Robust Global Asymptotic Stabilization of a 6-DOF Rigid Body by Quaternion-based Hybrid Feedback*

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***Abstract*— The problem of robust, global asymptotic stabilization of a rigid body is hampered by major topological obstructions. These obstructions prevent a continuous state feedback from solving the problem and also lead to robustness issues when (non-hybrid) discontinuous feedback is applied. In this paper, we extend a hybrid control strategy proposed in a companion paper for robust, global asymptotic stabilization of rigid body attitude to the case where translation is also considered. Through Lyapunov analysis, we develop quaternion-based hysteretic hybrid control laws in the kinematic and dynamic settings. In the dynamic setting, two control laws are derived: one derived from an energy-based Lyapunov function and one derived by backstepping. Robustness to measurement noise is asserted by employing recently developed stability theory for hybrid systems. A comparison between discontinuous and hysteretic feedback under measurement noise is shown in simulation.**

I. Introduction

The control of the translation and attitude of a rigid body has applications ranging from underwater vehicles [1] and robotic manipulators [2] to satellites [3], [4], [5]. While controlling the attitude of a rigid body is often addressed in the literature without mention of position control (e.g. [6], [7]), coupling terms present in the dynamics can complicate a separation in the design of attitude and position controllers. In this paper, we address attitude and position control simultaneously, motivated by the application to underwater vehicles in [1], [8], [9].

The problem of global rigid body stabilization is subject to major topological obstructions (see [7] for a rigorous description). First, any three-parameter parametrization of $SO(3)$ cannot be globally nonsingular [4], making controllers based on these parametrizations inherently non-global. Noted in [10], $SO(3)$, the configuration manifold for the rigid body, is compact, which precludes the existence of a globally stabilizing continuous feedback. Moreover, control schemes based on redundant parametrizations of $SO(3)$ may exhibit *unwinding*, where the attitude is rotated unnecessarily through large angles. Finally, when redundant parametrizations are used, it becomes necessary to stabilize a disconnected set of points, making global stabilization with state feedback (even discontinuous) that is robust to measurement noise impossible [7], [11].

In this paper, we extend current results for the robust global stabilization of the attitude of a rigid body [7] to the case where translation is also considered. As in [7], the results presented here use a quaternion-based hysteretic *hybrid* feedback that robustly, globally asymptotically stabilizes a desired rotation and translation of the rigid body. Results are presented in kinematic and dynamic settings. When dynamics are considered, two controllers are derived through Lyapunov analysis. One is developed from an energy-based Lyapunov function and the other is derived through backstepping and is similar to [3] for attitude-only regulation. As in [8], [1], the energy-based controller does not rely on backstepping and requires the use of an invariance principle. Interestingly, for the kinematic and energy-based control laws, the addition of translational motion does not add any complexity to the form of the hysteresis in [7]. However, when backstepping is applied, the form of the hysteresis can include coupling terms between position and rotation of the rigid body. In both cases, robustness of stability of the closed-loop system to measurement noise is asserted by a $\mathcal{KL}$ estimate provided by the results of [12].

This paper is organized as follows. Section II provides a review of the application of unit quaternions to rigid body stabilization, where quaternion algebra, kinematics, dynamics, and error coordinates on the appropriate state space are discussed. Section III provides introductory material on hybrid systems (those that allow continuous and discrete state evolution). Section IV develops the hybrid control strategy and presents the robust, global asymptotic stability results. Finally, Section V shows a simulation study, comparing (non-hybrid) discontinuous feedback to hysteretic hybrid feedback.

II. Quaternions and Rigid Body Stabilization

The position and attitude of a rigid body are represented by a pair $(p, R) \in \mathbb{R}^3 \times SO(3)$ where p is a vector representing the position of the rigid body, R is a rotation matrix representing the attitude of the rigid body (orientation),

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I, \det R = 1\}$$

is the *special orthogonal group of order three*, and $I \in \mathbb{R}^{3\times 3}$ denotes the identity matrix. We let

$$\mathfrak{so}(3) = \{S \in \mathbb{R}^{3\times 3} : S^\top = -S\}$$

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and define the map $S : \mathbb{R}^3 \to \mathfrak{so}(3)$ as

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Note that for two vectors, $x, y \in \mathbb{R}^3$, $S(x)y = x \times y$, where $\times$ denotes the vector cross product operation. We denote the n-dimensional sphere (embedded in $\mathbb{R}^{n+1}$) as

$$\mathcal{S}^n = \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}.$$

Then, given an angle $\theta \in \mathbb{R}$ and a rotation axis $\hat{n} \in \mathcal{S}^2$, a rotation matrix can be parametrized using the Rodrigues formula, $\mathcal{R} : \mathbb{R} \times \mathcal{S}^2 \to SO(3)$, defined as

$$\mathcal{R}(\theta, \hat{n}) = I + \sin(\theta) S(\hat{n}) + (1 - \cos(\theta)) S^2(\hat{n}). \quad (1)$$

Using the Rodrigues formula (1), we can define a parametrization of $SO(3)$ in terms of unit quaternions. A unit quaternion

$$q = \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} = \pm \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{n} \end{bmatrix} \in \mathcal{S}^3 \quad (2)$$

represents an element of $SO(3)$ by the map $\mathcal{R} : \mathcal{S}^3 \to SO(3)$ defined as

$$\mathcal{R}(q) = I + 2\eta S(\epsilon) + 2S^2(\epsilon). \quad (3)$$

Note that for every $R \in SO(3)$, there are exactly two unit quaternions, $\pm q$, such that $R = \mathcal{R}(q) = \mathcal{R}(-q)$.

