Dynamical Properties of Hybrid Systems Simulators

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Abstract

This paper analyzes the dynamical properties of a class of hybrid systems simulators. A hybrid system is a dynamical system with a state that can both flow and jump. Its simulator attempts to generate its solutions approximately. The paper presents mild regularity conditions on the hybrid system and its simulator to guarantee that simulated solutions are close to actual solutions on compact (hybrid) time intervals, and that asymptotically stable compact sets are preserved, in a semiglobal practical sense, under simulation. In fact, it is established that asymptotically stable compact sets are continuous in the integration step size parameter of the simulator; that is, as the step size of the simulator converges to zero, the asymptotically stable set observed in simulations approaches the asymptotically stable compact set of the true hybrid system. Examples are used to illustrate concepts and results.

Key words: numerical simulation, asymptotic stability, hybrid systems.

1 Introduction

The theory of numerical simulation for differential equations is well developed and several textbooks on the subject are available, including [32] and [3]. The properties of integration schemes for differential equations are often studied by treating them as dynamical systems. With this approach, one-step schemes (like Euler and Runge-Kutta methods), multi-step algorithms (like Adams method and backward differentiation), and their variable-step versions are shown to produce solutions that are close to the ideal solutions, on compact time intervals, for sufficiently small integration step size. This enables simulators to approximately reproduce the asymptotic stability properties, which are infinite time horizon attributes, of the true system. Results of this type for differential equations and inclusions can be found in the numerical analysis literature; see, e.g., [32], [10], and the references therein.

Over the last few decades, researchers have developed tools for analysis, design, and control of dynamical systems with both continuous and discrete dynamics, that is, *hybrid dynamical systems*. Among several studies on hybrid systems, including topics like stability, reachability, and robustness, numerous software tools for numerical simulation of hybrid systems have been recently developed. These include Matlab/Simulink, Modelica [11], Ptolemy [22], Charon [2], HYSDEL [34], and HyVisual [21]. More recently, special attention has been given to semantics for description and simulation of hybrid systems [19, 27, 21, 31]; event detection algorithms [23, 12, 20]; and solvers and error control [12, 8, 20, 1]. To the best of our knowledge, theoretical studies about the structural properties of simulators for hybrid systems that parallel those in the literature of simulation of differential and difference equations are not available.

In this paper, we treat simulations of hybrid systems as dynamical systems and provide a mathematical framework for the development of a theory of numerical simulations for hybrid systems. We follow the framework for hybrid systems introduced in [14], and further developed in [16] and [15], where the continuous dynamics or *flows* are given by a differential equation (or inclusion) and the discrete dynamics or *jumps* are given by a difference equation (or inclusion). Flows and jumps are only permitted on certain subsets of the state space. The right-hand side of the differential and difference equation/inclusion are called the *flow map* and *jump map*, respectively, while the subsets of the state space on which flows and jumps are allowed are called the *flow set* and

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jump set, respectively. In our context, the term *simulated solution* refers to the trajectory (or run) obtained from a discretization of the dynamics of a hybrid system. We establish conditions for the entity generating simulated solutions, which we call *hybrid simulator*, guaranteeing the following:

(1) On compact hybrid time domains, each simulated solution is close to some solution of the true hybrid system.
(2) Asymptotically stable compact sets for a hybrid system are semiglobally practically (in the integration step size) asymptotically stable compact sets for the hybrid simulator.
(3) Asymptotically stable compact sets for the simulator are continuous in the integration step size.

Item (1) is a fundamental upper semicontinuity property with respect to perturbations introduced when simulating hybrid systems. In fact, our proof relies on the property established in [16] stating that, under certain basic conditions, which we also use in the current work, for each solution to a perturbed hybrid system there exists a solution to the unperturbed system that is close to it. The converse of item (1), that is, the property of every solution to the true hybrid system being close to some simulated solution, relies on the converse of the said property asserted in [16]. Such a property has been studied for differential inclusions (see, e.g., [13, 4]), and more recently, for hybrid systems [6], where extra assumptions beyond the basic conditions of [16] were shown to be required. We will not address this property here.

The paper is organized as follows. In Section 2, we describe the hybrid systems framework considered. In Section 3, we present results on closeness of solutions and robust stability of compact sets for hybrid systems. Our main results build on these preliminaries. Section 4 introduces a framework for simulations of hybrid systems. We present results (1)-(3) in Section 5.

1.1 Notation

We use the following notation and definitions. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{B}$ denotes the closed unit ball in a Euclidean space. Given a set S, $\overline{S}$ denotes its closure. Given a set S, $\mathrm{con}\, S$ denotes the convex hull and $\overline{\mathrm{con}} S$ the closure of the convex hull. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$. Given sets $S_1, S_2 \subset \mathbb{R}^n$, $d_H(S_1, S_2)$ denotes the Hausdorff distance between S_1 and S_2, that is, $d_H(S_1, S_2) = \max\{\sup_{x \in S_1} |x|_{S_2}, \sup_{x \in S_2} |x|_{S_1}\}$. Given sets S_1, S_2 subsets of $\mathbb{R}^n$, $S_1 + S_2 := \{x_1 + x_2 \mid x_1 \in S_1, x_2 \in S_2\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if it is continuous, nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \searrow 0} \beta(s, r) = \lim_{r \to \infty} \beta(s, r) = 0$.

2 Hybrid Systems: Data and Solutions

Hybrid systems are dynamical systems with continuous and discrete dynamics. Several mathematical models for hybrid systems have appeared in the literature. These include the work of Tavernini [33], Michel and Hu [26], Lygeros et al. [24], Aubin et al. [5], among many others. In this paper, we consider the framework for hybrid systems outlined in [14], and further investigated in [16, 15], where a hybrid system $\mathcal{H}$ on a state space $\mathbb{R}^n$ is defined by the following objects:

- A set $C \subset \mathbb{R}^n$ called the *flow set*.
- A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ called the *flow map*.
- A set $D \subset \mathbb{R}^n$ called the *jump set*.
- A set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ called the *jump map*.

The flow map F defines the continuous dynamics on the flow set C, while the jump map G defines the discrete dynamics on the jump set D. These objects are referred to as the *data* of the hybrid system $\mathcal{H}$, which at times is explicitly denoted as $\mathcal{H} = (C, F, D, G)$.

The union of the flow and jump sets does not need to cover $\mathbb{R}^n$. For example, we may have $C \cup D = S \times Q \subset \mathbb{R}^n$ for some sets S and Q, where Q is some discrete subset of a Euclidean space. In this situation, the state of the hybrid system $\mathcal{H}$, denoted by x, would comprise both real-valued continuous states, perhaps denoted ξ, and integer-valued discrete (or logic) states, perhaps denoted q, i.e., $x = [\xi^\top,\ q^\top]^\top$. Example 2.6 and Example 2.7 deal with hybrid systems of this type.

The set-valued mappings F and G allow for the possibility of discontinuous flow and jump maps, which, after a regularization procedure like those in [30], become set valued. They also permit explicit modeling of perturbations in the system dynamics, a useful feature for robust stability analysis of dynamical systems in general and for the analysis of simulators in particular.

Hybrid systems $\mathcal{H} = (C, F, D, G)$ can be written as

$$\mathcal{H}: \qquad x \in \mathbb{R}^n \qquad \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{1}$$

Solutions to $\mathcal{H}$ are given on extended time domains, called *hybrid time domains*, by functions that satisfy the conditions suggested by (1) and are called *hybrid arcs*.

Definition 2.1 (hybrid time domain) *A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if*

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Hybrid time domains are similar to hybrid time trajectories in [24] and [5] but give a more prominent role to the number of jumps j (cf. hybrid time sets in [9]).

Definition 2.2 (hybrid arc) *A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a hybrid arc if $\operatorname{dom} x$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is absolutely continuous on the interval $I_j := \{t \mid (t, j) \in \operatorname{dom} x\}$.*

Hybrid arcs, and solutions to $\mathcal{H}$ in particular, are parameterized by pairs (t, j), where t is the ordinary-time component and j is the discrete-time component that keeps track of the number of jumps.

Definition 2.3 (solution) *A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ is a solution to the hybrid system $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and*

(S1) for each $j \in \mathbb{N}$ such that I_j has nonempty interior $int(I_j)$, $x(t, j) \in C$ for all $t \in int(I_j)$ and, for almost all $t \in I_j$, $\dot{x}(t, j) \in F(x(t, j))$;

(S2) for each $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)).$$

A hybrid arc x is said to be *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *maximal* if there does not exist a solution x' such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded. A hybrid system $\mathcal{H}$ is said to be *complete* (also called *forward complete* in [16] and [7]) if every maximal solution to $\mathcal{H}$ is complete. Note that, during flows, the absolute continuity property of the function $t \mapsto x(t, j)$ for each $j \in \mathbb{N}$ permits obtaining the value of the solution during flows from its derivative.

