Results on Input-to-Output and Input-Output-to-State Stability for Hybrid Systems and their Interconnections

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***Abstract*— We present results for the analysis of input/output properties of interconnections of a general class of hybrid systems. Input-to-output and input-output-to-state stability for hybrid systems are considered. Definitions of these notions and sufficient conditions are introduced for hybrid systems given by a flow set, a flow map, a jump set, a jump map, and an output function. An interconnection result in terms of a small gain theorem for the said notions is also presented.**

I. Introduction

Input-to-state stability (ISS) has wide applicability in analysis and design of nonlinear control systems. Introduced in [27], the recent article [29] summarizes the progress on the subject. The ISS concept is of particular importance in the study of interconnections of dynamical systems. It is well known that the feedback interconnection of ISS nonlinear systems is ISS when a small gain condition holds. Such a result can be asserted using the $\mathcal{KL}$ estimates involved in the definition of ISS for the individual systems. Small gain theorems in terms of $\mathcal{KL}$ estimates for interconnections of ISS systems for continuous-time systems appeared in [14], [35], [12]. Alternatively, sufficient conditions for ISS of interconnections in terms of Lyapunov functions have been shown to be powerful as they provide an ISS Lyapunov function for the entire interconnection. These exploit Lyapunov characterizations and sufficient conditions for ISS of the individual systems, results that were presented for continuous-time systems in [32], for discrete-time systems in [16], for switched systems in [21], [36], and for hybrid systems in [4]. A Lyapunov-based small gain theorem for interconnection of ISS systems appeared in [15] for continuous-time systems, and later extended to discrete-time and hybrid systems in [19] and [23] (see also [7]), respectively.

Classical notions relating inputs and outputs used in the study of input/output interconnections of linear systems [8] and certain classes of nonlinear systems [17] have been extended to the ISS framework of [28]. Input-to-output stability (IOS) for continuous-time systems in such a framework was introduced in [34], [26], along with several characterizations. Its output to state counterpart, output-to-state stability (OSS) [33], provides a tool to establish bounds on the state in terms of the system outputs as well as a link to detectability; see the recent extension to hybrid systems in [2]. Combined with ISS, OSS led to the concept of input-output-to-state stability (IOSS) in [31]. Instead of bounds on solutions in terms of inputs only, IOSS relates bounds on solutions to the size of both inputs and outputs. IOSS was recently extended to discrete-time systems in [3]. Lyapunov characterizations of IOSS were reported for continuous-time systems in [1], for discrete-time systems in [5], and for switched systems in [22]. Small gain results in terms of $\mathcal{KL}$ estimates were presented for interconnections of IOS/IOSS nonlinear continuous-time systems in [14], for IOS continuous and discrete-time systems [13], for input/output system models in [30], and for a class of systems with jumps in [24].

The purpose of this paper is to provide tools for the analysis of interconnections of hybrid systems using stability notions involving their inputs and outputs. To that end, we define the notions of IOS and IOSS to a class of hybrid systems with inputs and outputs specified by a flow set, a flow map, jump set, and a jump set. We present sufficient conditions for IOS and IOSS of individual hybrid systems as well as Lyapunov characterizations of these properties. We also show that, as for continuous and discrete-time systems [33], [18], certain IOSS properties are both necessary and sufficient for the existence of state-norm estimators for hybrid systems (a similar result was established in [2] for OSS). Tools for the analysis of interconnections of IOS and IOSS systems are given in terms of a small gain theorem. Following the ideas in [15] for continuous-time systems and in [23] for hybrid systems, we provide Lyapunov conditions guaranteeing IOS/IOSS of the interconnection of IOS/IOSS hybrid systems. To the best of the author's knowledge, there are no previous Lyapunov-based small gain theorems for IOS and IOSS, not even for continuous-time or discrete-time systems.

The remainder of the paper is organized as follows. Section II summarizes the framework for hybrid systems under study and gives basic definitions. In Section III, definitions of IOS and IOSS for hybrid systems are introduced. It presents sufficient conditions and Lyapunov characterizations. A small gain theorem for the analysis of interconnected IOS and IOSS hybrid systems is presented in Section IV.

II. Preliminaries

Before introducing the hybrid systems framework under study, we summarize the notation used throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space; $\mathbb{R}$ real numbers. $\mathbb{R}_{\geq 0}$ nonnegative real numbers; $\mathbb{N}$ natural numbers including 0; $\mathbb{B}$ the closed unit ball in a Euclidean space. Given a set S, $\overline{S}$ denotes its closure. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$. Given a set S, ess sup S denotes its essential supremum and $\text{int}(S)$ its interior. A function

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$\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if is continuous, zero at zero, and strictly increasing and to belong to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it belongs to class-$\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if is continuous, nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \searrow 0} \beta(s,r) = \lim_{r\to\infty} \beta(s,r) = 0$.

