loop flow map when $q = 4$.

To assess pre-asymptotic stability of the flow dynamics on C_3, C_5 and C_6 we use $W(x) = V(\xi)$. We obtain

$$\langle \nabla V(\xi), \Lambda(\kappa_q(x))\xi \rangle = \varsigma\xi_j \leq 0,$$

where $j = 2$ for $q = 5$ and $j = 1$ for $q \in \{3, 6\}$. This property establishes stability of $\mathcal{A}$, and since ξ_4 is bounded away from zero, there are no complete flowing solutions on C_3, C_5 or C_6, which implies that $\mathcal{A}$ is pre-asymptotically stable.

To assess pre-asymptotic stability of the combined flow and jump dynamics C_q, D_q, $q \in \{1, 2\}$, we use the Lyapunov function $W(x) = V(\xi)$. We obtain

$$\langle \nabla V(\xi), \Lambda(\kappa_q(x))\xi \rangle = -\xi_q^2 \leq -\frac{\mu}{\mu+1}(1 - \xi_4^2) < 0$$

$$\text{for all } x \in C_q \setminus \mathcal{A}.$$

In addition, $V(\xi)$ does not change during jumps. So, to establish pre-asymptotic stability of $\mathcal{A}$ we just need to rule out complete solutions that jump only and do not converge. This behavior is ruled out by the fact that a jump to mode $q \in \{1, 2\}$ from a point not in $\mathcal{A}$ means that $\xi_q^2 \geq \xi_{3-q}^2 > 0$ and $\hat{\mu}\xi_3^2 \leq \xi_q^2 + \xi_{3-q}^2$, which implies that

$$\begin{aligned} \mu(\xi_{3-q}^2 + \xi_3^2) &\leq \mu(\xi_{3-q}^2 + \hat{\mu}^{-1}(\xi_{3-q}^2 + \xi_q^2)) \\ &\leq \mu(1 + 2\hat{\mu}^{-1})\xi_q^2 < \xi_q^2. \end{aligned}$$

In other words, a jump from $D_{3-q} \setminus \mathcal{A}$ lands at a point not in D_q.

These calculations establish condition 2) of Theorem 32.

Conditions 3 and 4

We take $W(x) = 1 - \xi_4$, which satisfies Condition 3 with $\alpha_1(s) = \alpha_2(s) = 2\alpha_3(s) = 2s$ for all $s \geq 0$. Now we establish Condition 4. We start from a hybrid time where a

The concepts of average dwell-time switching and multiple Lyapunov functions, which are applicable to switched systems, extend to hybrid systems.

jump to $q=3$ has just occurred. By the time a jump to $q=4$ occurs, $W(x)$ has not increased and we have $\xi_1 = 0$ and $\xi_2^2 \le \hat{\mu}\xi_3^2$. When $q=4$, $W(x)$ increases, but we argue that the sequence of modes $q=4$, $q=5$, $q=6$ results in a decrease in $W(x)$. Since $W(x)$ is also strictly decreasing for $q \in \{1,2\}$, this implies that when $q=3$ is revisited, $W(x)$ has decreased. The key to showing this property is to establish that, with $\chi_{i,q}$ denoting ξ_i at the end of mode q, there exists a continuous function ρ that is less than one except when its argument is one, so that $|\chi_{3,5}| \le \rho(\chi_{4,3})|\chi_{3,3}|$. Indeed, using that $\chi_{1,6} = \chi_{2,5} = 0$ and $\chi_{2,6}^2 + \chi_{3,6}^2 = \chi_{2,5}^2 + \chi_{2,6}^2$, this gives

$$\begin{aligned}
1 - \chi_{4,6}^2 &= \chi_{1,6}^2 + \chi_{2,6}^2 + \chi_{3,6}^2 \\
&= \chi_{2,6}^2 + \chi_{3,6}^2 \\
&= \chi_{2,5}^2 + \chi_{3,5}^2 \\
&= \chi_{3,5}^2 \\
&\le \rho^2(\chi_{4,3})\chi_{3,3}^2 \\
&\le \rho^2(\chi_{4,3})(1 - \chi_{4,3}^2).
\end{aligned}$$

Since $\chi_{4,6}$ and $\chi_{4,3}$ are positive, this implies

$$1 - \chi_{4,6} \le \rho^2(\chi_{4,3})(1 - \chi_{4,3}).$$

Finally, we establish $|\chi_{3,5}| \le \rho(\chi_{4,3})|\chi_{3,3}|$. Let τ_q, $q \in \{4,5\}$, denote the time spent in mode q. Let ς_q, $q \in \{4,5\}$, denote the value of ς in mode q. A routine calculation involving the solution of a linear, two-dimensional oscillator and using that $\chi_{1,3} = \chi_{2,5} = 0$ gives

$$\begin{aligned}
\cos(\varsigma_4\tau_4)\chi_{3,5} = \cos(\varsigma_5\tau_5)[&(\cos(\varsigma_4\tau_4)^2 - \sin(\varsigma_4\tau_4)^2)\chi_{3,3} \\
&- 2\cos(\varsigma_4\tau_4)\sin(\varsigma_4\tau_4)\chi_{2,3}].
\end{aligned}$$