Next, we illustrate modeling a hybrid system as in (1).

Example 2.4 (bouncing ball) Consider a ball bouncing on the ground with relative height x_1 and vertical velocity x_2. The dynamics between bounces are modeled as $\dot{x}_1 = x_2, \dot{x}_2 = -\gamma$, when $x_1 \geq 0$, where $\gamma > 0$ is the gravity constant. The bouncing condition is $x_1 = 0$ and $x_2 \leq 0$ and the jump map is $x_1^+ = 0,\ x_2^+ = -\varrho x_2$, where $\varrho \in [0, 1)$ is the restitution coefficient. Then, the bouncing ball is a hybrid system, denoted $\mathcal{H}^{BB}$, with $F(x) = [x_2,\ -\gamma]^\top$, $C = \{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$, $G(x) = [0,\ -\varrho x_2]^\top$, and $D = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \leq 0\}$. Solutions to $\mathcal{H}^{BB}$ are unique and Zeno. △

Solutions to $\mathcal{H}$ may not be unique, not only due to the flow and jump maps being set valued but also due to having overlapping flow and jump sets. Indeed, there may exist points in $C \cap D$ from which it is possible both to flow and to jump.

Note that, in principle, various "semantics" can be enforced by modifying the data of the hybrid system. For example, *forcing* or *triggering semantics* [31] for the hybrid system (C, F, D, G) result by considering solutions to the hybrid system with data $(C \setminus D, F, D, G)$. However, the assumptions we impose next on the data of a hybrid system may not hold for the data $(C \setminus D, F, D, G)$, and in turn, as we discuss below, useful structural and robustness properties of the set of solutions to $\mathcal{H}$ may not be present. Alternatively, robustness of behavior under forcing semantics can be assessed by considering the system with data $(\overline{C \setminus D}, F, \overline{D}, G)$ at the price of potentially adding new solutions.

Assumption 2.5 (hybrid basic conditions [16]) *The data (C, F, D, G) of a hybrid system $\mathcal{H}$ satisfies:*

(A1) C and D are closed sets.
(A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, [1] *and $F(x)$ is nonempty and convex for all $x \in C$.*
(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $G(x)$ is a nonempty subset of $\mathbb{R}^n$ for all $x \in D$.

Under Assumption 2.5, given a bounded sequence of solutions to $\mathcal{H}$, there exists a subsequence that converges to a solution to $\mathcal{H}$ [16, Theorem 4.4]. Moreover, the set of solutions to a hybrid system $\mathcal{H}$ satisfying Assumption 2.5 can be shown to be equal to the set of solutions to it under the presence of perturbations with size converging to zero; see [30] for details.

Additional consequences of Assumption 2.5 that are relevant for analyzing simulators will be recalled in Section 3. Among other things, we desire for each simulated solution starting near a given point to be close, on a compact time domain, to some true solution emanating from that point. Presumably, the true system itself is the best possible simulator of its own solutions; thus, the stated

[1] A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for each sequence $\{x_i\}_{i=1}^{\infty}$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in F(x_i)$ converging to a point y, it holds that $y \in F(x)$; see [28, Definition 5.4]. (For locally bounded set-valued maps with closed values, outer semicontinuity coincides with what is usually called *upper semicontinuity*.) It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists $K' \subset \mathbb{R}^n$ such that $F(K) := \cup_{x \in K} F(x) \subset K'$.

property should also hold for solutions of the true system. Without Assumption 2.5, such a property may not hold, as the following example illustrates.

Example 2.6 (obstacle avoidance) The hybrid system in this example corresponds to an obstacle avoidance algorithm for a planar vehicle modeled as a fully-actuated point mass. We denote it as $\mathcal{H}^A$ and its state as $x = [\xi^\top, q]^\top \in \mathbb{R}^3$. The flow map is given by $F(x) = [1 - |q|,\ q,\ 0]^\top$. The flow and jump sets are given by

$$C = \bigcup_{q\in\{-1,0,1\}} (C_q \times \{q\}), \quad D = \bigcup_{q\in\{-1,0,1\}} (D_q \times \{q\}),$$

where $C_0 = \overline{([-3,3]\times[-3,3])\setminus\mathbb{B}}$, $C_1 = 2\mathbb{B}\cap(\mathbb{R}\times\mathbb{R}_{\geq -0.5})$, $C_{-1} = 2\mathbb{B}\cap(\mathbb{R}\times\mathbb{R}_{\leq 0.5})$, D_1 is the boundary of C_1, D_{-1} is the boundary of C_{-1} and $D_0 = S^1$, i.e., the unit circle in the plane, which includes the inner (but not the outer) boundary of C_0; see Figure 1. The jump map is $G(x) = [\xi^\top,\ \psi_q(\xi)]^\top$, where, for $q \in \{-1,0,1\}$, $\psi_q : D_q \rightrightarrows \{-1,0,1\}\setminus\{q\}$ is defined as follows: $\psi_0(\xi)$ equals the sign of ξ_2 unless $\xi_2 = 0$, in which case $\psi_0(\xi) = \{-1,1\}$; for $q\in\{-1,1\}$, $\psi_q(\xi) = \psi^-(\xi,q)\cup\psi^\circ(\xi)$, where $\psi^-(\xi,q) = \{-q\}$ when $|\xi_2| = 0.5$, $\psi^\circ(\xi) = \{0\}$ when $\xi \in 2S^1$, and ψ^- and ψ° are empty otherwise. From the initial condition $x^0 = [-3, 1+\varepsilon, 0]^\top$, we have

a) for $\varepsilon \in [0,2]$ there is a solution, with time domain $[0,6]\times\{0\}$, given by $x(t,j) = [-3+t, 1+\varepsilon, 0]^\top$ for all $(t,j)\in \operatorname{dom} x$;
b) for $\varepsilon\in(-0.5,0]$ there is a solution, with time domain $\cup_{j=0}^{2}([t_j,t_{j+1}]\times\{j\})$ with $0 = t_0 < t_1 < t_2 < t_3$, $x(t,j) = [-3+t, 1+\varepsilon, 0]^\top$ for all $(t,j)\in\operatorname{dom} x$ with $j = 0$, $x(t,j) = [-3+t_1, 1+\varepsilon+(t-t_1), 1]^\top$ for all $(t,j)\in\operatorname{dom} x$ with $j=1$, and $x(t,j) = [-3+t_1+t-t_2, 1+\varepsilon+t_2-t_1, 0]^\top$ for all $(t,j)\in\operatorname{dom} x$ with $j=2$.

When $\varepsilon = 0$, there are two solutions. This is caused by the "grazing" of the flowing solution at the boundary of D, which is allowed in the framework we consider. These solutions are depicted in Figure 1. For more on the grazing phenomenon, see, e.g., [9, 1]. Despite the lack of uniqueness for $\varepsilon = 0$, it can be said that each solution with small $\varepsilon \neq 0$ is close to some solution with $\varepsilon = 0$. More generally, it can be said that, for each $\varepsilon^*\in\mathbb{R}$, each solution with ε near ε^* is close to some solution with $\varepsilon = \varepsilon^*$.

Uniqueness of solutions for $\varepsilon = 0$ can be obtained by modifying the data as suggested above as a way to impose forcing semantics, that is, by removing D from C and obtaining a hybrid system with data $(C\setminus D, F, D, G)$. This amounts to removing the set S^1 from C_0 as it forces a jump when ξ reaches D_0. However, in this case, the closeness property described above can no longer be asserted. In particular, for all $\varepsilon > 0$, the solution will experience no jumps, but there are no such solutions for $\varepsilon = 0$. △

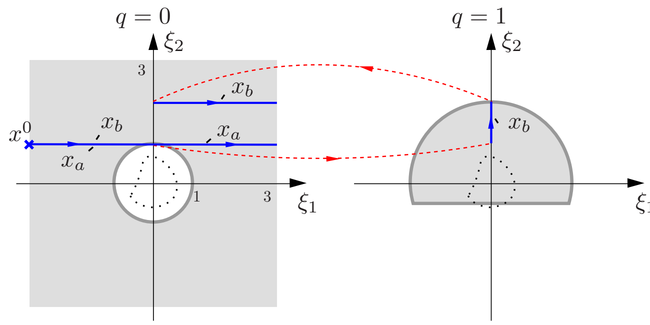


Fig. 1. Sets and ξ component of solutions in a) and b) to the system in Example 2.6. The obstacle is denoted with dotted line and is contained by the set D_0 given by S^1, which is the circle of unitary radius as the left plot shows. The set C_0 is depicted in gray in the left plot. The (truncated) circle of radius 2 in the right plot corresponds to the set C_1. Its boundary defines D_1. Sets C_{-1} and D_{-1} are given by vertical mirror of C_1 and D_1, respectively.

