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INTERCONNECTIONS OF HYBRID SYSTEMS: SOME CHALLENGES AND RECENT RESULTS

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Abstract. This paper provides an overview of the problem of interconnecting hybrid systems. Hybrid systems are given in terms of constrained differential and difference equations with inputs and outputs. Issues on existence of solutions and mismatch of time domains of their solutions are discussed. An input/output stability notion for such systems is presented and a small gain theorem for analyzing interconnections of hybrid systems is presented.

Keywords. hybrid systems, interconnections, input/output notions, stability.

1 Introduction

A supervisory algorithm selecting the most appropriate control law for the current operating conditions of a plant, an event-driven transmission of information between networked agents, and an (almost) instantaneous change of velocities in mechanical systems with impacts are just a few representative systems that can be studied using hybrid systems theory. The prevalent combination of continuous and discrete dynamics in systems in science and engineering makes hybrid systems a powerful modeling framework and setting for analysis, design, and simulation.

The understanding of the dynamical properties of hybrid systems has been possible using machinery inspired from control theory, as it has been demonstrated in [45, 27, 13]. Particular attention has been paid to the analysis of *closed hybrid systems*, that is, those without inputs. Tools for the analysis of such systems have permitted the study of Lyapunov stability, convergence, and robustness properties. These are particularly useful for the analysis of the closed-loop systems resulting from feedback control, in which the required properties of the control algorithm are inferred from the desired properties of the closed-loop system. *Open hybrid systems* are systems with inputs, such as disturbances and control inputs, and outputs. Recent results on input-to-state stability for hybrid systems [7] permit the study of bounds on the state trajectories in terms of the initial conditions and inputs, similarly to their counterparts for continuous-time systems [40], discrete-time systems [21], and switched systems [28, 44].

Tools for closed systems are applicable for the analysis of interconnected systems, but typically do not scale with the size of the interconnection. The understanding of the properties of open systems are particularly useful in the study of the properties of their interconnections. For instance, certain properties conferred by a controller to a closed-loop control system can be determined from the properties of the individual (interconnected) plant and controller. Recent results reported in [31] and [10] show that the small gain theorem in [20] for continuous-time systems can be formulated in the hybrid setting to assert that interconnections of input-to-state stable hybrid systems are input-to-state stable. This motivates the study of notions relating hybrid inputs and outputs, such as input-output boundedness [11, 22] and input-to-output stability [41, 37], as well as the generation of tools for the analysis of input/output properties of interconnections of hybrid systems.

The purpose of this paper is to provide an overview of the problem of interconnecting hybrid systems and to present some recent results for input/output analysis. More precisely, given hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with certain stability properties in an input/output (I/O) sense, we explore tools for the analysis of interconnections of hybrid systems, such as those in Figure 1. To this end, we introduce, in a tutorial tone, a framework for hybrid systems in Section 2 and key issues for the study of their interconnections in Section 3. Then, in Sections 4 and 5, we present results on input-to-output stability and a small gain theorem for the analysis of input-to-output stable interconnections. Examples throughout the paper illustrate the ideas.

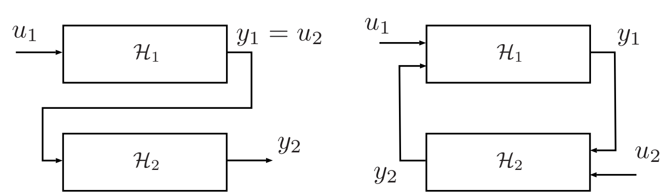

Figure 1: Series and feedback interconnections of two hybrid systems.

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2 Hybrid Systems

Hybrid systems combine continuous behavior, which is prevalent in physical systems, with discrete behavior, which is ubiquitous in digital and impulsive systems. Several working frameworks have been proposed in the literature. These include the automaton approach [16, 5, 43, 27] initiated in the computer science literature; switching systems [4, 25] and impulsive systems [2, 15] motivated by the field of dynamics and control systems; and more abstract mathematical descriptions such as measure-driven differential equations and inclusions [30, 36, 6], dynamics on time scales [3, 29], and differential inclusions [1, 9]. In this paper, the continuous behavior in hybrid systems is modeled by constrained differential equations while the discrete behavior is modeled by constrained difference equations.[1] In this setting, the continuous dynamics define the *flows* of the system while the discrete dynamics define the *jumps*. Flows and jumps are activated under certain conditions, which are implemented as constraints on the state and input of the system. More precisely, a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^n$, input $u \in \mathcal{U} \subset \mathbb{R}^m$, and output $y \in \mathbb{R}^p$ will be modeled as

$$\mathcal{H} \left\{ \begin{array}{rcll} \dot{x} & = & f(x,u) & (x,u) \in C \\ x^+ & = & g(x,u) & (x,u) \in D \\ y & = & h(x). & \end{array} \right. \tag{1}$$

The set $C \subset \mathbb{R}^n \times \mathcal{U}$ defines the *flow set*, which is the set of points (x,u) on which flows according to $\dot{x} = f(x,u)$ are allowed, where the function $f : C \to \mathbb{R}^n$ is the *flow map*. Similarly, the set $D \subset \mathbb{R}^n \times \mathcal{U}$ defines the *jump set*, which is where jumps are permitted. At jumps, the state x is updated via the difference equation $x^+ = g(x,u)$, where $g : D \to \mathbb{R}^n$ is the *jump map*. The output of the system is defined by the function $h : \mathbb{R}^n \to \mathbb{R}^p$, called the *output map*. Then, the data of a hybrid system $\mathcal{H}$ is given by (C, f, D, g, h).

The parameterization of the state trajectories or solutions to $\mathcal{H}$ will be symmetric and given by two parameters. The parameter t will take values in $\mathbb{R}_{\geq 0}$, which denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$, and will be used to keep track of the flows. A parameter j will take values in $\mathbb{N}$, which denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$, an will keep track of the number of jumps. Then, the "time domain" of the solutions to $\mathcal{H}$ will be subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ with appropriate structure, which are called *hybrid time domains*. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Solutions to $\mathcal{H}$ will be given by pairs of hybrid arcs and a hybrid inputs. A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a hybrid arc if $\operatorname{dom} x$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto x(t,j)$ is absolutely continuous on the interval $\{t \ : \ (t,j) \in \operatorname{dom} x \ \}$. A function $u : \operatorname{dom} u \to \mathcal{U}$ is a hybrid input if $\operatorname{dom} u$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t \ : \ (t,j) \in \operatorname{dom} u \ \}$. We will employ the following signal norm for general hybrid signals, in particular, for hybrid arcs and hybrid inputs. Denoting the Euclidean vector norm by $|\cdot|$, given a hybrid signal $r : \operatorname{dom} r \to \mathbb{R}^n$, let

$$\Gamma(r) := \{(t,j) \in \operatorname{dom} r \ : \ (t, j+1) \in \operatorname{dom} r \ \}.$$

Its $\mathcal{L}_\infty$ norm is given by

$$\|r\|_{(t,j)} \ := \ \max \left\{ \operatorname*{ess\,sup}_{(t',j') \in \operatorname{dom} r \setminus \Gamma(r), \ t'+j' \leq t+j} |r(t',j')|, \ \sup_{(t',j') \in \Gamma(r), \ t'+j' \leq t+j} |r(t',j')| \right\}.$$

For notational convenience, $\|r\|$ denotes $\lim_{t+j \to N} \|r\|_{(t,j)}$, where $N = \sup_{(t,j) \in \operatorname{dom} r} t + j \in [0, +\infty]$.