Let $q_1, q_2 \in \mathbb{R}^4$. Then, under the multiplication rule,

$$q_1 \otimes q_2 = \begin{bmatrix} \eta_1\eta_2 - \epsilon_1^\top \epsilon_2 \\ \eta_1\epsilon_2 + \eta_2\epsilon_1 + S(\epsilon_1)\epsilon_2 \end{bmatrix},$$

the unit quaternion inverse and identity are

$$q^{-1} = \begin{bmatrix} \eta \\ -\epsilon \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 \\ \mathbf{0}_{3\times 1} \end{bmatrix} \in \mathcal{S}^3.$$

Note that $\mathcal{R}$ is a group homomorphism, i.e.,

$$\mathcal{R}(q_1)\mathcal{R}(q_2) = \mathcal{R}(q_1 \otimes q_2)$$

and in particular, $\mathcal{R}^{-1}(q) = \mathcal{R}^\top(q) = \mathcal{R}(q^{-1})$. Note also that $\mathcal{R}(\mathbf{1}) = \mathcal{R}(-\mathbf{1}) = I$.

A. Kinematics, Dynamics, and Stabilization

The kinematics of a rigid body are given by

$$\left.\begin{aligned} \dot{p} &= Rv \\ \dot{R} &= RS(\omega) \end{aligned}\right\} \quad (p, R) \in \mathbb{R}^3 \times SO(3), \quad (4)$$

where R maps vectors in the body frame to the inertial frame and $v, \omega \in \mathbb{R}^3$ denote the rigid body's translational and angular velocities in the body frame, respectively. Written with unit quaternions, (4) becomes

$$\left.\begin{aligned} \dot{p} &= \mathcal{R}(q)v \\ \dot{q} &= \frac{1}{2} q \otimes \chi(\omega) \end{aligned}\right\} \quad (p, q) \in \mathbb{R}^3 \times \mathcal{S}^3 \quad (5)$$

where $\chi : \mathbb{R}^3 \to \mathbb{R}^4$ is defined as

$$\chi(\omega) = \begin{bmatrix} 0 \\ \omega \end{bmatrix}.$$

Let

$$U(q) = \begin{bmatrix} -\epsilon^\top \\ \eta I + S(\epsilon) \end{bmatrix}, \quad \Lambda(q) = \begin{bmatrix} \mathcal{R}(q) & 0_{3\times 3} \\ 0_{4\times 3} & \frac{1}{2}U(q) \end{bmatrix}.$$

Then, $\dot{q} = \frac{1}{2}U(q)\omega$ and (5) becomes

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \mathcal{R}(q) & 0 \\ 0 & \frac{1}{2}U(q) \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = \Lambda(q) \begin{bmatrix} v \\ \omega \end{bmatrix}. \quad (6)$$

Motivated by the application to underwater vehicles in [13], we assume the dynamic model,

$$\mathcal{M}\dot{\nu} + \mathcal{C}(\nu)\nu + \mathcal{D}(\nu)\nu + \xi(q) = \mathcal{F}, \quad (7)$$

where

$$\nu = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6,$$

$\mathcal{M} = \mathcal{M}^\top > 0$ is a matrix representing mass and inertia, $\mathcal{C}(\nu) = -\mathcal{C}^\top(\nu)$ is a skew-symmetric matrix containing Coriolis terms, $\mathcal{D}(\nu) = \mathcal{D}^\top(\nu) > 0$ is a matrix representing dissipative forces, $\xi(q)$ is a vector of known external forces (e.g. gravitational and buoyant forces) and $\mathcal{F}$ is a vector of control forces.

The control objective is stated in terms of appropriate error coordinates. Suppose that there is a desired position and attitude, $(p_d, R_d) \in \mathbb{R}^3 \times SO(3)$ for the rigid body and that $(p, R) \in \mathbb{R}^3 \times SO(3)$ denote the actual position and attitude. Then, error coordinates are obtained as $(p_e, R_e) = (p - p_d, R_d^\top R) \in \mathbb{R}^3 \times SO(3)$. Assuming that (p_d, R_d) is constant, the error coordinates have the kinematic equations $\dot{p}_e = Rv$ and $\dot{R}_e = R_e S(\omega)$. In this setting, the goal is to drive (p_e, R_e) to $(0, I)$ so that $(p, R) = (p_d, R_d)$. When written using unit quaternions, we see that if $R_e = I$, then the associated set of unit quaternions is $\pm\mathbf{1}$. Since the dynamic equations do not change when error coordinates are used, we henceforth drop the subscript e.