Beyond the properties illustrated in the previous example, we also do not want small inflations of the flow and jump sets to create new solutions with features that are dramatically different from the solutions to the original system. In the context of simulators, this is desired since a simulator approximates continuous change by discontinuous change through an integration step; thus, in a sense, a simulator inflates the flow set to include a continuum of points connecting the state value before an integration step to the state value after an integration step. The next example illustrates issues related to set inflation.

Example 2.7 (zero-crossing detection) Consider the task of driving a mobile robot, with position vector $\xi\in\mathbb{R}^2$, around a circle n times in the clockwise direction and then stopping. Rotations are to be counted by crossings of the positive ξ_2-axis. The count is stored in the variable ℓ, which remains constant during flows. The equations of motion for the robot are taken to be $\dot{\xi}_1 = \xi_2$, $\dot{\xi}_2 = -\xi_1$, and flows are restricted to the disk $\mathcal{X} := \{\xi\in\mathbb{R}^2 \mid |\xi|\in[1,2]\}$.

One possibility for the counting hybrid system is to take D to be the set where $\ell\in\{0,\ldots,n-1\}$ and ξ belongs to the disk $\mathcal{X}$ and the positive ξ_2-axis, since this is the set of points where the counter ℓ should be incremented. Then, C can be taken to be the complement of D in $\mathcal{X}\times\{0,\ldots,n-1\}$. This set C is not closed. Moreover, when C is replaced by its closure, which is an infinitesimally small inflation, we create solutions that always flow and never jump, so that the counter is never incremented. Such solutions do not appear with the original flow set C. However, they typically would appear with a simulator that does not use some type of zero-crossing detection algorithm since the set of points where jumps are allowed is a set of measure zero *and* flowing is allowed at any point not in the jump set.

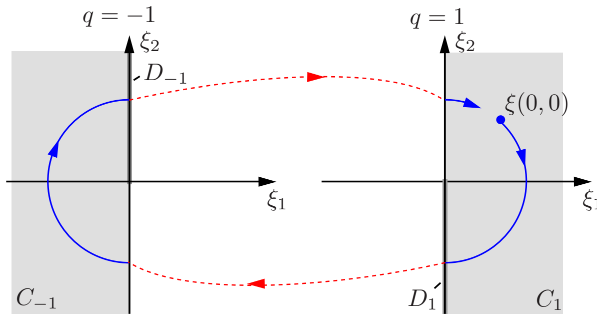


Fig. 2. Component ξ and q of a solution to and sets of $\mathcal{H}^{ZCD}$ in Example 2.7.

Despite this issue, the task can still be accomplished with a hybrid system, which we denote $\mathcal{H}^{ZCD}$, satisfying Assumption 2.5. We introduce an extra variable to define a zero-crossing detection algorithm. The extra variable is denoted q. Taking values in the two-point set $\{-1,1\}$, it is used to indicate whether ξ is to the left or to the right of the ξ_2-axis. We define $x=[\xi^\top,\ q,\ \ell]^\top$, $C_1:=\mathbb{R}_{\geq 0}\times\mathbb{R}$, $C_{-1}:=\mathbb{R}_{\leq 0}\times\mathbb{R}$, $D_1:=\{0\}\times\mathbb{R}_{\leq 0}$, $D_{-1}:=\{0\}\times\mathbb{R}_{\geq 0}$, $C := \cup_{q\in\{-1,1\}}\left((C_q\cap\mathcal{X})\times\{q\}\times\{0,1,\ldots,n-1\}\right)$, $D := \cup_{q\in\{-1,1\}}\left((D_q\cap\mathcal{X})\times\{q\}\times\{0,1,\ldots,n-1\}\right)$, and $G(x) := [\xi^\top,\ -q,\ \ell+(1-q)/2]^\top$. This hybrid system accomplishes the counting task while using data that satisfies Assumption 2.5. Figure 2 depicts a solution to $\mathcal{H}^{ZCD}$. Moreover, small inflations of C and D do not lead to solutions that never jump. Thus, based on upcoming results, the simulated solutions will either carry out the expected number of jumps or else terminate prematurely. The latter behavior can be ruled out by using a jump set in the simulator that is a slight inflation of the original D set. For an illustration of a "minimal" inflation, see Example 4.4. $\triangle$

3 Robustness results for hybrid systems

For hybrid systems with data satisfying Assumption 2.5, we recall some robustness results that are relevant for analyzing hybrid systems simulators. We consider results both on compact time domains and, under a stability assumption, on unbounded time domains. We start with the definition of the stability concept used.

A solution x to $\mathcal{H}$ is said to be bounded if there exists a compact set $K\subset\mathbb{R}^n$ such that $\{x(t,j)\mid (t,j)\in\operatorname{dom} x\}$ is contained in K. A compact set $\mathcal{A}\subset\mathbb{R}^n$ is said to be:

- *stable* if for each $\varepsilon>0$ there exists $\delta>0$ such that each solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}}\leq\delta$ satisfies $|x(t,j)|_{\mathcal{A}}\leq\varepsilon$ for all $(t,j)\in\operatorname{dom} x$;
- *globally pre-attractive* if every solution x to $\mathcal{H}$ with $x(0,0)\in C\cup D$ is bounded and if it is complete satisfies $\lim_{t+j\to\infty}|x(t,j)|_{\mathcal{A}}=0$;
- *globally pre-asymptotically stable* if stable and globally pre-attractive.

The prefix "pre" indicates that the concepts do not assume that every maximal solution is complete, that is, it is not required for $\mathcal{H}$ to be complete. In fact, in general, solutions to hybrid systems are not necessarily complete: a solution can reach a point from where neither flowing nor jumping is possible, that is, they are *blocking* [25]. When completeness of $\mathcal{H}$ is present, the prefix "pre" can be dropped and the classical stability and attractivity notions are recovered. For general results about asymptotically stable compact sets for hybrid systems $\mathcal{H}$ see [16, Section VI] and [29, Section VI and VII].

For the system in Example 2.6, the compact set $\mathcal{A} := (C\cup D)\cap\{x=(\xi,q)\mid \xi_1=3\ ,\ q=0\}$ is globally pre-asymptotically stable, and all solutions reach $\mathcal{A}$ in finite hybrid time. For the system in Example 2.7, the compact set $\mathcal{A} := (D_{-1}\cap\mathcal{X})\times\{1\}\times\{n\}$ is globally pre-asymptotically stable, and all solutions reach $\mathcal{A}$ in finite hybrid time. For the bouncing ball system in Example 2.4, invariance principles for hybrid systems, like the ones in [29], or the strict Lyapunov function suggested in [7], can be used to establish that the origin is globally asymptotically stable.

By augmenting $\mathcal{H}$ as in [7, Lemma 7.12] to a hybrid system for which $\mathcal{A}$ is globally asymptotically stable, the following $\mathcal{KL}$-stability result follows from [7, Lemma 7.11] and [7, Proposition 7.3]. It is a generalization of [16, Theorem 6.5] to the case of globally pre-asymptotically stable sets.

Theorem 3.1 ($\mathcal{KL}$-bound) *Assume that $\mathcal{H}$ satisfies Assumption 2.5. Let $\mathcal{A}$ be a globally pre-asymptotically stable compact set. There exists $\beta\in\mathcal{KL}$ s.t. for each solution x to $\mathcal{H}$*

$$|x(t,j)|_{\mathcal{A}}\leq\beta(|x(0,0)|_{\mathcal{A}},t+j)\qquad\forall(t,j)\in\operatorname{dom} x.\quad(2)$$

A result on how the bound (2) holds up under perturbations (see Theorem 3.5) will be relevant for statements about how stability holds up in simulators.

In the absence of stability, we discuss the closeness of perturbed solutions (perhaps coming from a simulator) to unperturbed solutions. As in [16], the distance between hybrid arcs (and solutions) is quantified by the distance between their graphs.