A hybrid system $\mathcal{H}$ with state x, input u, and output y is modeled as

$$\mathcal{H} \begin{cases} \dot{x} &= f(x,u) \qquad (x,u) \in C \\ x^+ &= g(x,u) \qquad (x,u) \in D \\ y &= h(x), \end{cases} \tag{1}$$

where $\mathbb{R}^n$ is the space for the state x, $\mathcal{U} \subset \mathbb{R}^m$ is the space for inputs u, the set $C \subset \mathbb{R}^n \times \mathcal{U}$ is the *flow set*, the function $f : C \to \mathbb{R}^n$ is the *flow map*, the set $D \subset \mathbb{R}^n \times \mathcal{U}$ is the *jump set*, $g : D \to \mathbb{R}^n$ is the *jump map*, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the *output map*. The data of the hybrid system $\mathcal{H}$ is given by (C, f, D, g, h).

Definition 2.1 (hybrid time domain): A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Solutions to hybrid systems $\mathcal{H}$ will be given in terms of hybrid arcs and hybrid inputs. These are parameterized by pairs (t, j), where t is the ordinary-time component and j is the discrete-time component that keeps track of the number of jumps.

Definition 2.2 (hybrid arc and input): A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a hybrid arc if $\operatorname{dom} x$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto x(t,j)$ is absolutely continuous on the interval $\{t \ : \ (t,j) \in \operatorname{dom} x \ \}$. A function $u : \operatorname{dom} u \to \mathcal{U}$ is a hybrid input if $\operatorname{dom} u$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t \ : \ (t,j) \in \operatorname{dom} u \ \}$.

We will employ the following signal norm for general hybrid signals, such as a hybrid arc or input.

Definition 2.3 ($\mathcal{L}_\infty$ norm of hybrid signals): Given a hybrid signal r, its $\mathcal{L}_\infty$ norm is given by

$$\|r\|_{(t,j)} := \max\left\{ \operatorname*{ess\,sup}_{(t',j')\in \operatorname{dom} r \setminus \Gamma(r),\ t'+j' \leq t+j} |r(t',j')|, \ \sup_{(t',j')\in\Gamma(r),\ t'+j'\leq t+j} |r(t',j')| \right\},$$

where $\Gamma(r) := \{(t,j) \in \operatorname{dom} r \ : \ (t, j+1) \in \operatorname{dom} r \ \}$. For notational convenience, $\|r\|$ denotes $\lim_{t+j\to N} \|r\|_{(t,j)}$, where $N = \sup_{(t,j)\in \operatorname{dom} r} t + j$.

With the definitions of hybrid time domain, and hybrid arc and input in Definitions 2.1 and 2.2, respectively, we define a concept of solution for hybrid systems $\mathcal{H}$.

Definition 2.4 (solution): Given a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ and an initial condition ξ, a hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ defines a *solution pair* (ϕ, u) *to the hybrid system* $\mathcal{H}$ if the following conditions hold:

(S0) $(\xi, u(0,0)) \in \overline{C} \cup D$ and $\operatorname{dom} \phi = \operatorname{dom} u$;

(S1) For each $j \in \mathbb{N}$ such that $I_j := \{t \ : \ (t,j) \in \operatorname{dom}(\phi, u) \ \}$ has nonempty interior $\operatorname{int}(I_j)$,

$$(\phi(t,j), u(t,j)) \in C \text{ for all } t \in \operatorname{int}(I_j),$$

and, for almost all $t \in I_j$,

$$\frac{d}{dt}\phi(t,j) = f(\phi(t,j), u(t,j));$$

(S2) For each $(t,j) \in \operatorname{dom}(\phi,u)$ such that $(t, j+1) \in \operatorname{dom}(\phi, u)$,

$$(\phi(t,j), u(t,j)) \in D,$$
$$\phi(t, j+1) = g(\phi(t,j), u(t,j)).$$

A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, u)$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}(\phi, u)$ onto $\mathbb{R}_{\geq 0}$ is bounded, *discrete* if their domain is $\{0\} \times \mathbb{N}$, and *maximal* if there does not exist another pair $(\phi, u)'$ such that (ϕ, u) is a truncation of $(\phi, u)'$ to some proper subset of $\operatorname{dom}(\phi, u)'$. Given $\xi \in \mathbb{R}^n$, $\mathcal{S}_\mathcal{H}(\xi)$ denotes the set of maximal solution pairs (ϕ, u) to $\mathcal{H}$ with $\phi(0,0) = \xi$ and u with finite $\|u\|$. For a solution pair $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$, we denote by $\phi(t, j, \xi, u)$ its value at $(t,j) \in \operatorname{dom}(\phi, u)$.

The following definition introduces a concept of stability for hybrid systems $\mathcal{H}$. It is stated for general compact sets of the state space.

Definition 2.5 (stability): A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution pair $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$ with $|\xi|_\mathcal{A} \leq \delta$ satisfies $|\phi(t,j,\xi,u)|_\mathcal{A} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}(\phi, u)$;
- *0-input stable* if it is *stable* with $u \equiv 0$;
- *pre-attractive* if there exists $\mu > 0$ such that every solution pair $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$ with $|\xi|_\mathcal{A} \leq \mu$ is bounded and if it is complete satisfies $\lim_{(t,j)\in\operatorname{dom}(\phi,u), t+j\to\infty} |\phi(t,j,\xi,u)|_\mathcal{A} = 0$;
- *0-input pre-attractive* if it is *pre-attractive* with $u \equiv 0$;
- *pre-asymptotically stable* if stable and pre-attractive;
- *0-input pre-asymptotically stable* if 0-input stable and 0-input pre-attractive.