We can now state our global stabilization goals. Then, the kinematic sub-problem is to robustly and globally asymptotically stabilize

$$\mathcal{A}_k = \{0\} \times \{\pm\mathbf{1}\} \subset \mathbb{R}^3 \times \mathcal{S}^1 \quad (8)$$

for the system (5) (equivalently, (6)). When dynamics are taken into account, the goal is to robustly and globally asymptotically stabilize

$$\mathcal{A}_d = \{0\} \times \{\pm\mathbf{1}\} \times \{0\} \subset \mathbb{R}^3 \times \mathcal{S}^1 \times \mathbb{R}^6 \quad (9)$$

for the system (5) (equivalently, (6)), (7).

III. Hybrid Systems Preliminaries

To break the topological obstructions to robust global stability discussed in Section I and [7], we employ the power of hybrid systems: dynamic systems where both continuous and discrete evolution of the state can occur. Following the framework presented in [12], [14], we let $x \in \mathbb{R}^n$ denote the state of a hybrid system $\mathcal{H} = (f, g, C, D)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the *flow map* that dictates continuous state evolution according to $\dot{x} = f(x)$, $g : \mathbb{R}^n \to \mathbb{R}^n$ is the *jump map* that dictates discrete evolution of the state according to $x^+ = g(x)$, $C \subset \mathbb{R}^n$ is the *flow set* that indicates where continuous evolution is possible, and D is the *jump set* that indicates where discrete evolution is possible. We write a hybrid system as

$$\mathcal{H}\begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D. \end{cases}$$

To reap the benefits of the robust stability theory in [12], the data of the hybrid system must satisfy some mild regularity conditions [12, A0–A3], which for the purposes of this paper reduce to f and g being continuous and C and D being closed sets.

The available robust stability theory in [12] largely depends on the notion of a solution to a hybrid system. We note here that a solution x to $\mathcal{H}$ is defined on a hybrid time domain, denoted $\operatorname{dom} x \subset [0,\infty) \times \{0,1,2,\dots\}$ and parametrized by t, the amount of time spent flowing, and j, the number of jumps that have occurred. The set of solutions to $\mathcal{H}$ with initial condition x_0 is denoted as $\mathcal{S}_{\mathcal{H}}(x_0)$. For further details, we refer the reader to [12], [14].

Defining stability and attractivity for compact sets is done in a familiar fashion. Let $\mathbb{B} = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ denote the closed n-dimensional unit ball and for some set $\mathcal{A}$, let $|\cdot|_{\mathcal{A}}$ denote the distance to $\mathcal{A}$. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x_0 \in \mathcal{A} + \delta\mathbb{B}$, each solution $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ satisfies $x(t,j) \in \mathcal{A} + \epsilon\mathbb{B}$ for all $(t,j) \in \operatorname{dom} x$. $\mathcal{A}$ is *attractive* with *basin of attraction* $\mathcal{B}_{\mathcal{A}}$ if $\exists \delta > 0$ such that $\forall x_0 \in \mathcal{B}_{\mathcal{A}} \supset \mathcal{A} + \delta\mathbb{B}$, every $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ is complete and satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. A compact set $\mathcal{A}$ is asymptotically stable if it is both stable and attractive and is *globally* asymptotically stable if $\mathcal{B}_{\mathcal{A}} = \mathbb{R}^n$. Note that $\mathbb{R}^n \setminus (C \cup D) \subset \mathcal{B}_{\mathcal{A}}$ since $\mathcal{S}_{\mathcal{H}}(\mathbb{R}^n \setminus (C \cup D)) = \emptyset$.

IV. Robust Global Asymptotic Stabilization: Kinematics and Dynamics

In this section, we derive a hybrid feedback that robustly globally asymptotically stabilizes $\mathcal{A}_k$ for (5). We then provide two extensions of this result into the dynamic setting: with a simple Lyapunov function requiring a recently developed invariance principle for hybrid systems and via backstepping. In all cases, the design of the flow and jump sets become critical for ensuring robust global asymptotic stability of the appropriate target sets. The design of these sets depends on a logic variable, which decides which pole of $\mathcal{S}^3$ q should be steered towards.

In the following sections, we let

$$\rho = \begin{bmatrix} p \\ \epsilon \end{bmatrix}, \quad L = \begin{bmatrix} I_{3\times 3} & 0 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & 0 & I_{3\times 3} \end{bmatrix}$$

so that

$$\rho = L\begin{bmatrix} p \\ q \end{bmatrix}, \quad \dot{\rho} = L\Lambda(q)\nu, \quad \rho = 0 \Leftrightarrow (p,q) \in \mathcal{A}_k.$$

Recall that $\nu = [v^\top \quad \omega^\top]^\top$.