Definition 3.2 ((T,J,ε)-closeness) *Given $T,J\geq 0$ and $\varepsilon>0$, two hybrid arcs $x_1:\operatorname{dom} x_1\to\mathbb{R}^n$ and $x_2:\operatorname{dom} x_2\to\mathbb{R}^n$ are (T,J,ε)-close if*

(a) for all $(t,j)\in\operatorname{dom} x_1$ with $t\leq T, j\leq J$ there exists s such that $(s,j)\in\operatorname{dom} x_2$, $|t-s|<\varepsilon$, and

$$|x_1(t,j)-x_2(s,j)|<\varepsilon,$$

(b) for all $(t,j)\in\operatorname{dom} x_2$ with $t\leq T, j\leq J$ there exists s such that $(s,j)\in\operatorname{dom} x_1$, $|t-s|<\varepsilon$, and

$$|x_2(t,j) - x_1(s,j)| < \varepsilon.$$

The types of perturbations we consider are detailed next. Given a sequence of (set-valued) mappings $\{F_i\}_{i=1}^{\infty}$, its outer graphical limit is the mapping F_0 such that $\operatorname{gph} F_0 = \lim_{i\to\infty} \operatorname{gph} F_i$; for details, see [28, Chapter 4].

Definition 3.3 (convergence property (CP) [16, Section 5]) *Given a hybrid system $\mathcal{H} = (C, F, D, G)$, a family of hybrid systems $\mathcal{H}_\delta$ with data $(C_\delta, F_\delta, D_\delta, G_\delta)$ is a perturbation of $\mathcal{H}$, with perturbation parameter $\delta > 0$, satisfying the convergence property (CP) if for any sequence $1 > \delta_1 > \delta_2 > \cdots > 0$ converging to 0, the sequences $\{F_i\}_{i=1}^{\infty}$, $\{C_i\}_{i=1}^{\infty}$, $\{G_i\}_{i=1}^{\infty}$, and $\{D_i\}_{i=1}^{\infty}$, where for each $i \in \{1, 2, \ldots\}$, $F_i := F_{\delta_i}$, $C_i := C_{\delta_i}$, $G_i := G_{\delta_i}$, and $D_i := D_{\delta_i}$, are such that the sequences $\{F_i\}_{i=1}^{\infty}$, $\{G_i\}_{i=1}^{\infty}$ are locally eventually bounded [2] and*

$$\limsup_{i\to\infty} C_i \subset C, \qquad \limsup_{i\to\infty} D_i \subset D \tag{3}$$

$$F_0(x) \subset F(x), \qquad G_0(x) \subset G(x) \qquad \forall x \in \mathbb{R}^n, \tag{4}$$

where F_0 and G_0 denote the outer graphical limits of $\{F_i\}_{i=1}^{\infty}$ and $\{G_i\}_{i=1}^{\infty}$, respectively, at the given $x \in \mathbb{R}^n$.

As pointed out in [16], in particular, the conditions in (3) hold when the sequences $\{C_i\}_{i=1}^{\infty}, \{D_i\}_{i=1}^{\infty}$ converge, the flow set C is equal to $\lim_{i\to\infty} C_i$, and the jump set D is equal to $\lim_{i\to\infty} D_i$. Analogous conditions on the sequences $\{F_i\}_{i=1}^{\infty}, \{G_i\}_{i=1}^{\infty}$ and F, G guarantee condition (4). Moreover, under the assumption that F_i is convex-valued and F is locally bounded, the condition on F in (4) implies that $\{F_i\}_{i=1}^{\infty}$ is locally eventually bounded.

The following result on closeness between solutions to unperturbed and perturbed hybrid systems is an extension of [16, Corollary 5.5] as it relaxes the completeness assumption therein to the "pre" case. It follows from [16, Corollary 5.5] after embedding the original hybrid system into an extended one with state and dynamics such that each solution is complete. (This is possible by appropriately allowing jumps from every point in $\mathbb{R}^n$.) Below, a hybrid system $\mathcal{H}$ is said to be *pre-complete* from K if each maximal solution starting from K is either bounded or complete.

Theorem 3.4 (closeness on compact domains) *Assume that $\mathcal{H}$ satisfies Assumption 2.5 and that its perturbation given by the family of hybrid systems $\mathcal{H}_\delta$ satisfies (CP). Let $K \subset \mathbb{R}^n$ be a compact set such that $\mathcal{H}$ is pre-complete from K. Then, for every $\varepsilon > 0$ and every $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $\delta^* > 0$ with the following property: for any $\delta \in (0, \delta^*]$ and any solution x_δ to $\mathcal{H}_\delta$ with $x_\delta(0,0) \in K + \delta\mathbb{B}$ there exists a solution x to $\mathcal{H}$ with $x(0,0) \in K$ such that x_δ and x are (T, J, ε)-close.*

[2] A sequence of mappings $\{F_i\}_{i=1}^{\infty}$ is locally eventually bounded if for any compact set $K \subset \mathbb{R}^n$ there exists $m > 0$ and $i_0 > 0$ such that for any $i > i_0$, $F_i(K) \subset m\mathbb{B}$.

The following $\mathcal{KL}$-stability result under perturbation generalizes [16, Theorem 6.6] to the case of globally pre-asymptotically stable sets. The proof technique uses ideas from [7, Lemma 7.12] and relies on [16, Theorem 6.6]. Its proof can be derived using [16, Theorem 6.6].

Theorem 3.5 ($\mathcal{KL}$-bound under perturbations) *Assume that $\mathcal{H}$ satisfies Assumption 2.5. Let $\mathcal{A}$ be a globally pre-asymptotically stable compact set. Let $\beta \in \mathcal{KL}$ be such that each solution x to $\mathcal{H}$ satisfies* (2). *Assume that the family of perturbed hybrid systems $\mathcal{H}_\delta$ satisfies (CP). Then, for each compact set $K \subset \mathbb{R}^n$ and each $\varepsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, each solution x_δ to $\mathcal{H}_\delta$ with $x_\delta(0,0) \in K$ satisfies for all $(t, j) \in \operatorname{dom} x_\delta$*

$$|x_\delta(t,j)|_{\mathcal{A}} \leq \beta(|x_\delta(0,0)|_{\mathcal{A}}, t + j) + \varepsilon. \tag{5}$$

4 A Simulation Framework for Hybrid Systems

Given a hybrid system $\mathcal{H} = (C, F, D, G)$, a *hybrid simulator for $\mathcal{H}$* is given by the family of systems $\mathcal{H}_s$ parameterized by the constant s satisfying $s \in (0, s^*]$, $s^* > 0$, which determines the step size of the simulator. Like in [32, Chapter 7], we consider the case of constant step size. The data of the hybrid simulator $\mathcal{H}_s$ is given by (C_s, F_s, D_s, G_s), where

- $C_s \subset \mathbb{R}^n$ is where integration of the flows is allowed;
- $F_s : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the integration scheme for flows of $\mathcal{H}$;
- $D_s \subset \mathbb{R}^n$ is where jumps are allowed;
- $G_s : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the jump mapping.

Following (1), the hybrid simulator $\mathcal{H}_s$ can be written as

$$\mathcal{H}_s: \qquad x \in \mathbb{R}^n \qquad \begin{cases} x^+ \in F_s(x) & x \in C_s \\ x^+ \in G_s(x) & x \in D_s. \end{cases} \tag{6}$$

Comparing (1) with (6), the dynamics for the flows of $\mathcal{H}$ have been replaced by the integration scheme $x^+ \in F_s(x)$, where F_s is constructed from F; this construction for particular integration schemes will be discussed in the next section. The jump map of $\mathcal{H}$ has been replaced by the mapping G_s while the flow and jump sets C and D have been replaced by the sets C_s and D_s, respectively.

Being the dynamics of the hybrid simulator $\mathcal{H}_s$ purely discrete, the solutions to $\mathcal{H}_s$ are given on discrete versions of hybrid time domains.

Definition 4.1 (discrete time domain) *A subset $E \subset \mathbb{N} \times \mathbb{N}$ is a compact discrete time domain if*

$$E = \bigcup_{j=0}^{J-1} \bigcup_{k=K_j}^{K_{j+1}} (k, j)$$

for some finite sequence $0 = K_0 \leq K_1 \leq K_2 ... \leq K_J$, $K_j \in \mathbb{N}$ *for every* $j \leq J$, $j \in \mathbb{N}$. *It is a discrete time domain if* $\forall (K, J) \in E$, $E \cap (\{0, 1, \ldots K\} \times \{0, 1, \ldots J\})$ *is a compact discrete time domain.*

Solutions to $\mathcal{H}_s$ are parameterized by the discrete variables j and k, where k keeps track of the steps of the integration scheme for flows and j counts the jumps.

Definition 4.2 (discrete arc) *A function* $x_s : \operatorname{dom} x_s \to \mathbb{R}^n$ *is a discrete arc if* $\operatorname{dom} x_s$ *is a discrete time domain.*

Recall that, in our context, the term *simulated solution* refers to a solution obtained from discretizing the dynamics of the hybrid system, that is, a solution to $\mathcal{H}_s$.