Due to the construction of the jump map from mode $q=3$, $\varsigma 4 = \mathrm{sgn}\,(\chi_{2,3}\chi_{3,3})$ when $\chi_{2,3}\chi_{3,3} \neq 0$; otherwise $\varsigma_4 \in \{-1,1\}$. Then, due to the value of φ in mode $q=4$, which limits τ_4, it follows that the sign of $\sin(\varsigma_4\tau_4)\chi_{2,3}$ is the same as the sign of $\chi_{3,3}$. Using $\tau_4 \le \pi/4$ and $\varsigma_4 \in \{-1,1\}$, it follows that

$$|\chi_{3,5}| \le \frac{\cos(\tau_4)^2 - \sin(\tau_4)^2}{\cos(\tau_4)}|\chi_{3,3}|.$$

The function involving τ_4 on the right-hand side takes values in the interval $[0,1)$ for all $\tau_4 \in (0, \pi/4]$. Now, by the definition of C_4, κ_4, and the fact that $\chi_{1,3} = 0$, τ_4 can be expressed as a continuous function of $\chi_{4,3}$ that is zero when $\chi_{4,3} = 1$ and is positive otherwise. In turn, this establishes the bound $|\chi_{3,5}| \le \rho(\chi_{4,3})|\chi_{3,3}|$ for a continuous, nonnegative-valued function ρ that is less than one except when $\chi_{4,3} = 1$. ■

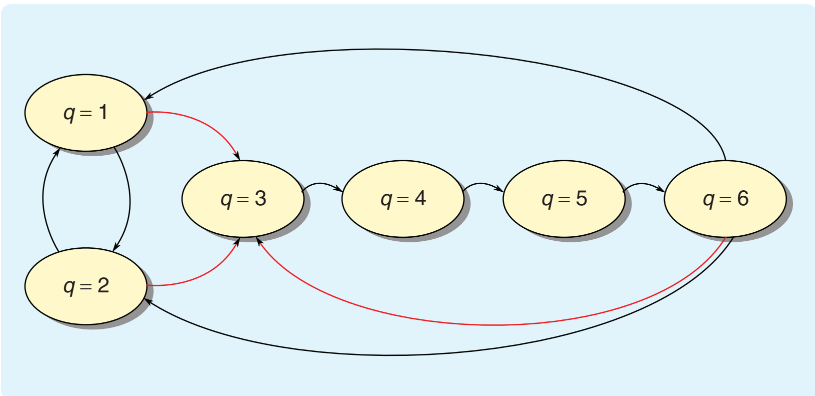

FIGURE 21 Possible mode transitions when stabilizing a point on the three-sphere by hybrid feedback that uses only two angular velocities, one at a time. Normal operation corresponds to jumps between modes $q=1$ and $q=2$. However, at times the sequence of modes $3 \rightarrow 4 \rightarrow 5 \rightarrow 6$ is required. The initiation of such a sequence is associated with an event in the hybrid system. Transitions associated with events are indicated by red arrows in the diagram. The quantity $V(\xi) = 1 - \xi_4$ does not increase along solutions except when $q=4$. When events are inhibited, so that the mode $q=4$ is reached no more than once, the resulting hybrid system has the correct pre-asymptotically stable set. With events included and $V(\xi)$ considered at event times, the resulting values of $V(\xi)$ constitute a strictly decreasing sequence. Thus, Theorem 32 can be used to establish pre-asymptotic stability for the hybrid system.

CONCLUSIONS

Hybrid dynamical systems combine flows and jumps. They can be modeled in a compact form, and they cover a fascinating variety of dynamic phenomena.

With the use of hybrid time domains and the notion of graphical convergence, sequential compactness of the space of solutions and semicontinuous dependence of solutions on initial conditions and perturbations can be established under mild conditions.

Basic questions about solutions to dynamical systems concern existence, uniqueness, and dependence on initial conditions and other parameters.

The properties of the space of solutions to a hybrid system have important consequences for stability theory. For example, these properties imply that asymptotic stability of a compact set is uniform and robust to perturbations. The properties also facilitate extensions of the classical invariance principle and converse Lyapunov theorems to the hybrid setting. The hybrid invariance principle and Lyapunov functions lend themselves to natural sufficient conditions for asymptotic stability in a hybrid system. Additional stability analysis tools, related to identifying and limiting events in a hybrid system, can also be developed.

The stability analysis tools can be used to predict the behavior of hybrid systems and to design hybrid control algorithms. The examples of hybrid control systems provided in this article only scratch the surface of what is possible using hybrid feedback control. The framework and tools presented in this article may help in the process of discovering new hybrid feedback control ideas.

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Advancing the Field

With no other choice but to live one day at a time and to keep intellectually alive, Tsien continued to work and teach and await the next step, whatever it might be, whenever it might be. He turned to other fields of research, such as the study of games and economic behavior. In 1954, he published a textbook titled *Engineering Cybernetics*, a book on systems of communication and control. It too would be well received.

Years later, Wallace Vander Velde, an MIT professor and renowned expert in cybernetics, would describe the book as "remarkable" and "an extraordinary achievement in its time." Wrote Vander Velde of the book:

> In 1954, a decent theory of feedback control for linear, time invariant systems existed and servomechanism design was an established practice. But Tsien was looking ahead to more complex control and guidance problems—notably the guidance of rocket-propelled vehicles. This stimulated his interest in the systems with time-varying coefficients, time lag and nonlinear behavior. All these topics are treated in this book.

But Tsien went further to deal with optimal control via the variational calculus, optimalizing control and fault-tolerant control systems among other topics! He visualized a theory of guidance and control which would be distinct from, and would support, the practice of these disciplines. This has certainly come to be, and his pioneering effort may be thought of as a major foundation stone of that effort which continues to this day.

—*Thread of the Silkworm*, by Iris Chang, BasicBooks, New York, 1994, pp. 175–176.