A. Stabilization of Kinematics

We consider the problem of stabilizing $\mathcal{A}_k$ for (5). We propose a *dynamic* feedback that depends on a logic variable $h \in \{-1,1\} =: H$. Let

$$G(q,h) = \begin{bmatrix} \mathcal{R}^\top(q) & 0 \\ 0 & hI \end{bmatrix}, \qquad K_\rho = \begin{bmatrix} K_p & 0 \\ 0 & k_\epsilon I \end{bmatrix},$$

where $K_p = K_p^\top > 0$ and $k_\varepsilon > 0$ (so that $K_\rho = K_\rho^\top > 0$). We define our velocity feedback as

$$\kappa(p,q,h) = -G(q,h)K_\rho\rho, \tag{10}$$

and let $\Phi_k(p,q) = \eta$, and $\delta \in (0,1)$. Note that for any $(q,h) \in \mathcal{S}^3 \times H$, $G(q,h)$ is orthogonal (i.e., omitting arguments, $G^\top G = GG^\top = I$). Then, we propose the hybrid control law,

$$\left.\begin{aligned} \dot{h} &= 0 \\ \nu &= \kappa(p,q,h) \end{aligned}\right\} \quad (p,q,h) \in C$$
$$h^+ = -h \qquad (p,q,h) \in D \tag{11}$$

where

$$\begin{aligned} C &= \{(p,q,h) \in \mathbb{R}^3 \times \mathcal{S}^3 \times H : h\Phi_k(p,q) \geq -\delta\} \\ D &= \{(p,q,h) \in \mathbb{R}^3 \times \mathcal{S}^3 \times H : h\Phi_k(p,q) \leq -\delta\}. \end{aligned} \tag{12}$$

Note that $C \cup D = \mathbb{R}^3 \times \mathcal{S}^3 \times H$. For compactness, we let

$$\mathcal{X} = \mathbb{R}^3 \times \mathcal{S}^3 \times H, \qquad x = (p,q,h) \in \mathcal{X}.$$

Then, with the hybrid feedback (11), closed-loop system becomes (in terms of x),

$$\begin{aligned} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{h} \end{bmatrix} &= \dot{x} = f(x) := \begin{bmatrix} \Lambda(q)\kappa(x) \\ 0 \end{bmatrix} \quad \Big\} \quad x \in C \\ \begin{bmatrix} p^+ \\ q^+ \\ h^+ \end{bmatrix} &= x^+ = g(x) := \begin{bmatrix} p \\ q \\ -h \end{bmatrix} \quad \Big\} \quad x \in D. \end{aligned} \tag{13}$$

Consider the Lyapunov function

$$V(x) = \frac{1}{2}p^\top K_p p + 2k_\varepsilon(1 - h\eta) \tag{14}$$

for analyzing the stability of the set

$$\mathcal{A} = \{x \in \mathcal{X} : p = 0,\ q = h\mathbf{1}\}.$$

Note that

$$\operatorname*{Proj}_{\mathbb{R}^3\times\mathcal{S}^1} \mathcal{A} = \mathcal{A}_k,$$

where $\operatorname{Proj}_Y X$ denotes the projection of a set X onto Y. Also note that $V(\mathcal{X} \setminus \mathcal{A}) > 0$, $V(\mathcal{A}) = 0$ and for every $\gamma \in V(\mathbb{R}^3 \times \mathcal{S}^3 \times H)$, the set $\{x \in \mathcal{X} : V(x) \leq \gamma\}$ is compact.

Recalling that for all $(q,h) \in \mathcal{S}^3 \times H$, $G(q,h)$ is orthogonal, we calculate change in V along flows as

$$\langle \nabla_x V(x), f(x) \rangle = \rho^\top K_\rho G^\top(q,h)\kappa(x) = -\rho^\top K_\rho^2 \rho,$$

Since K_ρ is a symmetric and positive definite matrix (and so is its square) it follows that $-\rho^\top K_\rho^2 \rho \leq 0$ and that

$\langle \nabla_x V(x), f(x)\rangle = 0$ if and only if $\rho = 0$. If $\rho = 0$, it follows that $p = 0$, $\eta = \pm 1$, and $\epsilon = 0$; however, we have the additional constraint that during flows ($x \in C$), $h\eta \geq -\delta > -1$, so it must follow that $\eta = h$ and so $x \in \mathcal{A}$. It follows that $\langle \nabla_x V(x), f(x)\rangle < 0$ for all $x \in C \setminus \mathcal{A}$. The change in V over jumps is

$$V(g(x)) - V(x) = 4k_\varepsilon h\eta = 4k_\varepsilon h\Phi_k(x_k).$$

When $x \in D$, $h\Phi_k(p,q) \leq -\delta$, so that $V(g(x)) - V(x) \leq -4k_\varepsilon\delta$. Hence, by [15, Corollary 7.7], $\mathcal{A}$ is globally asymptotically stable for the closed-loop system (13).

Theorem 4.1: The hybrid feedback (11), (12) *renders* $\mathcal{A}$ *globally asymptotically stable for* (13). *Moreover, there exists a class-*$\mathcal{KL}$ *function* β *such that for any* $\gamma > 0$ *and any compact set* $\mathcal{K} \subset \mathbb{R}^3$, *there exists* $\alpha > 0$ *such that for each measurable* $e = [e_p^\top \quad e_q^\top]^\top : \mathbb{R}_{\geq 0} \to \alpha\mathbb{B}$, *any solution* $x = (p,q,h)$ *to*

$$\left.\begin{aligned} &\hat{p} = p + e_p \\ &\hat{q} = q + e_q \\ &\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{h}\end{bmatrix} = \begin{bmatrix}\Lambda(q)\kappa(\hat{p},\hat{q},h)\\ 0\end{bmatrix}\end{aligned}\right\} \quad (\hat{p},\hat{q},h) \in C$$

$$\left.\begin{bmatrix}p^+\\ q^+\\ h^+\end{bmatrix} = \begin{bmatrix}p\\ q\\ -h\end{bmatrix}\quad\right\} \quad (\hat{p},\hat{q},h) \in D,$$

with initial condition $x(0,0) \in \mathcal{S}^3 \times \mathcal{K} \times H$ *satisfies*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma \quad \forall (t,j) \in \operatorname{dom} x.$$