The following mild assumptions on the data of $\mathcal{H}$ will be imposed in some of the results in this paper.

Assumption 2.6 ([11], [10]): The data (C, f, D, g, h) of the hybrid system $\mathcal{H}$ satisfies

(A1) C, D, and $\mathcal{U}$ are closed sets,

(A2) $f : C \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuous.

The conditions in Assumption 2.6 assure that closed hybrid systems $\mathcal{H}$ are well posed in the sense that they inherit several good structural properties of their solution sets. These include sequential compactness of the solution set, closedness of perturbed and unperturbed solutions, etc. We refer the reader to [10], [11] (see also [9]) and [25] for details on and consequences of these conditions.

III. IOS and IOSS for Hybrid Systems

This section introduces definitions of IOS and IOSS for hybrid systems $\mathcal{H}$ as well as sufficient conditions. We also define state-norm estimators for hybrid systems and establish their connections to IOSS for hybrid systems. Below, it is assumed that, given a compact set $\mathcal{A} \subset \mathbb{R}^n$, the output function $h : \mathbb{R}^n \to \mathbb{R}^p$ is such that $h(x) = 0$ for all $x \in \mathcal{A}$.

A. IOS definitions and results

The next input/output stability notion for hybrid systems follows those in [34], [26] for continuous-time systems.

Definition 3.1 (input-to-output stability): The hybrid system $\mathcal{H}$ is *input-to-output stable (IOS) with respect to a compact set* $\mathcal{A} \subset \mathbb{R}^n$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$ satisfies, for each $(t, j) \in \operatorname{dom}(\phi, u)$,

$$|h(\phi(t,j,\xi,u))| \leq \max\left\{\beta(|\phi(0,0,\xi,u)|_\mathcal{A}, t+j), \gamma(\|u\|_{(t,j)})\right\}. \quad (2)$$

It is said to be *locally input-to-output stable (locally IOS) with respect to a compact set* $\mathcal{A} \subset \mathbb{R}^n$ if there exist $\delta > 0$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$ with $|\xi|_\mathcal{A} \leq \delta$ and each solution pair $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$, $\|u\| \leq \delta$, we have that (2) holds for all $(t, j) \in \operatorname{dom}(\phi, u)$.

When the function h is given by the identity and $|h(\cdot)|$ is replaced by $|\cdot|_\mathcal{A}$ [1], then Definition 3.1 reduces to the definition of ISS in [4, Definitions 2.1 and 2.3].

The following proposition guarantees that, under growth conditions of the output function h, asymptotic stability with zero inputs guarantees that the IOS property holds for inputs with small enough size. It parallels the ISS results in [32, Lemma I.2] for continuous-time systems and [4, Proposition 2.3] for hybrid systems.

Proposition 3.2: Given a hybrid system $\mathcal{H}$ satisfying Assumption 2.6, if the compact set $\mathcal{A}$ is 0-input pre-asymptotically stable for $\mathcal{H}$ and there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$|h(x)| \leq \alpha_1(|x|_\mathcal{A}) + \alpha_2(|u|) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathcal{U}, \quad (3)$$

then $\mathcal{H}$ is locally IOS with respect to $\mathcal{A}$.

The following proposition is useful when dealing with ISS systems.

Proposition 3.3: Suppose that a hybrid system $\mathcal{H}$ is input-to-state stable with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$. If there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that (3) *holds then $\mathcal{H}$ is IOS with respect to $\mathcal{A}$.*

[1] Instead of $|\cdot|_\mathcal{A}$, using a proper indicator for $\mathcal{A}$ on $\mathbb{R}^n$ would be equivalent.

Lyapunov conditions asserting IOS for continuous-time systems have been introduced in [26]. Following [26, Definition 1.1], we define a class of IOS Lyapunov functions for $\mathcal{H}$.

Definition 3.4 (IOS Lyapunov function): A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an IOS Lyapunov function with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for $\mathcal{H}$ if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$ such that

$$\alpha_1(|h(x)|) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \mathbb{R}^n, \quad (4)$$

$$\langle \nabla V(x), f(x,u) \rangle \leq -\alpha_3(V(x)) \quad (5)$$
$$\forall (x,u) \in C, V(x) \geq \chi(|u|),$$

$$V(g(x,u)) - V(x) \leq -\alpha_3(V(x)) \quad (6)$$
$$\forall (x,u) \in D, V(x) \geq \chi(|u|).$$

The following proposition relates IOS Lyapunov functions in the form (5)-(6) to a dissipative inequality form.

Proposition 3.5: Given a hybrid system $\mathcal{H}$ satisfying Assumption 2.6, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an IOS Lyapunov function with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for the hybrid system $\mathcal{H}$ if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ such that

$$\alpha_1(|h(x)|) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \mathbb{R}^n, \quad (7)$$

$$\langle \nabla V(x), f(x,u) \rangle \leq -\alpha_3(V(x)) + \rho(|u|) \quad (8)$$
$$\forall (x,u) \in C,$$

$$V(g(x,u)) - V(x) \leq -\alpha_3(V(x)) + \rho(|u|) \quad (9)$$
$$\forall (x,u) \in D.$$

Next, it is established that the existence of an IOS Lyapunov function implies IOS.