Definition 4.3 (simulated solution to $\mathcal{H}$) *A discrete arc* $x_s : \operatorname{dom} x_s \to \mathbb{R}^n$ *is a simulated solution to the hybrid system* $\mathcal{H}$ *with a hybrid simulator* $\mathcal{H}_s$, $s > 0$, *if*

(S1$_s$) for all $k, j \in \mathbb{N}$ *such that* $(k, j), (k+1, j) \in \operatorname{dom} x_s$,
$$x_s(k, j) \in C_s, \quad x_s(k+1, j) \in F_s(x_s(k, j));$$

(S2$_s$) for all $k, j \in \mathbb{N}$ *such that* $(k, j), (k, j+1) \in \operatorname{dom} x_s$,
$$x_s(k, j) \in D_s, \quad x_s(k, j+1) \in G_s(x_s(k, j)).$$

The same solution types defined in Section 2 (i.e., nontrivial, maximal, etc.) apply for simulated solutions.

When the data of the simulator satisfies $F_s(C_s) \subset C_s \cup D_s$, $G_s(D_s) \subset C_s \cup D_s$, which simply require evaluating the mappings F_s and G_s on sets C_s and D_s, every simulated solution to $\mathcal{H}$ is complete. Note that when completeness of simulated solutions to $\mathcal{H}$ is not guaranteed by construction of $\mathcal{H}_s$, the discretization of the flows, which as in numerical simulation of differential/difference equations is determined by the integration scheme used (e.g., by Euler, Runge-Kutta, and multi-step methods), can lead to simulated solutions that, unlike the true solutions, end by leaving $C \cup D$ after a flowing step. A particular construction of G_s, C_s, and D_s preventing such a premature stopping of simulated solutions is given next.

Example 4.4 ("Minimal" inflation of jump set to keep simulator from stopping prematurely) Given $\mathcal{H} = (C, F, D, G)$, consider the simulator $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$, where $C_s := C$, $G_s(x) = G(x)$ for all $x \in D$ and, for all $x \in \mathbb{R}^n \setminus (C \cup D)$,

$$G_s(x) := \{g \in G(\xi) \mid \xi \in L_s(z, x) \cap D, z \in C, x \in F_s(z)\},$$

where, given a continuous function $\widetilde{\rho}$, $L_s(z, x)$ is given by

$$\{\xi \mid \xi = (1 - \lambda)z + \lambda(x + v), \lambda \in [0, 1], v \in s\widetilde{\rho}(s)\mathbb{B}\},$$

and let D_s be the domain of G_s, i.e., the set of points x for which $G_s(x)$ is not empty. In words, this construction is such that when a simulated solution reaches a point $x \notin C \cup D$, it checks if x is close enough to D that considering x to be in D would be reasonable. This is determined by first finding points in C from which the simulator may have generated x. These are points $z \in C$ such that $x \in F_s(z)$. Then, the simulator checks whether the set $L_s(z, x)$, which starts at z and ends in a neighborhood around x, intersects D. (Note that when $\widetilde{\rho}$ in L_s is taken to be zero, checking whether it intersects D just involves a line search.) With an affirmative answer, x is translated to an intermediate point where $L_s(z, x)$ intersects the set D and then a jump is executed from that intermediate point. With a negative answer, the simulation stops. $\triangle$

One way to translate a simulated solution x_s on a discrete time domain $\operatorname{dom} x_s$ to a hybrid arc ξ_s on a hybrid time domain $\operatorname{dom} \xi_s$ is by piecewise linear interpolation of the flows as follows. Given a simulated solution x_s to $\mathcal{H}$, its corresponding (linearly) *interpolated simulated solution* ξ_s is a hybrid arc as follows:

- For every (t, j) such that $(k, j), (k+1, j) \in \operatorname{dom} x_s$, $ks \leq t \leq (k+1)s$,

$$\xi_s(t, j) = x_s(k, j) + \frac{(t - ks)(x_s(k+1, j) - x_s(k, j))}{s}. \quad (7)$$

- For every (t, j) s.t. $(k, j), (k, j+1) \in \operatorname{dom} x_s$, $t = ks$,

$$\xi_s(t, j) = x_s(k, j). \quad (8)$$

Next, we illustrate this transformation in an example.

Example 4.5 (bouncing ball (revisited)) We propose a hybrid simulator $\mathcal{H}_s^{BB}$ for $\mathcal{H}^{BB}$ in Example 2.4 with step size $s > 0$ and data given by $F_s(x) = x + sF(x)$ for all $x \in C$ and C_s, G_s, D_s as in Example 4.4 with $\widetilde{\rho} \equiv 0$:

$$C_s = C, \ D_s = D \cup \{x \mid z \in C, \ x = z + sF(z) \notin C\},$$
$$G_s(x) = \begin{bmatrix} 0 \\ -\varrho\left(x_2 + \frac{x_1}{s}\left(1 - \frac{x_2}{x_2 + s\gamma}\right)\right) \end{bmatrix}.$$

The choice of F_s corresponds to the forward Euler rule. (This type of integration scheme has been used to simulate the continuous dynamics of hybrid systems in the literature, see e.g. [23, 27].) The data of $\mathcal{H}_s^{BB}$ is such that simulated solutions starting from $C_s \cup D_s$ are complete.

The exact solution x (not a simulated solution) to $\mathcal{H}^{BB}$ on a hybrid time domain starting from the initial condition $x_1(0, 0) = 6\text{m}, x_2(0, 0) = 0.1\text{m/s}$ is depicted in Figure 3(a) (exact computation of solutions to this system is possible by solving the for dynamics forward in hybrid time). A simulated solution along with its discrete domain using step size $s = 0.2\text{sec}$ are depicted in Figure 3(b) for $x_s(0, 0) = x(0, 0)$. Figure 3(b) also shows the

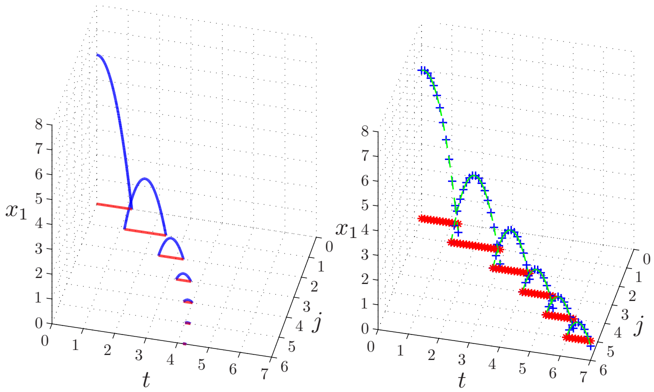


Fig. 3. Solution (solid curve) and simulated solution (dotted) to the bouncing ball system $\mathcal{H}^{BB}$. (a) A solution with $x_1(0,0) = 6$m, $x_2(0,0) = 0.1$m/s, $g = 9.8$m/s^2, and $\gamma = 0.6$. (b) Simulation with step size $s = 0.1$sec and same parameters as for the solution in (a). Discrete arc x_s denote with $+$ and hybrid time domains shown in red.

interpolated simulated solution ξ_s (depicted with dashed line) obtained from x_s using (7)-(8). $\triangle$

The stability definitions in Section 2 apply to hybrid simulators $\mathcal{H}_s$ by just passing from hybrid time domains to discrete time domains. Regarding closeness between solutions and simulated solutions, we say that a solution x and a simulated solution x_s to a hybrid system $\mathcal{H}$ are (T, J, ε)-close if x and the interpolated simulated solution ξ_s (associated with x_s) are (T, J, ε)-close.

The following conditions on the data of $\mathcal{H}_s$ are imposed.

Assumption 4.6 (data of the hybrid simulator) *The data of the hybrid simulator $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ for the hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies:*

(B0) F_s is such that, for each compact set $K \subset \mathbb{R}^n$, there exists $\rho \in \mathcal{K}_\infty$ and $s^* > 0$ such that for each $x \in C_s \cap K$ and each $s \in (0, s^*]$

$$F_s(x) \subset x + s\,\mathrm{con}\,F(x + \rho(s)\mathbb{B}) + s\rho(s)\mathbb{B};$$

(B1) G_s is such that $G_0(x) \subset G(x)$ for each $x \in \mathbb{R}^n$, where, for any positive sequence $\{s_i\}_{i=1}^{\infty}$ such that $s_i \searrow 0$, G_0 is the outer graphical limit of G_{s_i};

(B2) C_s and D_s are such that for any positive sequence $\{s_i\}_{i=1}^{\infty}$ such that $s_i \searrow 0$

$$\limsup_{i\to\infty} C_{s_i} \subset C, \quad \limsup_{i\to\infty} D_{s_i} \subset D,$$

where $\limsup_{i\to\infty} C_{s_i}, \limsup_{i\to\infty} D_{s_i}$ are the outer limits of the sequence of sets C_{s_i}, D_{s_i}, respectively.

