A feedback control motivation for generalized solutions to hybrid systems[†]

Ricardo G. Sanfelice, Rafal Goebel, and Andrew R. Teel

{rsanfelice,rafal,teel}@ece.ucsb.edu
Center for Control, Dynamical Systems, and Computation
Department of Electrical and Computer Engineering
University of California, Santa Barbara, CA 93106-9560.

Abstract. Several recent results in the area of robust asymptotic stability of hybrid systems show that the concept of a generalized solution to a hybrid system is suitable for the analysis and design of hybrid control systems. In this paper, we show that such generalized solutions are exactly the solutions that arise when measurement noise is present in the system.

1 Introduction

1.1 Motivation

Hybrid dynamical systems comprise a rich class of systems in which the state can both evolve continuously (flow) and discontinuously (jump). Over the last ten years or more, in research areas such as computer science, feedback control, and dynamical systems, researchers have given considerable attention to modeling and solution definitions for hybrid systems. Some notable references include [41, 38, 4, 9, 8, 28, 40].

In the paper [19], motivated by robust stability issues in hybrid control systems, the authors introduced the notion of a generalized solution to a hybrid system and outlined some stability theory consequences that followed from this solution concept. These included results on "for free" robustness of stability, a generalization of LaSalle's invariance principle, and the existence of smooth Lyapunov functions for asymptotically stable hybrid systems. More details about these results and generalizations were given in the subsequent conference papers [20] (see also [21]), [35] and [10], respectively.

The purpose of the current paper is to motivate further the use of generalized hybrid solutions by considering the effect of arbitrary small measurement noise

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in hybrid control systems. In this paper we show that, for hybrid systems arising from using hybrid feedback control, generalized hybrid solutions agree with the limits (in an appropriate sense) of solutions generated by arbitrarily small measurement noise in the hybrid control system. This result generalizes to hybrid systems a similar result for differential equations initially reported by Hermes in [23] and expanded upon by Hàjek in [22]. It contains, as a special case, an analogous result for difference equations that, to the best of our knowledge, has not appeared in the literature.

1.2 Controversy?

In continuous-time systems, generalized solutions to discontinuous differential equations are shunned at times because using such a solution precludes solving certain nonlinear control problems. For example, for asymptotically controllable nonlinear systems, it is possible to solve the stabilization problem by state feedback when using weak notions of solution for discontinuous differential equations (e.g., Caratheodory solutions, Euler solutions, etc.) (see [13]) but it is impossible to solve this problem in general when using generalized solutions such as those due to Krasovskii [25], Filippov [18], or Hermes [23]; for further details see [11].

The feedback stabilization problem does not provide the same motivation for avoiding generalized solutions to hybrid systems. Indeed, it is possible to robustly stabilize asymptotically controllable nonlinear systems using hybrid feedback and using generalized solutions to hybrid systems. See, for example, [31].

Despite our opinion that the use of generalized solutions to hybrid systems will never diminish the capabilities of hybrid control, we would not be surprised to see some resistance to the use of generalized hybrid solutions to hybrid control systems. We expect the main sticking point to be how the notion of generalized solutions affects the "semantics" of a hybrid control system. We now elaborate on what we mean.

For the purposes of this paper, a hybrid system is specified by the data $\mathcal{H} = (f, g, C, D, O)$ where the open set $O \subset \mathbb{R}^n$ is the state space of the hybrid system $\mathcal{H}$, f is a function from C to $\mathbb{R}^n$ called the "flow map", g is a function from D to $C \cup D$ called the "jump map", C is a subset of O called the "flow set" and indicates where in the state space flow may occur, D is a subset of O called the "jump set" and indicates from where in the state space jumps may occur. At times, we write the data in the suggestive form

$$\mathcal{H} \begin{cases} \dot{x} \;\; = f(x) & x \in C \\ x^+ = g(x) & x \in D \end{cases} \tag{1}$$

where x is the state of the hybrid system (with discrete modes already embedded in it). Several models for hybrid systems available in the literature (see e.g. [8], [28], [40]), under certain assumptions, can be fit in such framework. The particular concept of a solution to a hybrid system we use will be made precise in Section 2; it is not relevant for the discussion below.