Theorem 3.6: Given a hybrid system $\mathcal{H}$ satisfying Assumption 2.6, if there exists an IOS Lyapunov function with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for $\mathcal{H}$ then $\mathcal{H}$ is IOS with respect to $\mathcal{A}$.

Remark 3.7: Our proof technique for Theorem 3.6 differs from that one given in [26, Theorem 2.1], which uses a small gain result for output-Lagrange stability. Instead, it follows the proof of [4, Proposition 2.7], which is for hybrid systems that are not necessarily forward complete and, therefore, no underlying bounded-input/bounded-state property is required.

B. IOSS definitions and results

The following definition introduces the property of IOSS for hybrid systems.

Definition 3.8 (input-output-to-state stability): The hybrid system $\mathcal{H}$ is *input-output-to-state stable (IOSS) with respect to a compact set* $\mathcal{A} \subset \mathbb{R}^n$ if there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$ satisfies, for each $(t, j) \in \operatorname{dom}(\phi, u)$,

$$|\phi(t,j,\xi,u)|_\mathcal{A} \leq \max\left\{\beta(|\phi(0,0,\xi,u)|_\mathcal{A}, t+j), \gamma_1(\|u\|_{(t,j)}), \gamma_2(\|y\|_{(t,j)})\right\}. \quad (10)$$

It is *locally input-output-to-state stable (locally IOSS) with respect to a compact set* $\mathcal{A} \subset \mathbb{R}^n$ if there exist $\delta > 0$,

$\beta \in \mathcal{KL}$, and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$ with $|\xi|_{\mathcal{A}} \le \delta$ and each solution pair $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$, $\|u\| \le \delta$ and $\|y\| \le \delta$, we have that (10) holds for all $(t,j) \in \mathrm{dom}(\phi, u)$. When (10) holds for $u \equiv 0$, the system is said to be 0*-input (locally) IOSS*[2].

IOSS for hybrid systems can be guaranteed, at least locally, when a 0-input IOSS property holds.

Proposition 3.9: Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set and $\mathcal{H}$ be a hybrid system satisfying Assumption 2.6. If $\mathcal{A}$ is 0-input IOSS for $\mathcal{H}$, then $\mathcal{H}$ is locally IOSS with respect to $\mathcal{A}$.

Sufficient conditions for IOSS in terms of Lyapunov functions appeared in [18] and [3] for continuous and discrete-time systems, respectively. Following these references, we define IOSS Lyapunov functions for hybrid systems $\mathcal{H}$.

Definition 3.10 (IOSS Lyapunov function): A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is an *IOSS Lyapunov function with respect to a compact set* $\mathcal{A} \subset \mathbb{R}^n$ for the hybrid system $\mathcal{H}$ if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\rho_1, \rho_2 \in \mathcal{K}$, such that

$$\alpha_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n, \tag{11}$$

$$\begin{aligned}\langle \nabla V(x), f(x,u)\rangle &\le -\alpha_3(|x|_{\mathcal{A}}) + \rho_1(|u|) \\ &\quad + \rho_2(|h(x)|) \qquad \forall (x,u) \in C,\end{aligned} \tag{12}$$

$$\begin{aligned}V(g(x,u)) - V(x) &\le -\alpha_3(|x|_{\mathcal{A}}) + \rho_1(|u|) \\ &\quad + \rho_2(|h(x)|) \qquad \forall (x,u) \in D.\end{aligned} \tag{13}$$

The next proposition relates IOSS Lyapunov functions to a different inequality form; cf., [18, Definition 2.2] and [3, Definition 3.7].

Proposition 3.11: Given an IOSS Lyapunov V with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for the hybrid system $\mathcal{H}$, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi_1, \chi_2 \in \mathcal{K}$ such that (11) *holds and*

$$\begin{aligned}\langle \nabla V(x), f(x,u)\rangle &\le -\alpha_3(|x|_{\mathcal{A}}) \\ &\forall (x,u) \in C, |x|_{\mathcal{A}} \ge \max\{\chi_1(|u|), \chi_2(|h(x)|)\},\end{aligned} \tag{14}$$

$$\begin{aligned}V(g(x,u)) - V(x) &\le -\alpha_3(|x|_{\mathcal{A}}) \\ &\forall (x,u) \in D, |x|_{\mathcal{A}} \ge \max\{\chi_1(|u|), \chi_2(|h(x)|)\}.\end{aligned} \tag{15}$$

Definition 3.12: A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying, for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\rho_1, \rho_2 \in \mathcal{K}$, (11) and, for some $\varepsilon \in (0,1]$,

$$\langle \nabla V(x), f(x,u)\rangle \le -\varepsilon V(x) + \rho_1(|u|) + \rho_2(|h(x)|) \tag{16}$$

for all $(x,u) \in C$, and

$$V(g(x,u)) - V(x) \le -\varepsilon V(x) + \rho_1(|u|) + \rho_2(|h(x)|) \tag{17}$$

for all $(x,u) \in D$, is said to be an *exponential-decay IOSS Lyapunov function with respect to the compact set $\mathcal{A} \subset \mathbb{R}^n$* for the hybrid system $\mathcal{H}$.