Remark 4.7 Very often, the jump mapping G and the sets C and D can be implemented accurately in the hybrid simulator, i.e., it may be possible to take $G_s \equiv G$, $C_s = C$, and $D_s = D$. Condition (B2) is a condition on the perturbation by s of the flow and jump sets. Conditions (B1) and (B2) are satisfied when G_s, C_s, and D_s are contained in outer perturbations of G, C, and D, respectively. More precisely, the outer perturbation of G, C, and D for a state-dependent perturbation determined by a constant $\delta > 0$ and a continuous function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are given by

$$\hat{G}_\delta(x) := \{y \in \mathbb{R}^n \,|\, y \in \eta + \delta\alpha(\eta)\mathbb{B}, \eta \in G(x + \delta\alpha(x)\mathbb{B})\},$$
$$\hat{C}_\delta := \{x \in \mathbb{R}^n \mid (x + \delta\alpha(x)\mathbb{B}) \cap C \neq \emptyset\},$$
$$\hat{D}_\delta := \{x \in \mathbb{R}^n \mid (x + \delta\alpha(x)\mathbb{B}) \cap D \neq \emptyset\},$$

respectively, which satisfy (B1)-(B2) by Theorem 5.4 in [16] (see Example 5.3 in [16] for more details). Then, (B1)-(B2) hold if for each $\delta > 0$ there exists $s^* > 0$ such that for all $s \in (0, s^*]$

$$G_s(x) \subset \hat{G}_\delta(x) \qquad \forall x \in \mathbb{R}^n, \qquad C_s \subset \hat{C}_\delta, \qquad D_s \subset \hat{D}_\delta.$$

Outer perturbations allow to model, among other things, measurement error, unmodeled dynamics, as well as floating point errors. Assumption (B0) is a consistency condition on the integration scheme for flows (cf. [32, Definition 3.4.2]). It implies that, for every compact set $K \subset \mathbb{R}^n$ and step size $s > 0$, from every point $\xi \in C_s \cap K$ the new value given by the integration scheme for flows picked from $F_s(\xi)$ is close to a solution x to $\dot{x} \in F(x)$ starting from ξ and evaluated at time $t = s$.

The following examples illustrate that several widely used integration schemes for differential equations satisfy (B0) in Assumption 4.6.

Example 4.8 (forward Euler method) The simplest numerical method to approximate solutions to differential inclusions (or equations) $\dot{x} \in F(x)$ is the forward Euler rule [3, Chapter 3], [10, Chapter 2]. This method is based on the first-order Taylor expansion of the continuous right-hand side around $x \in \mathbb{R}^n$ and is given by $F_s(x) = x + sF(x)$. Then, condition (B0) is satisfied. $\triangle$

Example 4.9 (p-stage Runge-Kutta consistent methods) For differential inclusions (or equations) $\dot{x} \in F(x)$ with locally bounded $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the update law for p-stage Runge-Kutta integration schemes, $p \geq 1$, is

$$F_s(x) = x + s\sum_{i=1}^{p} b_i\xi_i, \tag{9}$$

where $b_i \in \mathbb{R}$ and $\xi_i \in F(Y_i)$, $i \in I := \{1, 2, \ldots, p\}$. The variables Y_i are called stage variables and are given by

$$Y_i = x + s\sum_{j=1}^{p} a_{ij}\xi_j, \quad \xi_j \in F(Y_j), \tag{10}$$

where $a_{ij} \in \mathbb{R}$, $(i,j) \in I \times I$. (When $a_{ij} = 0$ for all $j \geq i$, the method is called explicit since the stage variables can be solved without recursion.)

Provided that (10) is solvable, either in an explicit or implicit manner, for every compact set $K \subset \mathbb{R}^n$ there exists $\rho \in \mathcal{K}_\infty$ such that for each $x \in K$ the stage variables satisfy, for each $s > 0$, $Y_i \in x + s\rho(s)\mathbb{B}$ for all $i \in I$. Moreover, when the Runge-Kutta method is consistent (see, e.g., [32, Definition 3.4.2]), the coefficients b_i satisfy $\sum_{i=1}^p b_i = 1$ (see, e.g., [32, Section 3.4]). (This condition is usually required for stability of the Runge-Kutta integration method, see, e.g., [18] and [17].) Then, $\sum_{i=1}^p b_i \xi_i$ in (9) corresponds to a convex hull condition and Assumption (B0) is satisfied since for each $x \in K$ and $s > 0$, $F_s(x) \subset x + s\,\mathrm{con}_{i\in I} F(Y_i) \subset x + s\,\mathrm{con}\, F(x + \rho(s))$. $\triangle$

5 Main Results

In this section, we present results on closeness between solutions and simulated solutions to hybrid systems, and on asymptotic stability of compact sets for hybrid simulators. These extend results in [32] to hybrid systems, which include constrained differential and difference inclusions as special cases.

We treat the hybrid simulator $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ as a perturbation of the hybrid system $\mathcal{H} = (C, F, D, G)$ with perturbation parameter being the step size s. For a given compact set $K \subset \mathbb{R}^n$ and a simulation horizon $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, let $\mathrm{reach}_{T,J}^{\mathcal{H}_s}(K)$ be the reachable set of $\mathcal{H}_s$ from K up to (T, J), i.e.,

$$\begin{aligned}\mathrm{reach}_{T,J}^{\mathcal{H}_s}(K) := \{ & x_s(k,j) \mid x_s \text{ is a simulated solution} \\ & \text{to } \mathcal{H}, x_s(0,0) \in K, (k,j) \in \mathrm{dom}\, x_s, ks \leq T, j \leq J \}.\end{aligned}$$

Let $\rho \in \mathcal{K}_\infty$, s^* be given by (B0) in Assumption 4.6, and

$$\delta'(s) := \max_{\eta \in\ \mathrm{reach}_{(T,J)}^{\mathcal{H}_s}(K) \cap C_s} \mathrm{con}\, F(\eta + \rho(s)\mathbb{B}) + \rho(s)\mathbb{B}.$$

Then, for each $s \in (0, s^*]$, each simulated solution $x_s : \mathrm{dom}\, x_s \to \mathbb{R}^n$ to $\mathcal{H}$ with $\mathrm{dom}\, x_s \subset [0,T] \times \{0, \ldots, J\}$, $x_s(0,0) \in K$, and its associated hybrid arc $\xi_s : \mathrm{dom}\, \xi_s \to \mathbb{R}^n$ (obtained using (7)-(8)) are such that

- For all $(t,j) \in \mathrm{dom}\, \xi_s$, $ks \leq t \leq (k+1)s$, where $(k,j), (k+1,j) \in \mathrm{dom}\, x_s$, ξ_s satisfies

$$\begin{aligned}&\dot{\xi}_s(t,j) := \frac{x_s(k+1,j) - x_s(k,j)}{s},\\ &\dot{\xi}_s(t,j) \in \mathrm{con}\, F(\xi_s(ks,j) + \rho(s)\mathbb{B}) + \rho(s)\mathbb{B},\\ &|\xi_s(ks,j) - \xi_s(t,j)| \leq |x_s(k,j) - x_s(k+1,j)| \leq s\delta'(s),\end{aligned}$$

which implies

$$\begin{aligned}&\dot{\xi}_s(t,j) \in \overline{\mathrm{con}} F(\xi_s(t,j) + (s\delta'(s) + \rho(s))\mathbb{B}) + \rho(s)\mathbb{B},\\ &\xi_s(t,j) \in \{x \in \mathbb{R}^n \mid (x + s\delta'(s)\mathbb{B}) \cap C_s \neq \emptyset\}.\end{aligned}$$

- For all (t,j), $t = ks$, where $(k,j), (k,j+1) \in \mathrm{dom}\, x_s$, $\xi_s(t,j+1) \in G_s(\xi_s(t,j))$ and $\xi_s(t,j) \in D_s$.