With the above definitions, given a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ and an initial condition ξ, a hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ with $\phi(0,0) = \xi$ defines a *solution pair* (ϕ, u) *to the hybrid system* $\mathcal{H}$ if the following conditions hold:[2]

(S0) $(\xi, u(0,0)) \in \overline{C} \cup D$ and $\operatorname{dom} \phi = \operatorname{dom} u$;

(S1) For each $j \in \mathbb{N}$ such that $I_j := \{t \ : \ (t,j) \in \operatorname{dom}(\phi, u) \ \}$ has nonempty interior $\operatorname{int}(I_j)$,

$$(\phi(t,j), u(t,j)) \in C \text{ for all } t \in \operatorname{int}(I_j),$$

and, for almost all $t \in I_j$,

$$\frac{d}{dt}\phi(t,j) = f(\phi(t,j), u(t,j));$$

(S2) For each $(t,j) \in \operatorname{dom}(\phi, u)$ such that $(t, j+1) \in \operatorname{dom}(\phi, u)$,

$$\begin{array}{l} (\phi(t,j), u(t,j)) \in D \\ \phi(t, j+1) = g(\phi(t,j), u(t,j)). \end{array}$$

Above, $\operatorname{int}(I_j)$ denotes the interior of the interval I_j.

A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, u)$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}(\phi, u)$ onto $\mathbb{R}_{\geq 0}$ is bounded, *discrete* if

[1] We present the ideas in terms of equations, but they also extend to differential and difference inclusions.

[2] Given a set S, $\overline{S}$ denotes its closure.

its domain is $\{0\} \times \mathbb{N}$, and *maximal* if there does not exist another pair $(\phi, u)'$ such that (ϕ, u) is a truncation of $(\phi, u)'$ to some proper subset of $\operatorname{dom}(\phi, u)'$. Given $\xi \in \mathbb{R}^n$, $\mathcal{S}_{\mathcal{H}}(\xi)$ denotes the set of maximal solution pairs (ϕ, u) to $\mathcal{H}$ with $\phi(0,0) = \xi$ and u with finite $\|u\|$. For a solution pair $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$, we denote by $\phi(t, j; \xi, u)$ its value at $(t, j) \in \operatorname{dom}(\phi, u)$.

A concept of stability for hybrid systems $\mathcal{H}$ is introduced next. It is stated for general compact sets $\mathcal{A}$ of the state space. For a given set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution pair $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$ with $|\xi|_{\mathcal{A}} \le \delta$ satisfies $|\phi(t, j; \xi, u)|_{\mathcal{A}} \le \varepsilon$ for all $(t, j) \in \operatorname{dom}(\phi, u)$;
- *0-input stable* if it is *stable* with $u \equiv 0$;
- *pre-attractive* if there exists $\mu > 0$ such that every solution pair $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$ with $|\xi|_{\mathcal{A}} \le \mu$ is bounded and if it is complete satisfies $\lim_{(t,j) \in \operatorname{dom}(\phi,u), t+j \to \infty} |\phi(t, j; \xi, u)|_{\mathcal{A}} = 0$;
- *0-input pre-attractive* if it is *pre-attractive* with $u \equiv 0$;
- *pre-asymptotically stable* if stable and pre-attractive;
- *0-input pre-asymptotically stable* if 0-input stable and 0-input pre-attractive.

The following mild assumptions on the data of $\mathcal{H}$ will be imposed in some of the results in this paper. In such results, the data (C, f, D, g, h) of the hybrid system $\mathcal{H}$ will satisfy

(A1) C, D, and $\mathcal{U}$ are closed sets,

(A2) $f : C \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuous.

These conditions assure that closed hybrid systems $\mathcal{H}$ are well posed in the sense that they inherit several good structural properties of their solution sets. These include sequential compactness of the solution set, closedness of perturbed and unperturbed solutions, among others. We refer the reader to [13, 14] (see also [12]) and [35] for details on and consequences of these conditions.

3 On Interconnections of Hybrid Systems

An interconnection of several hybrid systems consists of an assignment relating the inputs and outputs of the individual systems. For example, the series interconnection in Figure 1 corresponds to the assignment

$$u_2 = y_1 \tag{2}$$

while the feedback interconnection therein corresponds to

$$\widetilde{u}_1 = y_2 \quad \text{and} \quad \widetilde{u}_2 = y_1, \tag{3}$$

where the input to the first system is decomposed as $[u_1^\top \ \widetilde{u}_1^\top]^\top$ and the input to the second system as $[u_2^\top \ \widetilde{u}_2^\top]^\top$. (For notational simplicity, at times we will use $(u_1, \widetilde{u}_1)$ instead of $[u_1^\top \ \widetilde{u}_1^\top]^\top$.) The dynamics of the resulting interconnection are governed by the "composed" dynamics of the individual systems. Since $\mathcal{H}_1$ allows flows from points x_1 that belong to C_1 (for given input) and $\mathcal{H}_2$ allows flows from points x_2 that belong to C_2, flows of the entire interconnection are only allowed when both conditions hold. Jumps of $\mathcal{H}_1$ are possible from points x_1 that belong to D_1 while jumps of $\mathcal{H}_2$ are possible from points x_2 that belong to D_2 (for given input). When jumps of $\mathcal{H}_1$ occur, x_1 is updated via the jump map g_1 while the update of x_2 depends on whether x_2 belongs to D_2 or not (for given input). Since updates of x_2 via g_2 are only allowed on D_2, then the jump map has to capture all the possibilities for the update of x_1 and x_2. Then, for the hybrid system $\mathcal{H}_1$ and $\mathcal{H}_2$ interconnected via (3), the resulting interconnection is denoted by $\mathcal{H}_1, \mathcal{H}_2$ and has dynamics

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, h_2(x_2), u_1) \\ \dot{x}_2 &= f_2(x_2, h_1(x_1), u_2) \end{aligned}\right\} \begin{aligned} &(x_1, h_2(x_2), u_1) \in C_1 \ \& \\ &(x_2, h_1(x_1), u_2) \in C_2 \end{aligned}$$

$$\left.\begin{aligned} x_1^+ &= g_1(x_1, h_2(x_2), u_1) \\ x_2^+ &= x_2 \end{aligned}\right\} \begin{aligned} &(x_1, h_2(x_2), u_1) \in D_1 \ \& \\ &(x_2, h_1(x_1), u_2) \notin D_2 \end{aligned}$$

$$\left.\begin{aligned} x_1^+ &= x_1 \\ x_2^+ &= g_2(x_2, h_1(x_1), u_2) \end{aligned}\right\} \begin{aligned} &(x_1, h_2(x_2), u_1) \notin D_1 \ \& \\ &(x_2, h_1(x_1), u_2) \in D_2 \end{aligned}$$