The following result relates IOSS and exponential-decay IOSS Lyapunov functions.

Proposition 3.13: Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set and $\mathcal{H}$ be a hybrid system satisfying Assumption 2.6. The hybrid system $\mathcal{H}$ admits an IOSS Lyapunov function with respect to $\mathcal{A}$ if and only if it admits an exponential-decay IOSS Lyapunov function with respect to $\mathcal{A}$.

[2]Equivalently, (locally) 0-OSS; see, e.g., [5].

C. State-norm estimators

State-norm estimators are useful for the purposes of control when the full state is not available for measurement, but rather, a function of the state defining an output. As shown in the literature (see, e.g., [33], [18]), their existence is linked to the OSS and IOSS properties of the system. A state-norm estimator will involve a hybrid system[3]

$$\mathcal{H}_\circ \begin{cases} \dot{\zeta} = f_\circ(\zeta, u, y) & (\zeta, u, y) \in C_\circ \\ \zeta^+ = g_\circ(\zeta, u, y) & (\zeta, u, y) \in D_\circ \end{cases} \tag{18}$$

with state ζ and inputs (u, y). A hybrid system $\mathcal{H}$, with input u and output y, and the hybrid system $\mathcal{H}_\circ$ define an interconnection, which we denote $\mathcal{H}, \mathcal{H}_\circ$. A state-norm estimator is defined as follows.

Definition 3.14: A *state-norm estimator for a hybrid system $\mathcal{H}$ with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$* consist of a function $\psi : \mathbb{R}^{n_\circ} \times \mathbb{R}^p \to \mathbb{R}$ and a hybrid system $\mathcal{H}_\circ$ as in (18), with state $\zeta \in \mathbb{R}^{n_\circ}$ and input (u, y), such that

- There exist $\hat{\rho}_1, \hat{\rho}_2 \in \mathcal{K}$, and $\hat{\beta} \in \mathcal{KL}$ such that, for every $z \in \mathbb{R}^{n_\circ}$, every solution pair $(\zeta, (u, y)) \in \mathcal{S}_{\mathcal{H}_\circ}(z)$ satisfies, for all $(t,j) \in \mathrm{dom}(\zeta, (u,y))$,

$$\begin{aligned}|\psi(\zeta(t,j,z,(u,y)), y(t,j))| &\le \\ \hat{\beta}(|z|, t+j) &+ \hat{\rho}_1(\|u\|_{(t,j)}) + \hat{\rho}_2(\|y\|_{(t,j)});\end{aligned} \tag{19}$$

- There exist $\widetilde{\rho} \in \mathcal{K}$ and $\widetilde{\beta} \in \mathcal{KL}$ such that, for every $(\xi, z) \in \mathbb{R}^n \times \mathbb{R}^{n_\circ}$, every solution pair $((\phi, \zeta), u) \in \mathcal{S}_{\mathcal{H}, \mathcal{H}_\circ}(\xi, z)$ satisfies, for all $(t,j) \in \mathrm{dom}((\phi, \zeta), u)$,

$$\begin{aligned}|\phi(t,j,\xi,u)|_{\mathcal{A}} &\le \widetilde{\beta}(|\xi|_{\mathcal{A}} + |z|, t+j) \\ &\quad + \widetilde{\rho}(|\psi(\zeta(t,j,z,u), h(\phi(t,j,\xi,u)))|).\end{aligned} \tag{20}$$

Input-output-to-state stability and the existence of a state-norm estimator are related as follows.

Proposition 3.15: Let $\mathcal{A} \subset \mathbb{R}^n$ be compact and consider a hybrid system $\mathcal{H}$. The following hold:

1) *If $\mathcal{H}$ satisfies Assumption 2.6 and admits an exponential-decay IOSS Lyapunov function with respect to $\mathcal{A}$ then $\mathcal{H}$ admits a state-norm estimator with respect to $\mathcal{A}$.*
2) *If $\mathcal{H}$ admits a state-norm estimator with respect to $\mathcal{A}$ then $\mathcal{H}$ is IOSS with respect to $\mathcal{A}$.*

Proposition 3.15 permits to establish that hybrid systems admitting an IOSS Lyapunov function with respect to a compact $\mathcal{A}$ as in Definition 3.10 are IOSS with respect to $\mathcal{A}$.

Theorem 3.16: Given a hybrid system $\mathcal{H}$ satisfying Assumption 2.6, if there exists an IOSS Lyapunov function with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for $\mathcal{H}$ then $\mathcal{H}$ is IOSS with respect to $\mathcal{A}$.

[3]A family of state-norm estimators for hybrid systems was also introduced in [2, Definition 12].