Given $\delta > 0$, let $s > 0$ be small enough such that $s\delta'(s) + \rho(s) \leq \delta$. Then, let

$$\begin{aligned} F_\delta(x) &:= \overline{\mathrm{con}} F(x + \delta\mathbb{B}) + \delta\mathbb{B},\\ C_\delta &:= \left\{x \in \mathbb{R}^n \;\middle|\; (x + \delta\mathbb{B}) \cap C_{s(\delta)} \neq \emptyset\right\}, \\ G_\delta(x) &:= G_{s(\delta)}(x),\ D_\delta := D_{s(\delta)},\end{aligned} \tag{11}$$

where the dependence of s on δ is explicitly indicated and is such that s approaches zero as $\delta \searrow 0$. From the properties of ξ_s above, interpolated simulated solutions ξ_s to $\mathcal{H}$ that start from K are solutions to the perturbed hybrid system $\mathcal{H}_\delta = (C_\delta, F_\delta, D_\delta, G_\delta)$ on hybrid time domains that are subset of $[0,T] \times \{0, 1, \ldots, J\}$.

Lemma 5.1 (convergence property for $\mathcal{H}_\delta$) *Assume $\mathcal{H}$ satisfies Assumption 2.5 and $\mathcal{H}_s$ satisfies Assumption 4.6. Then, for every compact set $K \subset \mathbb{R}^n$ and every simulation horizon $(T,J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, the family of hybrid systems $\mathcal{H}_\delta = (C_\delta, F_\delta, D_\delta, G_\delta)$ with data as in* (11) *satisfies (CP).*

Proof. For every sequence $\{\delta_i\}_{i=1}^\infty$ with $1 > \delta_1 > \delta_2 > \ldots > 0$ converging to zero, by construction and (B1)-(B2) in Assumption 4.6, the sequences $C_i := C_{\delta_i}$, $D_i := D_{\delta_i}$ satisfy the conditions in (CP). Locally eventually boundedness of $F_i := F_{\delta_i}$, $G_i := G_{\delta_i}$ follows by construction and (A2)-(A3). The proof that F_i, G_i satisfy (4) follows from [16, Lemma 5.4] by the fact that F_i, G_i are outer perturbations of F, G, respectively. $\square$

From the construction of $\mathcal{H}_\delta$, the following closeness result between solutions and simulated solutions holds.

Theorem 5.2 (closeness of simulated solutions on compact domains) *Assume that $\mathcal{H}$ satisfies Assumption 2.5 and that the family of hybrid systems $\mathcal{H}_s$ satisfies Assumption 4.6. Then, for every compact set $K \subset \mathbb{R}^n$, every $\varepsilon > 0$, and every simulation horizon $(T,J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $s^* > 0$ with the following property: for any $s \in (0, s^*]$ and any simulated solution x_s to $\mathcal{H}_s$ with $x_s(0,0) = x_s^0 \in K + \delta\mathbb{B}$ there exists a solution x to $\mathcal{H}$ with $x(0,0) \in K$ such that x_s and x are (T,J,ε)-close.*

Proof. Since $\mathcal{H}$ satisfies Assumption 2.5 and $\mathcal{H}_s$ satisfies Assumption 4.6, by Lemma 5.1 and Theorem 3.4, there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, each $s > 0$ such that $s\delta'(s) + \rho(s) \leq \delta$, for each solution x_δ to

$\mathcal{H}_\delta$ with $x_\delta(0,0) \in K + \delta\mathbb{B}$ there exists a solution x to $\mathcal{H}$ with $x(0,0) \in K$ such that x_δ and x are (T, J, ε)-close. Then, since hybrid arcs ξ_s obtained from simulated solutions x_s to $\mathcal{H}$ are solutions to the perturbed hybrid system $\mathcal{H}_\delta$, it follows that for every such hybrid arc ξ_s with $\operatorname{dom} \xi_s \subset [0,T] \times \{0,\ldots,J\}$, there exists a solution x to $\mathcal{H}$ with $x(0,0) \in K$ such that ξ_s and x are (T, J, ε)-close. The proof concludes by picking $s^* > 0$ to be the largest s satisfying $s\delta'(s) + \rho(s) \le \delta^*$ and by the relationship between simulated solutions x_s and interpolated simulated solutions ξ_s in (7)-(8). $\square$

When a compact set is globally pre-asymptotically stable for $\mathcal{H}$, then its hybrid simulator has the same set semiglobally practically pre-asymptotically stable.

Theorem 5.3 (semiglobal practical stability) *Assume that the hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies Assumption 2.5 and that $\mathcal{A}$ is a globally pre-asymptotically stable compact set for $\mathcal{H}$. Assume that the family of hybrid systems $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ satisfies Assumption 4.6. Then, $\mathcal{A}$ is semiglobally practically pre-asymptotically stable for $\mathcal{H}_s$, i.e., there exists $\beta \in \mathcal{KL}$ such that, for every compact set $K \subset \mathbb{R}^n$, every $\varepsilon > 0$, and every simulation horizon $(T, J) \in \mathbb{R}_{\ge 0} \times \mathbb{N}$ there exists $s^* > 0$ such that, for each $s \in (0, s^*]$, every simulated solution x_s to $\mathcal{H}$ with $x_s(0,0) \in K$ satisfies for all $(k, j) \in \operatorname{dom} x_s$*

$$|x_s(k,j)|_{\mathcal{A}} \le \beta(|x_s(0,0)|_{\mathcal{A}}, ks + j) + \varepsilon.$$

Proof. By Theorem 3.1, there exists $\beta \in \mathcal{KL}$ such that for each solution x to $\mathcal{H}$

$$|x(t,j)|_{\mathcal{A}} \le \beta(|x(0,0)|_{\mathcal{A}}, t + j) \qquad \forall (t,j) \in \operatorname{dom} x.$$

Given a compact set $K \subset \mathbb{R}^n$ and a simulation horizon $(T, J) \in \mathbb{R}_{\ge 0} \times \mathbb{N}$, let $\mathcal{H}_\delta = (C_\delta, F_\delta, D_\delta, G_\delta)$ be a family of hybrid systems given by the perturbation of $\mathcal{H}$ with data as in (11) (with appropriately chosen parameter δ). By the assumptions, Lemma 5.1 implies that $\mathcal{H}_\delta$ satisfies (CP). Then, by Theorem 3.5 using K above, for each $\varepsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$, every solution x_δ to $\mathcal{H}_\delta$ with $x_\delta(0,0) \in K + \delta\mathbb{B}$ satisfies

$$|x_\delta(t,j)|_{\mathcal{A}} \le \beta(|x_\delta(0,0)|_{\mathcal{A}}, t + j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} x_\delta.$$

Pick $s^* > 0$ to be the largest s satisfying $s\delta'(s) + \rho(s) \le \delta^*$. Then, since interpolated simulated solutions ξ_s are solutions to the perturbed hybrid system $\mathcal{H}_\delta$, the result follows by recovering simulated solutions x_s via sampling of the interpolated simulated solutions ξ_s. $\square$

Note that the property in Theorem 5.3 holds for small enough step size s. The step size bound s^* decreases with the desired level of closeness to $\mathcal{A}$, which is given by ε.

Our main result is as follows.

Theorem 5.4 (continuity of asymptotically stable sets) *Assume that the hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies Assumption 2.5 and that $\mathcal{A}$ is a globally pre-asymptotically stable compact set for $\mathcal{H}$. Assume that the family of hybrid systems $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ satisfies Assumption 4.6. Then, there exists $s^* > 0$ such that for each $s \in (0, s^*]$, the hybrid simulator $\mathcal{H}_s$ has a semiglobally pre-asymptotically stable compact set $\mathcal{A}_s$ satisfying*

$$d_H(\mathcal{A}_s, \mathcal{A}) \to 0 \quad \text{as } s \searrow 0. \tag{12}$$

Proof. Let K be any compact set such that for some $\varepsilon > 0$, $\mathcal{A} + 2\varepsilon\mathbb{B} \subset K \subset \mathbb{R}^n$. With $\mathcal{H}_\delta$ as in (11), using K as above and an arbitrary simulation horizon $(T, J) \in \mathbb{R}_{\ge 0} \times \mathbb{N}$, define $\widetilde{\mathcal{H}}_\delta = (F_\delta, C_\delta, \widetilde{G}_\delta, \widetilde{D}_\delta)$, where

$$\widetilde{G}_\delta(x) = \begin{cases} G_\delta(x) \cup \mathcal{A} & \text{if } x \in D_\delta \\ \mathcal{A} & \text{if } x \in O \setminus D_\delta \end{cases}, \qquad \widetilde{D}_\delta = \mathbb{R}^n.$$

Using K and ε as above, by Theorem 3.5, there exists $\beta \in \mathcal{KL}$ such that solutions $\widetilde{x}_\delta$ to $\widetilde{\mathcal{H}}_\delta$ satisfy (5).