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$$\begin{aligned} y_1 &= h_1(x_1) \\ y_2 &= h_2(x_2). \end{aligned}$$

Its state is $x := (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, its input is $u := (u_1, u_2) \in \mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$, and its output is $y := (y_1, y_2)$. The interconnection $\mathcal{H}_1, \mathcal{H}_2$ can be written as $\mathcal{H}$ with

$$\begin{aligned} C \;:=\; & \{(x, u) \ : \ (x_1, h_2(x_2), u_1) \in C_1\} \\ & \cap \{(x, u) \ : \ (x_2, h_1(x_1), u_2) \in C_2\}, \end{aligned}$$

$$f(x, u) := [f_1(x_1, h_2(x_2), u_1)^\top \ f_2(x_2, h_1(x_1), u_2)^\top]^\top,$$

$$\begin{aligned} D \;:=\; & \{(x, u) \ : \ (x_1, h_2(x_2), u_1) \in D_1 \ \} \\ & \cup \{(x, u) \ : \ (x_2, h_1(x_1), u_2) \in D_2 \ \}, \end{aligned}$$

$$g(x, u) \;:=\; [\widetilde{g}_1(x_1, h_2(x_2), u_1)^\top \ \widetilde{g}_2(x_2, h_1(x_1), u_2)^\top]^\top,$$

where

$$\widetilde{g}_1 := \begin{cases} g_1(x_1, h_2(x_2), u_1) & (x_1, h_2(x_2), u_1) \in D_1 \\ x_1 & \text{otherwise,} \end{cases}$$

$$\widetilde{g}_2 := \begin{cases} g_2(x_2, h_1(x_1), u_2) & (x_2, h_1(x_1), u_2) \in D_2 \\ x_2 & \text{otherwise,} \end{cases}$$

and $h(x) := (h_1(x_1), h_2(x_2))$. As discussed above, the data of the interconnection $\mathcal{H}_1, \mathcal{H}_2$ is such that flows occur when the flow conditions imposed by $\mathcal{H}_1$ and by $\mathcal{H}_2$ hold simultaneously, while jumps occur when either the jump condition of $\mathcal{H}_1$ or $\mathcal{H}_2$, or both hold. The construction of the jump map g and the functions $\widetilde{g}_1$ and $\widetilde{g}_2$ are such that x_1 is updated via g_1 only when the jump condition imposed by $\mathcal{H}_1$ holds; similarly for the update of x_2.

As already suggested by the interconnection $\mathcal{H}_1, \mathcal{H}_2$ and illustrated next, interconnected hybrid systems can have totally different dynamics (and solutions) from the individual hybrid systems. This is due to the coupling through the interconnection assignment of the flows and jumps of the individual systems.

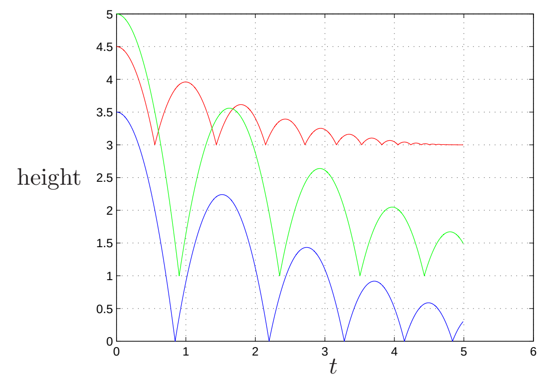

Figure 2: Three bouncing balls with height of floor u at 3 (top), 1 (middle), and 0 (bottom).

3.1 Solutions stop at Zeno time

The interconnection between several hybrid systems may have Zeno solutions that were not part of the set of solutions to every subsystem. The simplest case is perhaps a vacuous interconnection between a system with Zeno solutions and a system with continuous complete solutions. By "vacuous interconnection" we mean that the systems are both in the same model, but do not share inputs and outputs. The time domain of the solutions of the "interconnection" may be dramatically different from those of the individual systems. For instance, consider a ball bouncing on a floor at a height determined by u, with state $x = [x_1 \ x_2]^\top \in \mathbb{R}^2$, where x_1 denotes the height of the ball relative to the floor and x_2 the vertical velocity. The data of the associated hybrid system is given by

$$\begin{aligned} C &= \{(x,u) \ : \ x_1 \geq u\}, \\ f(x,u) &= \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \qquad \forall (x,u) \in C, \\ D &= \{(x,u) \ : \ x_1 = u, x_2 \leq 0\}, \\ g(x,u) &= \begin{bmatrix} x_1 \\ -e x_2 \end{bmatrix} \qquad \forall (x,u) \in D, \\ h(x) &= x_1 \qquad \forall x \in \mathbb{R}^2, \end{aligned} \tag{4}$$

where $\gamma > 0$ is the gravity constant and $e \in (0,1)$ is the restitution coefficient. It is easy to check that every solution pair $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$, to this system with u constant and $\xi_1 > u$ is complete and has a time domain such that

$$\sup\{t \ : \ (t,j) \in \operatorname{dom}\phi\} = \frac{\xi_2 + \sqrt{\xi_2^2 + 2\gamma(\xi_1 - u)}}{\gamma} + \frac{2e\sqrt{\xi_2^2 + 2\gamma(\xi_1 - u)}}{\gamma(1-e)}, \tag{5}$$

which is known as the Zeno time. This hybrid system will prevent any purely continuous-time system jointly modeled with it from having solutions that are defined for all $t \in [0, +\infty)$. An example of the (vacuous) interconnection between two such systems is the model of a bouncing ball as in (4) and a timer counting the elapsed flow time

$$\dot{\tau} = 1 \qquad \tau \geq 0.$$

The timer system alone has complete solutions with hybrid time domain $[0, +\infty) \times \{0\}$, but when interconnected with the bouncing ball inherits Zeno.

Another example of an interconnection leading to Zeno solutions that do not belong to the set of solutions of all of the individual subsystems is given by a model of multiple bouncing balls. When the balls start from different initial conditions or when they bounce at different floor heights, the solution to such model may not allow all of the balls to reach their own Zeno time as the original model of each bouncing ball would. Figure 2 shows a simulation of the heights of three bouncing balls modeled as in (4), up to the Zeno time of one of the balls, which is at around 5 sec, where infinitely many impacts accumulate. Note that the other two balls are still far from being at rest and, hence, far from their own Zeno times.