IV. Interconnections of Two Hybrid Systems

We consider the interconnection of two hybrid systems, $\mathcal{H}_1$ and $\mathcal{H}_2$, given, for each $i = 1, 2$, by

$$\mathcal{H}_i \left\{ \begin{array}{rcll} \dot{x}_i &=& f_i(x_i, u_i, v_i) & (x_i, u_i, v_i) \in C_i \\ x_i^+ &=& g_i(x_i, u_i, v_i) & (x_i, u_i, v_i) \in D_i \\ y_i &=& h_i(x_i), & \end{array} \right. \quad (21)$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathcal{U}_i$ and $v_i \in \mathcal{V}_i$ are inputs, $\mathcal{U}_i \times \mathcal{V}_i \subset \mathbb{R}^m$, u_i corresponding to inputs for interconnection between the systems while v_i are exogenous inputs. The system resulting from the interconnection condition

$$u_1 = y_2, \ \ u_2 = y_1$$

is given by $\mathcal{H}_1, \mathcal{H}_2$

$$\left.\begin{array}{l} \dot{x}_1 = f_1(x_1, h_2(x_2), v_1) \\ \dot{x}_2 = f_2(x_2, h_1(x_1), v_2) \end{array}\right\} \begin{array}{l} (x_1, h_2(x_2), v_1) \in C_1 \ \& \\ (x_2, h_1(x_1), v_2) \in C_2 \end{array}$$

$$\left.\begin{array}{l} x_1^+ = g_1(x_1, h_2(x_2), v_1) \\ x_2^+ = x_2 \end{array}\right\} \begin{array}{l} (x_1, h_2(x_2), v_1) \in D_1 \ \& \\ (x_2, h_1(x_1), v_2) \notin D_2 \end{array}$$

$$\left.\begin{array}{l} x_1^+ = x_1 \\ x_2^+ = g_2(x_2, h_1(x_1), v_2) \end{array}\right\} \begin{array}{l} (x_1, h_2(x_2), v_1) \notin D_1 \ \& \\ (x_2, h_1(x_1), v_2) \in D_2 \end{array}$$

$$\left.\begin{array}{l} x_1^+ = g_1(x_1, h_2(x_2), v_1) \\ x_2^+ = g_2(x_2, h_1(x_1), v_2) \end{array}\right\} \begin{array}{l} (x_1, h_2(x_2), v_1) \in D_1 \ \& \\ (x_2, h_1(x_1), v_2) \in D_2 \end{array}$$

$$y_1 = h_1(x_1), \ \ y_2 = h_2(x_2).$$

Its state is $x := (x_1, x_2)$, its input is $v := (v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2 =: \mathcal{V}$, and its output is $y := (y_1, y_2)$. The interconnection $\mathcal{H}_1, \mathcal{H}_2$ can be written as $\mathcal{H}$ with

$$f(x, v) := [f_1(x_1, h_2(x_2), v_1)^\top \ f_2(x_2, h_1(x_1), v_2)^\top]^\top,$$

$$C := \{(x, v) \ : \ (x_1, h_2(x_2), v_1) \in C_1, (x_2, h_1(x_1), v_2) \in C_2 \ \},$$

$$\begin{array}{l} D := \{(x, v) \ : \ (x_1, h_2(x_2), v_1) \in D_1 \ \} \\ \qquad \cup \{(x, v) \ : \ (x_2, h_1(x_1), v_2) \in D_1 \ \}, \end{array}$$

$$g(x, v) := [\widetilde{g}_1(x_1, h_2(x_2), v_1)^\top \ \widetilde{g}_2(x_2, h_1(x_1), v_2)^\top]^\top,$$

where, omitting the arguments of the functions,

$$\widetilde{g}_1 := \left\{ \begin{array}{ll} g_1 & (x_1, h_2(x_2), v_1) \in D_1 \\ x_1 & \text{otherwise}, \end{array} \right.$$

$$\widetilde{g}_2 := \left\{ \begin{array}{ll} g_2 & (x_2, h_1(x_1), v_2) \in D_2 \\ x_2 & \text{otherwise}, \end{array} \right.$$

and $h(x) := [h_1(x_1)^\top \ h_2(x_2)^\top]^\top$.

In the same hybrid systems framework as above, interconnections of two hybrid systems and several hybrid systems were considered in [23] and [7], respectively, for the study of their ISS properties. Interconnections of two systems with jumps at pre-specified times and with state-triggered jumps were considered in [24] and [20], respectively.

Note that the solutions to the interconnection $\mathcal{H}_1, \mathcal{H}_2$ may have different hybrid time domains than the solutions of the individual systems $\mathcal{H}_1$ and $\mathcal{H}_2$. Furthermore, in general, there is no guarantee that solutions to $\mathcal{H}_1, \mathcal{H}_2$ are complete, or even exist.

Before we present a Lyapunov-based small gain result, we state the following key result from [14].

Lemma 4.1: Let $\chi_1, \chi_2 \in \mathcal{K}_\infty$ satisfy $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$. Then, there exists $\rho \in \mathcal{K}_\infty$ such that

1) *$\chi_1(s) < \rho(s)$ for all $s > 0$;*
2) *$\chi_2(s) < \rho^{-1}(s)$ for all $s > 0$;*
3) *ρ is continuously differentiable on $(0, \infty)$ and $\frac{d\rho}{ds}(s) > 0$ for all $s > 0$.*

Let X_1, X_2, and X be the projection of the closure of $C \cup D \cup (g(D, \mathcal{V}) \times \mathcal{V})$ onto $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and $\mathbb{R}^n$, respectively. Below, for a locally Lipschitz function V, $V^\circ(x, w)$ denotes the Clarke generalized derivative of V at x in the direction w [6].