It is important to choose $\delta \in (0,1)$. When $\delta > 0$, switching h becomes a hysteretic decision that yields a strict decrease of the Lyapunov function over jumps and provides robustness to noise. When $\delta < 1$, we avoid making the point $p = 0$, $q = -h\mathbf{1}$ an unstable equilibrium point and ensure a strict decrease in the Lyapunov function along flows. Since $\eta \in [-1,1]$, setting $\delta \geq 1$ would cause h to *never* change and induce the unwinding phenomenon [10]. Together, the logic variable h and the hysteresis half-width, δ, manage a trade-off between robustness to noise and unwinding. Finally, we note that the $\mathcal{KL}$ estimate of Theorem 4.1 applies to solutions of the perturbed system starting from initial conditions with q anywhere in $\mathcal{S}^3$. Such cannot be said for certain discontinuous control laws (see [7], [11] for examples).

One might note that the Lyapunov function appearing here uses the term $V_q(q) = 1 - h\eta$ to define an appropriate potential function. Comparing this with [1, Table 1], one can see that $V_q(q)$ resembles the first few entries. As noted in the table, the h variable is used to select which of the two points in the quaternion space to stabilize. In some sense, one can think of $h\eta$ as a generalization of $|\eta|$ (first entry of [1, Table 1]). Indeed, only for $|\eta| \leq \delta$ can one possibly have $h \neq \operatorname{sgn}(\eta)$. With this observation, one might propose other Lyapunov functions like those in [1, Table 1], but dependent on a logic variable h that selects which equilibrium point to stabilize.

B. Stabilization of Dynamics

In this section, we propose two controllers for stabilizing the set $\mathcal{A}_d$ for (5), (7). The proposed hybrid feedback will take the form

$$\begin{aligned} &\left.\begin{aligned}\dot{h} &= 0\\ \mathcal{F} &= \mathcal{F}_i(x_d,h)\end{aligned}\right\} \quad (x_d,h) \in \tilde{C}_i \\ &\quad h^+ = -h \qquad (x_d,h) \in \tilde{D}_i.\end{aligned} \tag{15}$$

Let

$$\tilde{\mathcal{X}} = \mathbb{R}^3 \times \mathcal{S}^1 \times \mathbb{R}^6 \times H, \qquad \tilde{x} = (p,q,\nu,h) \in \tilde{\mathcal{X}},$$

and $e : \mathbb{R}_{\geq 0} \to \mathbb{R}^{13}$. Then, applying the hybrid feedback (15) to (5), (7), the closed-loop system subjected to measurement noise $e = [e_p^\top \quad e_q^\top \quad e_\nu^\top]^\top$ becomes

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{f}(\tilde{x}, \mathcal{F}_i(\hat{x})) & \hat{x} \in \tilde{C}_i\\ \tilde{x}^+ &= \tilde{g}(\tilde{x}) & \hat{x} \in \tilde{D}_i.\end{aligned} \tag{16}$$

where $\hat{x} = (p + e_p, q + e_q, \nu + e_\nu, h)$ and

$$\tilde{f}(\tilde{x},\mathcal{F}) = \begin{bmatrix}\mathcal{R}(q)v\\ \frac{1}{2}q \otimes \chi(\omega)\\ \mathcal{M}^{-1}(\mathcal{F} - \xi(q) - \mathcal{C}(\nu)\nu - \mathcal{D}(\nu)\nu)\\ 0\end{bmatrix}$$

$$\tilde{g}(\tilde{x}) = \begin{bmatrix}p\\ q\\ \nu\\ -h\end{bmatrix}.$$

At this point, we will not define the flow and jump sets for our Lyapunov-based control designs, as they will depend on our choice of Lyapunov function.

1) Energy-based Lyapunov Function: Consider the Lyapunov function,

$$\tilde{V}_1(\tilde{x}) = V(x) + \frac{1}{2}\nu^\top \mathcal{M}\nu,$$

for analyzing the stability of the compact set

$$\tilde{\mathcal{A}} = \{\tilde{x} \in \tilde{\mathcal{X}} : p = 0,\ q = h\mathbf{1},\ \nu = 0\}.$$

Note that $\operatorname{Proj}_{\mathcal{X}_d}\tilde{\mathcal{A}} = \mathcal{A}_d$. Also note that $\tilde{V}_1(\tilde{\mathcal{X}} \setminus \tilde{\mathcal{A}}) > 0$, $\tilde{V}_1(\tilde{\mathcal{A}}) = 0$ and for every $\gamma \in \tilde{V}_1(\tilde{\mathcal{X}})$, the set $\{\tilde{x} \in \tilde{\mathcal{X}} : \tilde{V}_1(x) \leq \gamma\}$ is compact.