Interconnections generating "new" Zeno solutions are possible in the context of hybrid control of continuous-time plants. Consider a distributed network system with N nodes or agents in which each agent has an internal state $\xi_i \in \mathbb{R}^m$ storing some value with continuous dynamics $\dot{\xi}_i = u_i^1$ and discrete dynamics $\xi_i^+ = u_i^2$, and a controller designed with the goal of achieving consensus among those internal values, i.e., guarantee that the internal states satisfy $\xi_i = \xi_j$ for all i, j. Suppose that to accomplish this task, a sample-and-hold value of the neighbors' state is available to each agent, that is, the i-th agent has sample-and-hold values z_k available to it, where $k \in N_i \subset \{1, 2, \dots, N\}$ defines the index of the neighbors to the i-th agent. A simple hybrid control algorithm for consensus that, during flows follows [33] leads to continuous dynamics of the form

$$u_i^1 = -\sum_{k \in N_i} (\xi_i - z_k). \tag{6}$$

and at jumps follows [23, 46]

$$\xi_i^+ = z_i^+ = u_i^2, \quad u_i^2 = \xi_i - \alpha \sum_{k \in N_i} (\xi_i - z_k) \quad \text{for each } i, \tag{7}$$

where α is a positive constant. When the timer state τ in the network reaches a threshold, the internal states and their samples are reset to the same value given by a function of the local information. Otherwise, flows are allowed. Let $x := [\xi_1, z_1, \xi_2, z_2, \ldots, \xi_N, z_N, \tau]^\top$. A general construction of such sets is given by

$$C := \{x \ : \ \tau \le \varphi(x) \ \}, \qquad D := \{x \ : \ \tau \ge \varphi(x) \ \}. \quad (8)$$

Figure 3 shows the solutions for the case of four agents ($N = 4$) with $N_1 = \{2\}$, $N_2 = \{1, 3, 4\}$, $N_3 = \{2, 4\}$, and $N_4 = \{2, 3\}$ for which the solutions are Zeno. Evidently, the signal generated by the controller does not permit the agents from having solutions that exist for all $t \in [0, +\infty)$.

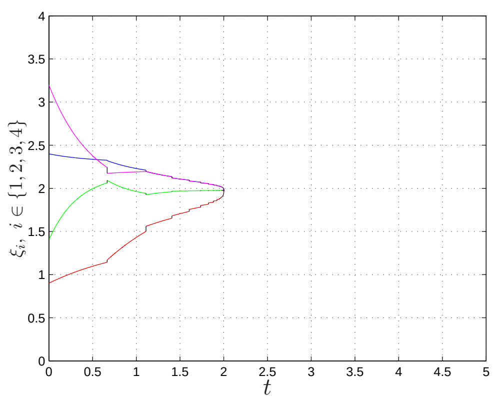

Figure 3: Solution with a simple hybrid consensus algorithm controlling four continuous-time agents.

3.2 Interconnection with empty sets

A more dramatic situation may emerge due to interconnecting hybrid systems having solutions with radically different time domains, such as continuous and discrete. For instance, the dynamics of a continuous-time system

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, u_1) \\ y_1 & = & h_1(x_1) \end{array} \quad (9)$$

can be modeled as a hybrid system $\mathcal{H}_1$ with sets

$$C_1 = \mathbb{R}^{n_1} \times \mathbb{R}^{m_1}, \quad D_1 = \emptyset \quad (10)$$

and an arbitrary function g_1. A purely discrete-time system

$$\begin{array}{rcl} x_2^+ & = & \widetilde{g}_2(x_2, u_2) \\ y_2 & = & h_2(x_2) \end{array} \quad (11)$$

with state $x_2 \in \mathbb{R}^{n_2}$ can be modeled as a hybrid system $\mathcal{H}_2$ with sets

$$C_2 = \emptyset, \quad D_2 = \mathbb{R}^{n_2} \times \mathbb{R}^{m_2}, \quad (12)$$

$g_2 = \widetilde{g}_2$, and an arbitrary function f_2. The feedback interconnection between these two systems, which may correspond to the situation where a discrete-time algorithm measures the output of a continuous-time system and updates its input, i.e., sample-and-hold control, is such that there are no nontrivial solutions. While, for any hybrid input, basic regularity properties of f_1 and g_2 would guarantee that the solutions to $\mathcal{H}_1$ are continuous and complete and the solutions to $\mathcal{H}_2$ are discrete and complete (see [14, Proposition 2.4]), the solutions to the interconnection can only be discrete. This is due to the fact that the sets of the resulting interconnection will be

$$C = \emptyset, \qquad D = \{(x_1, x_2) \ : \ (x_2, h_1(x_1)) \in D_2 \ \}. \quad (13)$$

3.3 Interconnection with not meaningful solutions

The rather simple illustrations above suggest that the solutions of the individual systems may not necessarily be preserved in the resulting interconnection. In fact, as indicated above by the vacuous interconnection of a hybrid system having Zeno solutions and a purely continuous-time system, it is not the case that the set of solutions of the interconnection "contains" the set of solutions of the individual hybrid systems. As a consequence, it is necessary to check whether the set of solutions of the interconnection is meaningful, and if not, redefine the interconnection conditions or the models of the subsystems. For instance, if the interconnection of the hybrid system $\mathcal{H}_1$ in (9)-(10) with the hybrid system $\mathcal{H}_2$ in (11)-(12) was supposed to model the closed-loop system resulting from controlling a continuous-time plant with a discrete-time algorithm, then its set of solutions does not capture the desired behavior. One way to obtain an interconnection capturing the desired behavior is to remove the undesired discrete solutions by augmenting the state of the control algorithm model with a timer and redefining its sets to trigger jumps with intervals of flow time of nonzero length in between. More precisely, the state will be given by $x_2 = [x_{21}^\top \ x_{22}]^\top$, where x_{21} is the state of the control algorithm and x_{22} is the timer. The continuous dynamics of the system will be given by

$$\begin{array}{rcl} \dot{x}_{21} & = & 0 \\ \dot{x}_{22} & = & 1 \end{array} \quad (14)$$

when the timer state is no larger than, say, a given threshold T, that is,

$$C_2' = \{(x_2, u_2) \ : \ x_{22} \in [0, T] \ \}.$$

The discrete dynamics of the system will be given by

$$\begin{array}{rcl} x_{21}^+ & = & \widetilde{g}_2(x_{21}, u_2) \\ x_{22}^+ & = & 0 \end{array} \quad (15)$$

when the timer reaches the threshold T, that is,

$$D_2' = \{(x_2, u_2) \ : \ x_{22} = T \ \}.$$

With the output given by $y_2 = h_2(x_2)$, the resulting interconnection is given by

$$\mathcal{H}_1, \mathcal{H}_2 \left\{ \begin{array}{l} \left. \begin{array}{l} \dot{x}_1 = f_1(x_1, h_2(x_2)) \\ \dot{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \right\} (x_2, h_1(x_1)) \in C_2' \\ \left. \begin{array}{l} x_1^+ = x_1 \\ x_2^+ = \begin{bmatrix} \widetilde{g}_2(x_2, h_1(x_1)) \\ 0 \end{bmatrix} \end{array} \right\} (x_2, h_1(x_1)) \in D_2' \end{array} \right.$$

Its solutions are complete with jumps occurring every T seconds (except the first jump if $x_{22}(0,0) \neq 0$).