Theorem 4.2: Suppose that for $i = 1, 2$ there exist continuously differentiable functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ such that:

A) *There exist functions $\alpha_{i1}, \alpha_{i2} \in \mathcal{K}_\infty$ and $\phi_{i1}, \phi_{i2} : \mathbb{R}^{n_i} \to \mathbb{R}^{p_i}$ such that for all $x_i \in X_i$*

$$\alpha_{i1}(|\phi_{i1}(x_i)|) \ \leq \ V_i(x_i) \ \leq \ \alpha_{i2}(|\phi_{i2}(x_i)|) \quad (22)$$

B) *There exist functions $\chi_i, \gamma_i, \varphi_i \in \mathcal{K}_\infty$, positive definite functions α_i and λ_i satisfying $\lambda_i(s) < s$ for all $s > 0$ such that:*
   - *For all $(x, v) \in C$ and $V_1(x_1) \geq \max\{\chi_1(V_2(x_2)), \gamma_1(|v_1|), \varphi_1(|h_1(x_1)|)\}$:*

   $$\langle \nabla V_1(x_1), f_1(x_1, h_2(x_2), v_1) \rangle \leq -\alpha_1(V_1(x_1)) \quad (23)$$

   and, for all $(x, v) \in C$ and $V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(|v_2|), \varphi_2(|h_2(x_2)|)\}$:

   $$\langle \nabla V_2(x_2), f_2(x_2, h_1(x_1), v_2) \rangle \leq -\alpha_2(V_2(x_2)) \quad (24)$$

   - *For all $(x, v) \in D$ we have*

   $$\begin{array}{rl} V_1(\widetilde{g}_1(x_1, h_2(x_2), v_1)) \leq & \max\{\lambda_1(V_1(x_1)), \\ & \chi_1(V_2(x_2)), \gamma_1(|v_1|), \varphi_1(|h_1(x_1)|)\} \end{array} \quad (25)$$

   and

   $$\begin{array}{rl} V_2(\widetilde{g}_2(x_2, h_1(x_1), v_2)) \leq & \max\{\lambda_2(V_2(x_2)), \\ & \chi_2(V_1(x_1)), \gamma_2(|v_2|), \varphi_2(|h_2(x_2)|)\}. \end{array} \quad (26)$$

C) *The following holds*

$$\chi_1 \circ \chi_2(s) < s \qquad \forall s > 0. \quad (27)$$

Let $\rho \in \mathcal{K}_\infty$ be generated by Lemma 4.1 using χ_1, χ_2 and

$$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}. \quad (28)$$

Then:

1) *There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in X$,*

$$\begin{array}{rl} \alpha_1(|(\phi_{11}(x_1), \phi_{21}(x_2))|) & \leq \ V(x) \\ & \leq \ \alpha_2(|(\phi_{12}(x_1), \phi_{22}(x_2))|); \end{array} \quad (29)$$

2) *There exist a positive definite function α, and functions $\widetilde{\gamma}_1, \widetilde{\varphi}_1 \in \mathcal{K}$ such that, for all $(x, v) \in C$ and $V(x) \geq \max\{\widetilde{\gamma}_1(|v|), \widetilde{\varphi}_1(|h(x)|)\}$, we have*

$$V^\circ(x, f(x, v)) \leq -\alpha(V(x)); \quad (30)$$

3) *There exist a positive definite function λ, $\lambda(s) < s$ for all $s > 0$, and functions $\widetilde{\gamma}_2, \widetilde{\varphi}_2 \in \mathcal{K}$ such that, for all $(x,v) \in D$,*

$$V(g(x,v)) \leq \max\{\lambda(V(x)), \widetilde{\gamma}_2(|v|), \widetilde{\varphi}_2(|h(x)|)\}. \quad (31)$$

Remark 4.3: The proof of Theorem 4.2 uses the ideas of the proof of [15, Theorem 3.1] for continuous-time systems, which have been recently exploited to establish small gain results for ISS hybrid systems in [23, Theorem 2.1] and [7, Theorem 3.1]. Note that Theorem 4.2 does not provide a smooth function V. It allows for different subsystems with different compact sets of interest through ϕ_{i1}, ϕ_{i2}. The following special cases are of interest:

i) For each $i = 1, 2$, $\phi_{i,1} = \phi_{i,2}$ and $\varphi_i \equiv 0$.
ii) For each $i = 1, 2$, $\phi_{i,1} = h_i$ and $\phi_{i,2}$ is the identity function.
iii) For each $i = 1, 2$, $\phi_{i,1}$ and $\phi_{i,2}$ are given by identity functions.

The special case i) coincides with [23, Theorem 2.1]. The conditions in Theorem 4.2 for cases ii) and iii) are in terms of IOS and IOSS Lyapunov functions in the forms given by (5)-(6) and (15)-(14), respectively. Note that the interconnection $\mathcal{H}_1, \mathcal{H}_2$ does not rule out the possibility of solutions that are discrete, or that eventually, after some (t,j), become discrete. This includes interconnections having solutions with one of the state components, say x_2, constant through the second option in the definition of the update law $\widetilde{g}_2$, for which it would be difficult to satisfy conditions (25),(26).