The change in $\tilde{V}_1$ along flows is

$$\begin{aligned}\left\langle \nabla_{\tilde{x}}\tilde{V}_1(\tilde{x}), \tilde{f}(\tilde{x},\mathcal{F})\right\rangle = \nu^\top(&\mathcal{F} - \mathcal{C}(\nu)\nu - \mathcal{D}(\nu)\nu - \xi(q)\\ &+ G(q,h)K_\rho\rho).\end{aligned}$$

Let

$$\mathcal{F} = \mathcal{F}_1(\tilde{x}) := \xi(q) - G(q,h)K_\rho\rho - K_\nu\nu, \tag{17}$$

where $K_\nu = K_\nu^\top \geq 0$. Recalling that $x^\top S x = 0$ for any $S \in \mathfrak{so}(3)$, it follows that

$$\left\langle \nabla_{\tilde{x}}\tilde{V}_1(\tilde{x}), \tilde{f}(\tilde{x},\mathcal{F}_1(\tilde{x}))\right\rangle = -\nu^\top(\mathcal{D}(\nu) + K_\nu)\nu.$$

Since there is no change in x_d during jumps, we find that

$$\tilde{V}_1(\tilde{g}(\tilde{x})) - \tilde{V}_1(\tilde{x}) = 4k_\varepsilon h\eta.$$

Let $\Phi_1(x_d) = \eta$ and $\delta \in (0,1)$, then we define

$$\tilde{C}_1 = \{\tilde{x} \in \tilde{\mathcal{X}} : h\Phi_1(x_d) \geq -\delta\}$$
$$\tilde{D}_1 = \{\tilde{x} \in \tilde{\mathcal{X}} : h\Phi_1(x_d) \leq -\delta\}.$$

With these definitions, it follows that

$$\begin{aligned} \left\langle \nabla_{\tilde{x}} \tilde{V}_1(\tilde{x}), \tilde{f}(\tilde{x}, \mathcal{F}_1(\tilde{x})) \right\rangle \leq 0 \quad \forall \tilde{x} \in \tilde{C}_1 \\ \tilde{V}_1(\tilde{g}(\tilde{x})) - \tilde{V}_1(\tilde{x}) < 0 \quad \forall \tilde{x} \in \tilde{D}_1. \end{aligned} \tag{18}$$

Then, it follows from [15, Theorem 7.6] that $\tilde{\mathcal{A}}$ is stable; however, we must apply an invariance principle for hybrid systems to assert the attractivity of $\tilde{\mathcal{A}}$ (and hence, global asymptotic stability). Since $\langle \nabla_{\tilde{x}} \tilde{V}_1(\tilde{x}), \tilde{f}(\tilde{x}, \mathcal{F}_1(\tilde{x}))\rangle = 0$ if and only if $\nu = 0$ and $\{\tilde{x} \in \tilde{D}_1 : \tilde{V}_1(\tilde{g}(\tilde{x})) - \tilde{V}_1(\tilde{x}) = 0\} = \emptyset$, it follows from [15, Theorem 4.7] that solutions converge to the largest invariant set contained in

$$W = \{\tilde{x} \in \tilde{\mathcal{X}} : h\eta \geq -\delta,\ \nu = 0\}.$$

Examining the closed-loop system while holding $\nu \equiv 0$, we see that $0 = G(q,h)K_\rho\rho$, which implies that $p = 0$ and $\epsilon = 0$ (i.e. $\eta = \pm 1$). Also, since $h\eta \geq -\delta > -1$, it follows that $\eta = h$ and so $q = h\mathbf{1}$. Since solutions are complete and bounded, they must then converge to $\tilde{\mathcal{A}}$.

2) Backstepping-based Lyapunov Function: In this section, we employ a backstepping procedure to construct a Lyapunov function and control law that does not require the use of an invariance principle to complete the stability proof.

Recalling that the feedback $\nu = \kappa(x) = -G(q,h)K_\rho\rho$ derived in Section IV-A resulted in a decrease in $V(x)$ along flows of (13), we, omitting arguments for readability, introduce the backstepping variable

$$z = \nu + GK_\rho\rho.$$

Let

$$\widehat{G}(q,\omega) = \dot{G}(q,\omega) = \begin{bmatrix} (\mathcal{R}(q)S(\omega))^\top & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}.$$

Then,

$$\begin{aligned} \mathcal{M}\dot{z} &= \mathcal{M}\dot{\nu} + \mathcal{M}\left(\dot{G}K_\rho\rho + GK_\rho\dot{\rho}\right) \\ &= \mathcal{M}\dot{\nu} + \mathcal{M}\left(\widehat{G}K_\rho\rho + GK_\rho L\Lambda(q)\nu\right) \\ \left\langle \nabla_{\tilde{x}} V(\tilde{x}), \tilde{f}(\tilde{x}, \mathcal{F}) \right\rangle &= -\rho^\top K_\rho^2 \rho + z^\top GK_\rho\rho. \end{aligned}$$