Now, let $B_\varepsilon := \overline{\operatorname{reach}_\infty^{\widetilde{\mathcal{H}}_\delta}(\mathcal{A} + 2\varepsilon\mathbb{B})}$, where $\operatorname{reach}_\infty^{\widetilde{\mathcal{H}}_\delta}(\mathcal{A} + 2\varepsilon\mathbb{B})$ is the reachable set of $\widetilde{\mathcal{H}}_\delta$ from $\mathcal{A} + 2\varepsilon\mathbb{B}$. Since solutions to $\widetilde{\mathcal{H}}_\delta$ satisfy (5), then B_ε is bounded. Since B_ε is closed by definition, it follows that it is compact. We show that it is forward invariant. Let $\widetilde{x}_\delta$ be a solution to $\widetilde{\mathcal{H}}_\delta$ from $\widetilde{x}_\delta(0,0) \in B_\varepsilon$. Assume that there exists $(t', j') \in \operatorname{dom} \widetilde{x}_\delta$ for which $\widetilde{x}_\delta(t', j') \notin B_\varepsilon$. By definition of B_ε, since $\widetilde{x}_\delta(0,0) \in B_\varepsilon$, the solution $\widetilde{x}_\delta$ belongs to B_ε for all $(t,j) \in \operatorname{dom} \widetilde{x}_\delta$. This is a contradiction. To show that solutions to $\widetilde{\mathcal{H}}_\delta$ starting from K converge to B_ε uniformly, note that, (5) implies that for the given K and ε, there exists $N > 0$ such that for every solution $\widetilde{x}_\delta$ to $\widetilde{\mathcal{H}}_\delta$, $\widetilde{x}_\delta(0,0) \in K$, and all $(t,j) \in \operatorname{dom} \widetilde{x}_\delta$, $t + j \ge N$:

$$|\widetilde{x}_\delta(t,j)|_{\mathcal{A}} \le \beta(|\widetilde{x}_\delta(0,0)|_{\mathcal{A}}, t + j) + \varepsilon \le 2\varepsilon.$$

Then, since B_ε is compact, forward invariant, and uniformly attractive from K, by [16, Proposition 6.1], B_ε is an asymptotically stable set for $\widetilde{\mathcal{H}}_\delta$. Pre-asymptotic stability of B_ε for $\mathcal{H}_s$ follows.

Finally, note that $B_0 = \mathcal{A}$ and that as $\varepsilon \searrow 0$, $d_H(B_\varepsilon, B_0) \to 0$. From the statements above, $\varepsilon \searrow 0$ implies $\delta \searrow 0$. Since for a given δ, s needs to satisfy $s\delta'(s) + \rho(s) \le \delta$, we have $s \searrow 0$ as $\varepsilon \searrow 0$. With some abuse of notation, the claim holds with $\mathcal{A}_s = B_\varepsilon$. $\square$

We illustrate our results in examples 2.6 and 2.4.

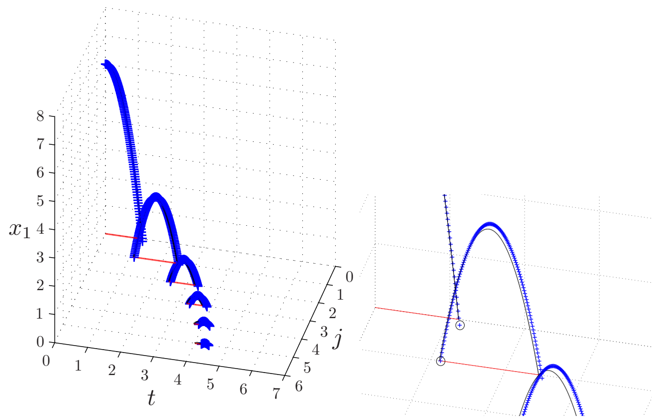
Example 5.5 (Obstacle avoidance (revisited)) For the hybrid system in Example 2.6, we propose a hybrid simulator $\mathcal{H}_s^A$ with data given by

$$F_s(x) = x + sF(x) \ \forall x \in C, \ C_s = C, \ G_s = G, D_s = D.$$

One particular type of simulated solutions to $\mathcal{H}_s^A$ are those hitting the jump set tangentially (or due to the discretization, almost hitting it). For example, there exist simulated solutions starting from $x^0 = [-3, 1+\varepsilon, 0]^\top$ with $\varepsilon = 0$ that, depending on the step size s, either hit the set D_0 or not and never jump. In either case, the obtained simulated solution is close to the solution to $\mathcal{H}^A$ given in item a) of Example 2.6, which, as pointed out in the discussion therein, is a grazing solution. $\triangle$

Example 5.6 (bouncing ball (re-revisited)) Consider the ball bouncing system $\mathcal{H}^{BB}$ and the hybrid simulator $\mathcal{H}_s^{BB} = (F_s, C_s, G_s, D_s)$ in Example 4.5. From its construction and the discussion in Example 4.8, it can be checked that $\mathcal{H}_s^{BB}$ satisfies Assumption 4.6. To illustrate the closeness property in Theorem 5.2, for a given finite simulation horizon (T, J) and a level of closeness ε, Figure 4(a) depicts the first component of a solution x and the first component of a simulated solution x_s to $\mathcal{H}^{BB}$ for a particular step size s. Note that solutions to $\mathcal{H}^{BB}$ can be computed exactly by solving its dynamics forward in hybrid time. Since solutions to $\mathcal{H}^{BB}$ are unique, the solution to which each simulated solution is close to is uniquely defined. The level of closeness between the solution x and the simulated solution x_s is satisfied for the time horizon (T, J) for which the step size s was chosen for (the step size s used for the simulated solution in Figure 4(a) was heuristically chosen). Figure 4(b) shows a zoomed version to indicate that at points $(k, j) \in \operatorname{dom} x_s$ (denoted by $\circ$) for which $x_s(k, j)$ enters the set D_s, closeness between x and x_s is not possible for a particular $t = ks$. As the notion of closeness defined in Definition 3.2 states, the desired level of closeness is obtained by considering the distance between graphs of x and x_s. (See also the end of Section 4.)

As indicated earlier, the origin, i.e., the compact set $\mathcal{A} = \{(0, 0)\}$, is globally asymptotically stable for $\mathcal{H}^{BB}$. By Theorem 5.3, the set $\mathcal{A}$ is semiglobally practically asymptotically stable for $\mathcal{H}_s^{BB}$. This is depicted in Figure 5. We emphasize that this is an infinite horizon result saying that, for arbitrarily large time, the state x is close to $\mathcal{A}$. It is not a statement that, for arbitrarily large time, the hybrid time domains of the simulated solutions are close to the hybrid time domains of the true solutions. Indeed, the true solutions exhibit Zeno behavior whereas the simulated solutions do not. $\triangle$



(a) (b)

Fig. 4. Closeness between solutions and simulated solutions. (a) Discrete arc x_s (+), exact hybrid arc solution x (black) and exact hybrid time domain (red) are shown. Their graphs are close until some finite hybrid time (T', J'). Parameters: $s = 0.01$sec, $x_1(0,0) = 6$m, $x_2(0,0) = 0.1$m/s, $\gamma = 9.8\text{m/s}^2$, and $\varrho = 0.6$. The circled points in x_s are not close to x at the same hybrid time instant – the closeness property is between the graphs of x_s and x.

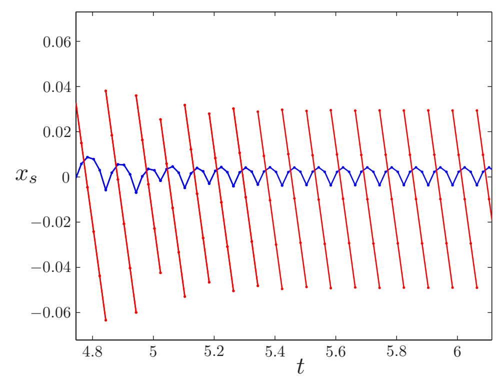


Fig. 5. Projection to the t axis of a simulated (and interpolated) solution to the bouncing ball nearby $\mathcal{A}$.

6 Conclusion

We introduced a framework for simulation of hybrid systems $\mathcal{H} = (C, F, D, G)$ that features a simulator model $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ as a function of the step size s. We have given conditions on the data (C_s, F_s, D_s, G_s) of the hybrid simulator $\mathcal{H}_s$ that guarantee structural properties of simulated solutions to hybrid systems including closeness between solutions and simulated solutions on compact time domains, and semiglobal practical stability of compact sets under simulations. We have shown how these new results enable us to establish that, for the proposed framework, asymptotically stable compact sets are continuous in the step size. We illustrated the concepts and results in several examples.

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