V. Conclusion

For a general class of hybrid systems, we presented input-to-output and input-output-to-state stability notions, sufficient conditions for those to hold, and a small gain result for the study of an interconnection of two such systems. The nature of the results and the general hybrid systems framework under study, which cover classical continuous and discrete-time systems, suggest wide applicability of the new tools.

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References

[1] D. Angeli, B. Ingalls, E. D. Sontag, and Y. Wang. Separation principles for input-output and integral-input-to-state stability. *SIAM J. Control Optim.*, 43:256–276, 2004.
[2] C. Cai and A. R. Teel. A globally detectable hybrid system admits a smooth OSS-Lyapunov function. In *American Control Conference, 2006*, pages 4049–4054, 2006.
[3] C. Cai and A. R. Teel. Input-output-to-state stability for discrete-time systems. *Automatica*, 44:326–336, 2008.
[4] C. Cai and A. R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58:47–53, 2009.
[5] C.Cai and A. R. Teel. Asymptotic characterization of input-output-to-state stability for discrete-time systems. *Sys. & Cont. Letters*, 56:408–415, 2007.
[6] F.H. Clarke. *Optimization and Nonsmooth Analysis*. SIAM's Classic in Applied Mathematics, 1990.
[7] S. Dashkovskiy and M. Kosmykov. Stability of networks of hybrid ISS systems. In *Proc. Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, 2009.
[8] C. A. Desoer and M. Vidyasagar. *Feedback systems: input-output properties*. New York, Academic Press, 1975.
[9] R. Goebel, J.P. Hespanha, A.R. Teel, C. Cai, and R.G. Sanfelice. Hybrid systems: generalized solutions and robust stability. In *Proc. 6th IFAC Symposium in Nonlinear Control Systems*, pages 1–12, 2004.
[10] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, pages 28–93, 2009.
[11] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573–587, 2006.
[12] A. Isidori. *Nonlinear Control Systems II*. Springer-Verlag, London, 1999.
[13] Z. Jiang and Y. Wang. Small gain theorems on input-to-output stability. *Dynamics of Continuous Discrete and Impulsive Systems, Series B–Applications & Algorithms*, pages 220–224, 2003.
[14] Z.-P. Jiang, A.R.Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Syst.*, 7:95–120, 1994.
[15] Z. P. Jiang, I. M. Y. Mareels, and Y. Wang. A lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica-London*, 32(8):1211–1216, 1996.
[16] Z-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.
[17] H.K. Khalil. *Nonlinear Systems 3rd. Edition*. Prentice Hall, 2002.
[18] M. Krichman, E. D. Sontag, and Y. Wang. Input-output-to-state stability. *SIAM J. Control Optim.*, 39:1874–1928, 2000.
[19] D. S. Laila and D. Nesic. Lyapunov-based small-gain theorem for parameterized discrete-time interconnected iss systems. *IEEE Transactions of Automatic Control*, 48:1783–1788, 2003.
[20] D. Liberzon and D. Nesic. Stability analysis of hybrid systems via small-gain theorems. In *Hybrid systems: Computation and Control*, pages 421–435. Springer, 2006.
[21] J. L. Mancilla-Aguilar and R. A. Garcia. On converse Lyapunov theorems for ISS and iISS switched nonlinear systems. *Systems & Control Letters*, 42(1):47–53, 2001.
[22] M. A. Muller and D. Liberzon. Input/output-to-state stability of switched nonlinear systems. In *To appear in Proc. 2010 American Control Conference*, 2010.
[23] D. Nesic and A. R. Teel. A lypaunov-based small-gain theorem for hybrid ISS systems. In *Proc. 47th IEEE Conference on Decision and Control*, 2008.
[24] D. Nesic and A.R. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49:1650–1667, 2004.
[25] R.G. Sanfelice, R. Goebel, and A.R. Teel. Generalized solutions to hybrid dynamical systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(4):699–724, 2008.
[26] E. Sontag and Y. Wang. Lyapunov characterizations of input to output stability. *SIAM J. Control Optimization*, 39:226–249, 2000.
[27] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34:435–443, 1989.
[28] E. D. Sontag. On the input-to-state stability property. *European J. Control*, 1:24–36, 1995.
[29] E. D Sontag. Input to state stability: Basic concepts and results. *Lecture notes in Mathematics, Springer-Verlang*, 1932:163, 2008.
[30] E. D. Sontag and B. Ingalls. A small-gain theorem with applications to input/output systems, incremental stability, detectability, and interconnections. *Journal of the Franklin Institute*, 339:211–229, 2002.
[31] E. D. Sontag and M. Krichman. An example of a gas system which can be destabilized by an integrable perturbation. *IEEE Transactions on Automatic Control*, 46(6):1046–1049, 2003.
[32] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.
[33] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Sys. & Cont. Letters*, 29:279–290, 1997.
[34] E. D. Sontag and Y. Wang. Notions of input to output stability. *Sys. & Cont. Letters*, 38:235–248, 1999.
[35] A. R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control*, 41(9):1256–1270, September 1996.
[36] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43:639–646, 2007.