We then define the Lyapunov function

$$\tilde{V}_2(\tilde{x}) = V(x) + \frac{1}{2} z^\top \mathcal{M} z,$$

which satisfies $\tilde{V}_2(\tilde{\mathcal{X}} \setminus \tilde{\mathcal{A}}) > 0$ and $\tilde{V}_2(\tilde{\mathcal{A}}) = 0$. The change in $\tilde{V}_2$ along flows is given by

$$\left\langle \nabla_{\tilde{x}} \tilde{V}_2(\tilde{x}), \tilde{f}(\tilde{x}, \mathcal{F}) \right\rangle = -\rho^\top K_\rho^2\rho + z^\top\left(\mathcal{M}\dot{z} + GK_\rho\rho\right).$$

Let $\mathcal{F} = \mathcal{F}_2(\tilde{x})$, where

$$\begin{aligned} \mathcal{F}_2(\tilde{x}) = \xi(q) + \mathcal{C}(\nu)\nu + \mathcal{D}(\nu)\nu - GK_\rho\rho - K_z z \\ - \mathcal{M}\left(\widehat{G}K_\rho\rho + GK_\rho L\Lambda(q)\nu\right) \end{aligned}$$

and $K_z = K_z^\top > 0$. Then,

$$\left\langle \nabla_{\tilde{x}} \tilde{V}_2(\tilde{x}), \tilde{f}(\tilde{x}, \mathcal{F}_2(\tilde{x})) \right\rangle = -\rho^\top K_\rho^2 \rho - z^\top K_z z.$$

Letting $G^+ = G(q, -h)$, the change in $\tilde{V}_2$ along jumps is

$$\begin{aligned} \tilde{V}_2(\tilde{g}(\tilde{x})) - \tilde{V}_2(\tilde{x}) = 4k_\varepsilon h\eta + \nu^\top\mathcal{M}(G^+ - G)K_\rho\rho \\ + \frac{1}{2}\rho^\top K_\rho\left(G^{+T}\mathcal{M}G^+ - G^\top\mathcal{M}G\right)K_\rho\rho. \end{aligned} \tag{19}$$

Let

$$\tilde{I} = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & I \end{bmatrix} \quad \mathcal{M} = \begin{bmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_2^\top & \mathcal{M}_3 \end{bmatrix}$$

and

$$\Gamma(q) = \begin{bmatrix} 0 & \mathcal{R}(q)\mathcal{M}_2 \\ \mathcal{M}_2^\top\mathcal{R}^\top(q) & 0 \end{bmatrix}.$$

Then,

$$G^+ - G = -2h\tilde{I} \tag{20}$$
$$G^{+\top}\mathcal{M}G - G^\top\mathcal{M}G = -2h\Gamma(q). \tag{21}$$

Let

$$\Phi_2(x_d) = \eta - \frac{1}{2k_\varepsilon}\nu^\top\mathcal{M}\tilde{I}K_\rho\rho - \frac{1}{4k_\varepsilon}\rho^\top K_\rho\Gamma(q)K_\rho\rho^\top.$$

Then, from (19), (20), and (21), it follows that

$$\tilde{V}_2(\tilde{g}(\tilde{x})) - \tilde{V}_2(\tilde{x}) = 4k_\varepsilon h\Phi_2(x_d).$$

By defining flow and jump sets as

$$\tilde{C}_2 = \{\tilde{x} \in \tilde{\mathcal{X}} : h\Phi_2(x_d) \geq -\delta\}$$
$$\tilde{D}_2 = \{\tilde{x} \in \tilde{\mathcal{X}} : h\Phi_2(x_d) \leq -\delta\},$$

it follows that $\tilde{V}_2(\tilde{g}(\tilde{x})) - \tilde{V}_2(\tilde{x}) \leq -4k_\varepsilon\delta$ when $\tilde{x} \in \tilde{D}_2$. Since $\delta < 1$, it follows that $\tilde{V}_2$ is strictly decreasing along flows for all $\tilde{x} \in \tilde{C}_2 \setminus \tilde{\mathcal{A}}$ and By [15, Corollary 7.7], $\tilde{\mathcal{A}}$ is globally asymptotically stable.

Theorem 4.2: For each $i \in 1, 2$, applying the hybrid feedback (15) *to* (5), (7) *with $e = 0$ renders $\tilde{\mathcal{A}}$ globally asymptotically stable for the closed-loop system* (16)*. Moreover, there exists a class-$\mathcal{KL}$ function β_i such that for every $\gamma > 0$ and any compact set $\mathcal{K} \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, there exists $\alpha_i > 0$ such that for each measurable $e : \mathbb{R}_{\geq 0} \to \alpha_i\mathbb{B}$, solutions to* (16) *starting from $(q_0, p_0, v_0, \omega_0, h_0) \in \mathcal{S}^3 \times \mathcal{K} \times H$ satisfy*

$$|\tilde{x}(t,j)|_{\tilde{\mathcal{A}}} \leq \beta_i\left(|\tilde{x}(0,0)|_{\tilde{\mathcal{A}}}, t + j\right) + \gamma \quad \forall (t,j) \in \operatorname{dom}\tilde{x}.$$