The examples above suggest that solutions may radically change when systems are interconnected. Conditions for existence of solutions to interconnections can be readily obtained from [14, Proposition 2.4], which require viability properties of flows and jumps to hold. Unfortunately, as suggested by the examples above, the satisfaction of those conditions by the individual hybrid systems does not imply that they hold for the resulting interconnection.

4 Input-to-Output Stability for Hybrid Systems

Classical stability notions relating inputs and outputs of dynamical systems are aimed at guaranteeing that "well-behaved" inputs generate "well-behaved" outputs. Different characterizations of "well-behaved" signals have been proposed in the literature, involving boundedness, integrability, and convergence properties of the signals and using a diversity of norms; see, e.g., [11] and [22]. In this section, we consider the stability notion called *input-to-output stability*, IOS for short. Along with several characterizations, IOS for continuous-time systems around the origin was introduced in [41, 37] and consists of the following property:[3]

- There exist a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}$ function α such that every input $t \mapsto u(t)$ and associated solutions $t \mapsto \phi(t;\xi,u)$ satisfy

$$|h(\phi(t;\xi,u))| \leq \max\{\beta(|\xi|,t), \alpha(\|u\|)\} \qquad \forall t \geq 0. \tag{16}$$

The bound (16) implies the property that a bounded input will generate a bounded output $y = h(x)$ with overshoot depending on the initial condition. The norm of the input can be replaced by the norm up to t, sometimes denoted $\|u\|_{[0,t]}$. Note that, in principle, the property is required to hold for all $t \geq 0$, which implicitly requires the solutions to be complete. Also note that when the output function is the identity the definition of IOS reduces to input-to-state stability as defined in [38].

The purpose of the remainder of this section is to present an extension of the notion of IOS for hybrid systems $\mathcal{H}$ and provide Lyapunov characterizations. The new notion will be used in the analysis of interconnections of hybrid systems in the next section.

4.1 Definitions and Basic Properties

Input-to-output stability for hybrid systems $\mathcal{H} = (C, f, D, g, h)$ is defined with respect to compact sets $\mathcal{A}$ and without insisting on completeness of solutions. It is assumed that, given a compact set $\mathcal{A} \subset \mathbb{R}^n$, the output function $h : \mathbb{R}^n \to \mathbb{R}^p$ is such that $h(x) = 0$ for all $x \in \mathcal{A}$. Following the definition of IOS for continuous-time systems in [41, 37], a hybrid system $\mathcal{H}$ is said to be input-to-output stable with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ if there exist a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}$ function γ such that, for each $\xi \in \mathbb{R}^n$, each $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$ satisfies

$$|h(\phi(t,j;\xi,u))| \quad \leq \quad \max\left\{\beta(|\xi|_\mathcal{A}, t+j), \gamma(\|u\|_{(t,j)})\right\} \tag{17}$$

for all $(t,j) \in \operatorname{dom}(\phi, u)$. When the function h is given by the identity and $|h(\cdot)|$ is replaced by[4] $|\cdot|_\mathcal{A}$, then IOS reduces ISS as in [7, Definitions 2.1 and 2.3].

It is expected that under mild assumptions on the data of the hybrid system, asymptotic stability with zero input guarantees the IOS property to hold locally. A local version of the IOS property above consists of the existence of $\delta > 0$, a class-$\mathcal{KL}$ function β, and a class-$\mathcal{K}$ function α such that, for each $\xi \in \mathbb{R}^n$ with $|\xi|_\mathcal{A} \leq \delta$ and each solution pair $(\phi, u) \in \mathcal{S}_\mathcal{H}(\xi)$, $\|u\| \leq \delta$, we have that (17) holds for all $(t,j) \in \operatorname{dom}(\phi, u)$. With this definition, it follows that asymptotic stability with zero inputs guarantees that the IOS property holds for inputs with small enough size. It parallels the ISS results in [40, Lemma I.2] for continuous-time systems and [7, Proposition 2.3] for hybrid systems. In fact:

($\star$) Given a hybrid system $\mathcal{H}$ satisfying (A1)-(A2), if the compact set $\mathcal{A}$ is 0-input pre-asymptotically stable for $\mathcal{H}$ and there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$|h(x)| \leq \alpha_1(|x|_\mathcal{A}) + \alpha_2(|u|) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathcal{U}, \tag{18}$$

then $\mathcal{H}$ is locally IOS with respect to $\mathcal{A}$.

The output bound (18) combined with input-to-state stability implies input-to-output stability. In fact, input-to-state stability with respect to $\mathcal{A}$ implies the existence

[3] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if it is continuous, nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \searrow 0} \beta(s,r) = \lim_{r \to \infty} \beta(s,r) = 0$.

[4] Instead of $|\cdot|_\mathcal{A}$, using a proper indicator for $\mathcal{A}$ on $\mathbb{R}^n$ would be equivalent.

of a class-$\mathcal{KL}$ function $\widetilde{\beta}$ and class-$\mathcal{K}$ function $\widetilde{\gamma}$ such that for each $\xi \in \mathbb{R}^n$, each solution pair $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)$ satisfies

$$\begin{aligned} |\phi(t,j;\xi,u)|_{\mathcal{A}} \quad &\leq \\ \max\Big\{\widetilde{\beta}(|\phi(0,0;\xi,u)|_{\mathcal{A}}, t+j), \widetilde{\gamma}(\|u\|_{(t,j)})\Big\} \end{aligned} \tag{19}$$

for all $(t,j) \in \mathrm{dom}(\phi, u)$. Then, with (18), it is straightforward to check that IOS holds with $\beta := 2\alpha_1 \circ \widetilde{\beta}$ and $\gamma := 2(\alpha_1 \circ \widetilde{\gamma} + \alpha_2)$. This leads to the following result:

($\star$) Suppose that a hybrid system $\mathcal{H}$ is input-to-state stable with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$. If there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that (18) holds then $\mathcal{H}$ is IOS with respect to $\mathcal{A}$.

4.2 A Lyapunov characterization

Lyapunov conditions asserting IOS for continuous-time systems have been introduced in [37]. Following [37, Definition 1.1], we define a class of IOS Lyapunov functions for $\mathcal{H}$. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an IOS Lyapunov function with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for $\mathcal{H}$ if there exist class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a class-$\mathcal{K}$ function χ such that

$$\alpha_1(|h(x)|) \;\leq\; V(x) \;\leq\; \alpha_2(|x|_{\mathcal{A}}) \;\; \forall x \in \mathbb{R}^n, \tag{20}$$

$$\begin{aligned} \langle \nabla V(x), f(x,u)\rangle \;&\leq\; -\alpha_3(V(x)) \\ &\quad \forall (x,u) \in C,\; V(x) \geq \chi(|u|), \end{aligned} \tag{21}$$

$$\begin{aligned} V(g(x,u)) - V(x) \;&\leq\; -\alpha_3(V(x)) \\ &\quad \forall (x,u) \in D,\; V(x) \geq \chi(|u|). \end{aligned} \tag{22}$$

It turns out that, under (A1)-(A2), the conditions (21) and (22) are equivalent to the dissipative form

$$\begin{aligned} \langle \nabla V(x), f(x,u)\rangle \;&\leq\; -\alpha_3(V(x)) + \rho(|u|) \\ &\quad \forall (x,u) \in C, \end{aligned} \tag{23}$$

$$\begin{aligned} V(g(x,u)) - V(x) \;&\leq\; -\alpha_3(V(x)) + \rho(|u|) \\ &\quad \forall (x,u) \in D, \end{aligned} \tag{24}$$

for some class-$\mathcal{K}_\infty$ function α_3 and class-$\mathcal{K}$ function ρ.

It turns out that the existence of an IOS Lyapunov function implies IOS with respect to a compact set $\mathcal{A}$.

($\star$) Given a hybrid system $\mathcal{H}$ satisfying (A1)-(A2), if there exists an IOS Lyapunov function with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for $\mathcal{H}$ then $\mathcal{H}$ is IOS with respect to $\mathcal{A}$.

Note that the converse of this result does not necessarily hold without further assumptions. In particular, it has been already shown in [7] that, for the case of h being the identity, ISS does not imply the existence of an ISS-Lyapunov function. Moreover, as pointed out in [41, 37], bounds (21) and (22) explicitly depending on $|x|$ are expected to be required.

5 A Small Gain Theorem for Input-to-Output Stability

One of the main tools for analysis of interconnections of dynamical systems is the small gain theorem. It is well known that the feedback interconnection of ISS nonlinear systems is ISS when a small gain condition holds. Such a result can be asserted using the $\mathcal{KL}$ estimates involved in the definition of ISS for the individual systems. Small gain results in terms of $\mathcal{KL}$ estimates were presented for interconnections of ISS systems in [19, 42, 17], for IOS/IOSS nonlinear continuous-time systems in [19], for IOS continuous and discrete-time systems [18], for input/output system models in [39], and for a class of systems with jumps in [32]. Sufficient conditions for ISS of interconnections in terms of Lyapunov functions have been shown to be powerful as they provide an ISS Lyapunov function for the entire interconnection. These exploit Lyapunov characterizations and sufficient conditions for ISS of the individual systems, results that were presented for continuous-time systems in [40], for discrete-time systems in [21], for switched systems in [28, 44], and for hybrid systems in [7]. A Lyapunov-based small gain theorem for interconnections of ISS systems appeared in [20] for continuous-time systems, and later extended to discrete-time and hybrid systems in [24] and [31] (see also [26] and [10]), respectively.

5.1 Main Result

In this section, a small gain theorem for the analysis of interconnections of IOS systems using Lyapunov functions is given. Let X_1, X_2, and X be the projection of the closure of $C \cup D \cup (g(D) \times \mathcal{U})$ onto $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and $\mathbb{R}^n$, $n = n_1 + n_2$, respectively. We consider the interconnection of two hybrid systems $\mathcal{H}_1, \mathcal{H}_2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be compact subsets of $\mathbb{R}^{n_1}, \mathbb{R}^{n_2}$, respectively. Below, for a locally Lipschitz function V, $V^\circ(x,w)$ denotes the Clarke generalized derivative of V at x in the direction w [8]. A small gain result for interconnected IOS hybrid systems is as follows:

($\star$) Suppose that for $i = 1, 2$ there exist continuously differentiable functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ such that

A) There exist class-$\mathcal{K}_\infty$ functions α_{i1}, α_{i2} such that for all $x_i \in X_i$

$$\alpha_{i1}(|h_i(x_i)|) \;\leq\; V_i(x_i) \;\leq\; \alpha_{i2}(|x_i|_{\mathcal{A}_i}) \tag{25}$$

B) There exist class-$\mathcal{K}_\infty$ functions χ_i, γ_i, positive definite functions α_i and λ_i satisfying $\lambda_i(s) < s$ for all $s > 0$ such that:

* For all $(x,u) \in C$ and $V_1(x_1) \geq \max\{\chi_1(V_2(x_2)), \gamma_1(|u_1|)\}$:

$$\begin{aligned} \langle \nabla V_1(x_1), f_1(x_1, h_2(x_2), u_1)\rangle \;&\leq \\ -\alpha_1(V_1(x_1)) \end{aligned} \tag{26}$$

and, for all $(x,u) \in C$ and $V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(|u_2|)\}$:

$$\langle \nabla V_2(x_2), f_2(x_2, h_1(x_1), u_2)\rangle \leq -\alpha_2(V_2(x_2)) \quad (27)$$

$*$ For all $(x,u) \in D$ we have

$$V_1(\widetilde{g}_1(x_1, h_2(x_2), u_1)) \leq \max\{\lambda_1(V_1(x_1)), \chi_1(V_2(x_2)), \gamma_1(|u_1|)\} \quad (28)$$

and

$$V_2(\widetilde{g}_2(x_2, h_1(x_1), u_2)) \leq \max\{\lambda_2(V_2(x_2)), \chi_2(V_1(x_1)), \gamma_2(|u_2|)\}. \quad (29)$$

C) The following holds

$$\chi_1 \circ \chi_2(s) < s \qquad \forall s > 0. \quad (30)$$

Let a class-$\mathcal{K}_\infty$ function ρ be such that[5] $\chi_1(s) < \rho(s)$ for all $s > 0$, $\chi_2(s) < \rho^{-1}(s)$ for all $s > 0$, and continuously differentiable on $(0,\infty)$ with $\frac{d\rho}{ds}(s) > 0$ for all $s > 0$. Then, let

$$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}. \quad (31)$$

It follows that

1) There exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that, for all $x \in X$,

$$\alpha_1(|(h_1(x_1), h_2(x_2))|) \leq V(x) \leq \alpha_2(|(x_1, x_2)|_{\mathcal{A}_1 \times \mathcal{A}_2}); \quad (32)$$

2) There exist a positive definite function α and a class-$\mathcal{K}$ function $\widetilde{\gamma}_1$ such that, for all $(x,u) \in C$ and $V(x) \geq \widetilde{\gamma}_1(|u|)$, we have

$$V^\circ(x, f(x,u)) \leq -\alpha(V(x)); \quad (33)$$

3) There exist a positive definite function λ, $\lambda(s) < s$ for all $s > 0$, and a class-$\mathcal{K}$ function $\widetilde{\gamma}_2$ such that, for all $(x,u) \in D$,

$$V(g(x,u)) \leq \max\{\lambda(V(x)), \widetilde{\gamma}_2(|u|)\}. \quad (34)$$

This small gain result can be established following the ideas of the proof of [20, Theorem 3.1] for continuous-time systems, which have been recently exploited to establish small gain results for ISS hybrid systems in [31, Theorem 2.1] and [10, Theorem 3.1]. Note that the result does not provide a smooth IOS Lyapunov function. Note that the interconnection $\mathcal{H}_1, \mathcal{H}_2$ does not rule out the possibility of solutions that are discrete, or that eventually, after some (t,j), become discrete. This includes interconnections having solutions with one of the state components, say x_2, constant through the second option in the definition of the update law $\widetilde{g}_2$, for which it would be difficult to satisfy conditions (28) and (29).