V. Simulation Study

In this section, we present a brief simulation study of the non-backstepping control law derived in Section IV-B.1. As in [1],

$$\begin{aligned} \mathcal{M} &= \operatorname{diag}(215, 265, 265, 40, 80, 80) \\ \mathcal{D}(\nu) &= \operatorname{diag}(70, 100, 100, 30, 50, 50) \\ &\quad + \operatorname{diag}(100, 200, 200, 50, 100, 100)\operatorname{diag}(|\nu|), \end{aligned}$$

where $|\nu| = (|\nu|_1, \cdots, |\nu|_6)$. $\mathcal{C}(\nu)$ is calculated as

$$\mathcal{C}(\nu) = \begin{bmatrix} 0_{3\times3} & -S(\mathcal{M}_1 v + \mathcal{M}_2\omega) \\ -S(\mathcal{M}_1 v + \mathcal{M}_2\omega) & -S(\mathcal{M}_3\omega + \mathcal{M}_2^\top) \end{bmatrix}$$

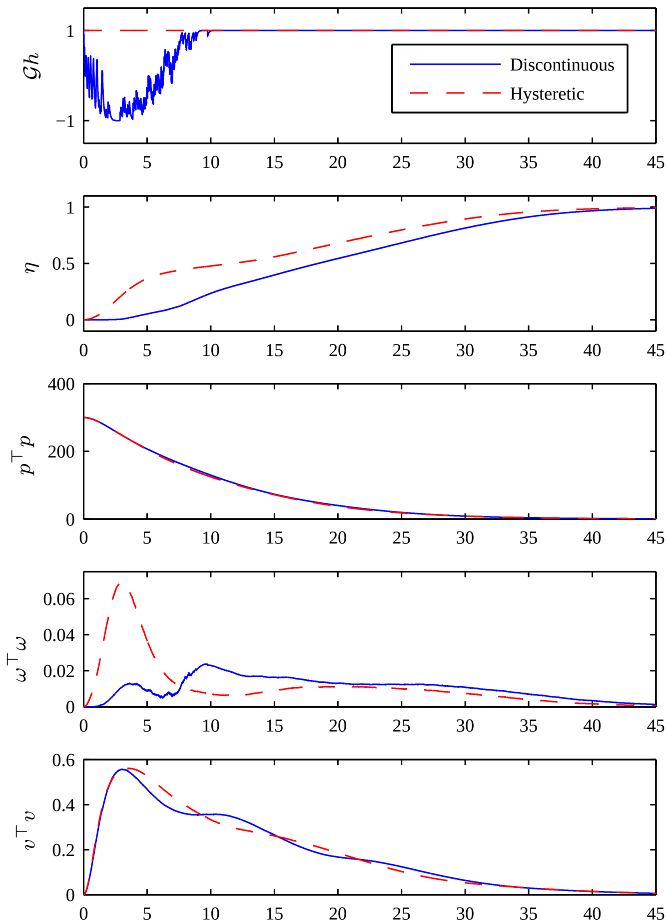

Fig. 1. A comparison between discontinuous and hysteretic control with initial conditions close to the discontinuity. The discontinuous control exhibits noise-induced chattering on the discontinuity and creates a lag in the angular response. On the other hand, the hysteretic hybrid feedback is impervious to the bounded noise.

In this simulation, the disturbance torque $\xi(q)$ is assumed to be zero.

In this simulation, we compare the hysteretic feedback with $\delta = 0.25$ to the discontinuous feedback ($\delta = 0$). The control gains are selected as $K_p = 10I$, $k_\varepsilon = 10$, and $K_\nu = 0$. Initial conditions are selected as $q(0,0) = (0, x/\|x\|)$, where $x^\top = \begin{bmatrix}3 & -4 & 5\end{bmatrix}$, $p^\top(0,0) = \begin{bmatrix}10 & 10 & 10\end{bmatrix}$, $\nu(0,0) = 0$, and $h(0,0) = 1$.

To demonstrate how discontinuous control is susceptible to noise, we inflicted measurement noise only upon the q state in the following way. Let q_{meas} denote the measurement of q. Then, $q_{\text{meas}} = (q + e_1 + e_2(q))/\|q + e_1 + e_2(q)\|$, where the direction of e_1 is selected from a normal distribution and the magnitude is selected from a uniform distribution and bounded by 0.16. The noise $e_2(q) = \begin{bmatrix}-0.08\,\mathrm{sgn}(\eta) & 0^\top\end{bmatrix}^\top$ depends on q and is designed to confuse the control law. In this setting, the randomly generated noise can have twice the magnitude of the adversarial noise, e_2.

Fig. 1 shows how the discontinuous control can chatter about the discontinuity when measurement noise is present. This creates a response lag and wastes control energy. Due to the immense amount of chattering, Fig. 1 shows a filtered version of h, $\mathcal{G}h$, where $\mathcal{G} = 10/(s+10)$. The filter was given an initial condition of 1 – the same as the initial condition on h. The hysteretic hybrid feedback ignores this noise and responds to the error signal immediately. While this particular noise profile only delayed the discontinuous control response for approximately 10 seconds, there exists a noise profile that keeps the attitude 180° away from the desired attitude for all time (see [7]).

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