[5] Such a function always exists since χ_1 and χ_2 are class-$\mathcal{K}_\infty$ functions satisfying $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$; see [19].

5.2 Illustrative Example

The small gain result is illustrated in the next example. Consider two hybrid systems $\mathcal{H}_i$, $i \in \{1,2\}$, given by

$$\mathcal{H}_i \left\{ \begin{array}{rcll} \dot{x}_i & = & -a_i x_i + b_i \widetilde{u}_i + u_i & (x_i, \widetilde{u}_i, u_i) \in C_i \\ x_i^+ & = & \widetilde{u}_i & (x_i, \widetilde{u}_i, u_i) \in D_i \\ y_i & = & x_i, & \end{array} \right.$$

where[6]

$$\begin{array}{rcl} C_i & := & \{(x_i, \widetilde{u}_i, u_i) \ : \ \widetilde{u}_i\,(x_i - \varepsilon_i \widetilde{u}_i) \leq 0 \ \}, \\ D_i & := & \{(x_i, \widetilde{u}_i, u_i) \ : \ \widetilde{u}_i\,(x_i - \varepsilon_i \widetilde{u}_i) \geq 0 \ \}, \end{array}$$

$a_i, b_i, \varepsilon_i > 0$ and $x_i, \widetilde{u}_i, u_i \in \mathbb{R}$. Let $V_i(x_i) = \frac{1}{2}x_i^2$ and note that, for every $i \in \{1,2\}$, (25) holds with

$$\alpha_{i1}(s) = \alpha_{i2}(s) := \frac{1}{2}s^2 \qquad \forall s \geq 0$$

and that, on C_i,

$$\begin{array}{rcl} \langle \nabla V_i(x_i), f_i(x_i, \widetilde{u}_i, u_i)\rangle & = & -a_i x_i^2 + b_i x_i \widetilde{u}_i + x_i u_i \\ & \leq & -a_i x_i^2 + b_i \varepsilon_i \widetilde{u}_i^2 + x_i u_i \end{array} \quad (35)$$

since for points in C_i we have $\widetilde{u}_i x_i \leq \varepsilon_i \widetilde{u}_i^2$. Consider the input assignment

$$\widetilde{u}_1 = y_2, \qquad \widetilde{u}_2 = y_1. \quad (36)$$

We start checking the flow conditions. From the definition of $\mathcal{H}_i$, $f_i(x_i, \widetilde{u}_i, u_i) := -a_i x_i + b_i \widetilde{u}_i + u_i$, $g_i(x_i, \widetilde{u}_i, u_i) := \widetilde{u}_i$, and $h_i(x_i) := x_i$. With the assignment (36) and using the bounds

$$x_i u_i \leq \frac{a_i}{2} x_i^2 + \frac{1}{2a_i} u_i^2,$$

and (25), it follows that

$$\begin{array}{rcl} \langle \nabla V_1(x_1), f_1(x_1, h_2(x_2), u_1)\rangle & \leq & -a_1 x_1^2 + b_1 \varepsilon_1 x_2^2 + x_1 u_1 \\ & \leq & -a_1 V_1(x_1) \\ & & +2b_1 \varepsilon_1 V_2(x_2) + \frac{1}{2a_1} u_1. \end{array}$$

Then

$$\langle \nabla V_1(x_1), f_1(x_1, h_2(x_2), u_1)\rangle \leq -\frac{a_1}{2} V_1(x_1) \quad (37)$$

for all $(x_1, h_2(x_2), u_1) \in C_1$ when

$$V_1(x_1) \geq \max\left\{ \frac{8 b_1 \varepsilon_1}{a_1} V_2(x_2), \frac{2}{a_1^2} u_1^2 \right\} \quad (38)$$

Similarly,

$$\begin{array}{rcl} \langle \nabla V_2(x_2), f_2(x_2, h_1(x_1), u_2)\rangle & \leq & -a_2 x_2^2 + b_2 \varepsilon_2 x_1^2 + x_2 u_2 \\ & \leq & -a_2 V_2(x_2) \\ & & +2b_2 \varepsilon_2 V_1(x_1) + \frac{1}{2a_2} u_2. \end{array}$$

[6] The data of this hybrid system was inspired from an example in [31].

Then

$$\langle \nabla V_2(x_2), f_2(x_2, h_1(x_1), u_2)\rangle \leq -\frac{a_2}{2} V_2(x_2) \quad (39)$$

when

$$V_2(x_2) \geq \max\left\{\frac{8b_2\varepsilon_2}{a_2} V_1(x_1), \frac{2}{a_2^2} u_2^2\right\}. \quad (40)$$

Then, since (38) implies (37), and (40) implies (39), we have that (26) and (27) hold for

$$\chi_1(s) = \frac{8b_1\varepsilon_1}{a_1} s,\ \alpha_1(s) = \frac{a_1}{2} s,\ \gamma_1 = \frac{2}{a_1^2} s \quad \forall s \geq 0;$$

$$\chi_2(s) = \frac{8b_2\varepsilon_2}{a_2} s,\ \alpha_2(s) = \frac{a_2}{2} s,\ \gamma_2 = \frac{2}{a_2^2} s \quad \forall s \geq 0.$$

We now check the jump conditions. With the assignment (36), it follows that for every $(x, u) \in D$ such that $(x_1, h_2(x_2), u_1) \in D_1$, $(x_2, h_1(x_1), u_2) \notin D_2$, we get

$$\begin{aligned} V_1(\widetilde{g}_1(x_1, h_2(x_2), u_1)) &= V_2(\widetilde{g}_2(x_2, h_1(x_1), u_2)) \\ &= \frac{1}{2} x_2^2 \leq \frac{x_1 x_2}{\varepsilon_1} \\ &\leq \frac{1}{\varepsilon_1}\frac{x_1^2}{2} + \frac{1}{\varepsilon_1}\frac{x_2^2}{2} \\ &\leq \frac{2}{\varepsilon_1} \max\{V_1(x_1), V_2(x_2)\} \end{aligned} \quad (41)$$

since such points satisfy $\varepsilon_1 x_2^2 \leq x_1 x_2$. For every $(x, u) \in D$ such that $(x_1, h_2(x_2), u_1) \notin D_1$, $(x_2, h_1(x_1), u_2) \in D_2$, we get

$$\begin{aligned} V_1(\widetilde{g}_1(x_1, h_2(x_2), u_1)) &= V_2(\widetilde{g}_2(x_2, h_1(x_1), u_2)) \\ &= \frac{1}{2} x_1^2 \leq \frac{x_1 x_2}{\varepsilon_2} \\ &\leq \frac{1}{\varepsilon_2}\frac{x_1^2}{2} + \frac{1}{\varepsilon_2}\frac{x_2^2}{2} \\ &\leq \frac{2}{\varepsilon_2} \max\{V_1(x_1), V_2(x_2)\} \end{aligned} \quad (42)$$

since such points satisfy $\varepsilon_2 x_1^2 \leq x_1 x_2$. For every $(x, u) \in D$ such that $(x_1, h_2(x_2), u_1) \in D_1$, $(x_2, h_1(x_1), u_2) \in D_2$, we get

$$V_1(\widetilde{g}_1(x_1, h_2(x_2), u_1)) \leq \frac{2}{\varepsilon_1} \max\{V_1(x_1), V_2(x_2)\} \quad (43)$$

$$V_2(\widetilde{g}_2(x_2, h_1(x_1), u_2)) \leq \frac{2}{\varepsilon_2} \max\{V_1(x_1), V_2(x_2)\}. \quad (44)$$

Combining (41)-(44), we obtain

$$V_1(\widetilde{g}_1(x_1, h_2(x_2), u_1)) \leq \max\left\{\tfrac{2}{\varepsilon_1}, \tfrac{2}{\varepsilon_2}\right\} \max\{V_1(x_1), V_2(x_2)\}, \quad (45)$$

$$V_2(\widetilde{g}_2(x_2, h_1(x_1), u_2)) \leq \max\left\{\tfrac{2}{\varepsilon_1}, \tfrac{2}{\varepsilon_2}\right\} \max\{V_1(x_1), V_2(x_2)\}. \quad (46)$$

Then, (28) and (29) hold with

$$\lambda_1(s) = \chi_1(s) = \lambda_2(s) = \chi_2(s) = \max\left\{\tfrac{2}{\varepsilon_1}, \tfrac{2}{\varepsilon_2}\right\} s \ \forall s \geq 0,$$

$$\gamma_1 = \gamma_2 \equiv 0.$$

Then, for $\lambda_i(s) < s$ to hold we require

$$\max\left\{\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}\right\} < \frac{1}{2}.$$

Then, the functions χ_1 and χ_2 are defined as

$$\begin{aligned} \chi_1(s) &= \max\left\{\frac{8b_1\varepsilon_1}{a_1}, \frac{2}{\varepsilon_1}, \frac{2}{\varepsilon_2}\right\} s, \\ \chi_2(s) &= \max\left\{\frac{8b_2\varepsilon_2}{a_2}, \frac{2}{\varepsilon_1}, \frac{2}{\varepsilon_2}\right\} s. \end{aligned}$$

Finally, the small gain condition (30) holds when

$$\max\left\{\frac{8b_1\varepsilon_1}{a_1}, \frac{2}{\varepsilon_1}, \frac{2}{\varepsilon_2}\right\} \max\left\{\frac{8b_2\varepsilon_2}{a_2}, \frac{2}{\varepsilon_1}, \frac{2}{\varepsilon_2}\right\} < 1.$$

It follows that items 1)-3) of the small gain result hold with V as in (31). As suggested by the discussion in Section 3, existence of solutions to the resulting interconnection needs to be checked separately.

Figure 4 shows a solution to the interconnection when the inputs u_1 and u_2 are set to zero. At jumps due to the state hitting D_1, the component x_1 is reset to x_2. Figure 5 shows the evolution of x_1 and x_2 over hybrid time for different values of the inputs u_1 and u_2. The plots illustrate that the size of the bound on the solutions decreases with the size of the applied input.

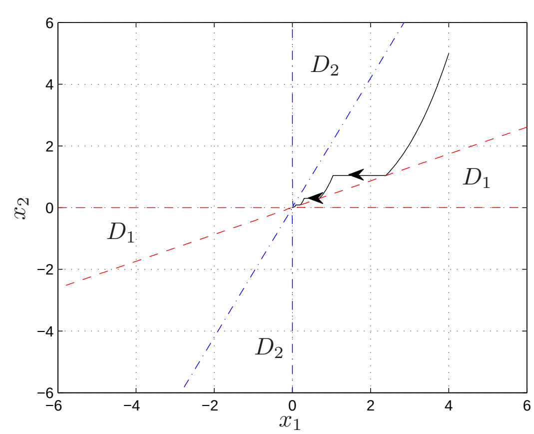

Figure 4: A solution in the plane to the interconnection in Section 5.2 for initial condition $x(0,0) = [4\ 5]^\top$ and zero inputs. The solution hits the set D_1 and jumps to a point in the flow set C from where it flows to D_1 and jumps again. Parameters: $a_1 = 15$, $b_1 = 0.5$, $\varepsilon_1 = 2.3$, $a_2 = 45$, $b_2 = 0.3$, $\varepsilon_2 = 2.1$.

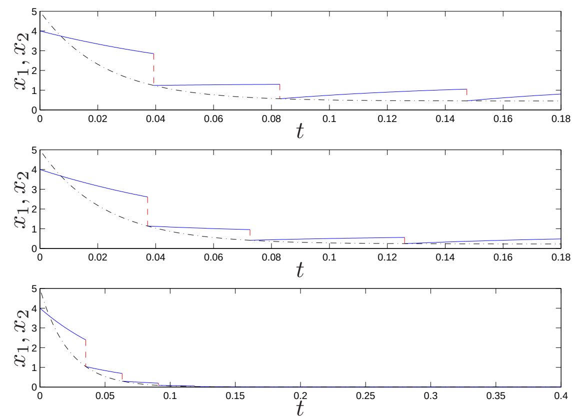

Figure 5: Solution components x_1 (solid) and x_2 (dashed-dot) to the interconnection in Section 5.2. Initial conditions and parameters are as in the plot in Figure 4. Top plot: $u_1 = u_2 = 20$; middle plot: $u_1 = u_2 = 10$; bottom plot: $u_1 = u_2 = 0$.

6 Conclusion

For a general class of hybrid systems, we discussed difficulties in the analysis of interconnection of hybrid systems. An input-to-output stability notion was presented, along with basic properties and a Lyapunov characterizations. A small gain result for the study of an interconnection of two hybrid systems was presented and illustrated in a simple example. The nature of the results and the general hybrid systems framework under study, which cover classical continuous and discrete-time systems, suggest wide applicability of these tools. Further details on the IOS results outline here and results on input-output-to-state stability for hybrid systems were reported in [34].

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