Generalized solutions to $\mathcal{H}$ are solutions to a hybrid system with regularized data $\overline{\mathcal{H}} = (\overline{f}, \overline{g}, \overline{C}, \overline{D}, O)$, where $\overline{f}$ and $\overline{g}$ are constructed from f and g in a manner that will be made precise later (see Definition 3) and $\overline{C}$ and $\overline{D}$ denote the closures of C and D, respectively, relative to O. In particular this means that if $C \cup D = O$ then $\overline{C} \cap \overline{D}$ is not empty[1] even if $C \cap D$ is empty. It turns out that many models of hybrid systems insist on having $C \cap D$ empty. For example, one has $C = O \backslash D$ in the definition of state-dependent impulsive systems in [5] (see also [6] and [12]). The condition $C = O \backslash D$ is also used in many of the hybrid models considered in [8]. Making $C \cap D$ empty is one way to guarantee that jumps are enforced in the jump set rather than simply enabled. (Some researchers use the phrases " 'as is' semantics" and "enabling semantics" for these two respective situations, see [36].) Moreover, it is a way to guarantee that solutions, if they exist, are unique when the flow map f is locally Lipschitz. See, for example, [29].

As we pointed out in [19], changing C and D to their relative closures can have a dramatic effect on the solutions to the hybrid control system. For example, if D has measure zero, perhaps being a surface on which jumps are enforced, and $C = O \backslash D$ (see, for example, the model of reset control systems used in [6] and the references therein) then the relative closure of C will be equal to the entire state space. This may enable solutions that never jump, circumventing the reason for hybrid control in the first place. However, the point we are making in this paper is that the new behavior that appears when taking the relative closures can manifest itself due to measurement noise in a feedback control system. In this sense, this new behavior should be taken into account.

There are many motivations for not taking the flow set C and the jump set D to be sets that are closed relative to O in the definition of a hybrid system. However, in the context of hybrid control systems, we hope that the robust stability motivation given in [19], the solution properties reported in [20], the stability theory corollaries reported in [20] and [35], and the new results reported here on the equivalence between generalized solutions and the limit of solutions due to measurement noise continue to motivate the development of hybrid control system models that use jump and flow sets that are closed relative to the state space. An example in this direction is the work of [42, 30] which revisits the reset control systems considered in [6] and finds a natural definition of the flow set and jump set so that they are closed and yet still force jumps at the appropriate locations in the state space.

2 Definition of generalized solutions

In what follows we write $\mathbb{R}_{\geq 0}$ for $[0, +\infty)$, $\mathbb{N}$ for $\{0, 1, 2, ...\}$, and $|\cdot|$ for the Euclidean vector norm.

2.1 Generalized time domain

In what could be described as the "classical" approach to hybrid systems, a solution to $\mathcal{H} = (f, g, C, D, O)$ is, vaguely, a piecewise continuous function ξ that

[1] This is true unless either C is empty or D is empty, in which case the original system was not truly hybrid in the first place.

is left-continuous and such that, on each interval of continuity satisfies $\xi(t) \in C$ and $\dot{\xi} = f(\xi(t))$, while at each point τ of discontinuity satisfies $\lim_{t\to\tau^-} \xi(t) \in D$ and $\xi(\tau) = g(\lim_{t\to\tau^-} \xi(t))$ (so, more compactly, $\xi^- \in D$, $\xi^+ = g(\xi^-)$). By design, such concept of a solution excludes multiple jumps at a single time instant. Furthermore, it makes it troublesome (or impossible) to discuss limits of solutions; see Example 1. These issues can be overcome by using a "generalized" time domain, as defined below.

Definition 1 (hybrid time domain). *A subset* $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ *is a* compact hybrid time domain *if*

$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

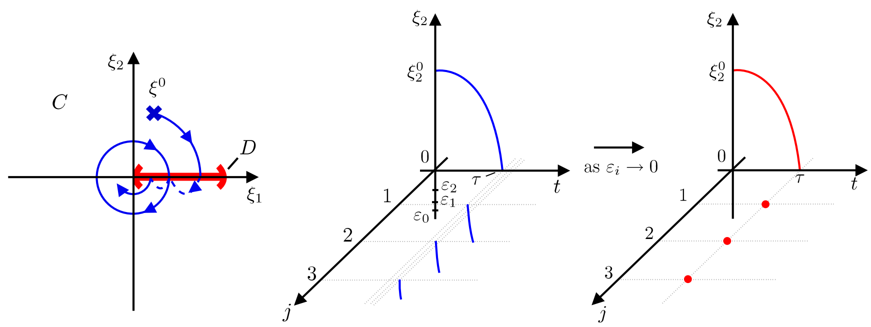
for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. *It is a* hybrid time domain *if for all* $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, ...J\})$ *is a compact hybrid domain.*

Hybrid time domains are similar to hybrid time trajectories in [28],[29], and [3], and to the concept of time evolution in [40], but give a more prominent role to the number of jumps j (c.f. the definition of hybrid time set by Collins in [15]). On each hybrid time domain there is a natural ordering of points: we write $(t, j) \preceq (t', j')$ for $(t, j), (t', j') \in \mathcal{D}$ if $t \leq t'$ and $j \leq j'$.

Definition 2 (hybrid arc). *A* hybrid arc *is a pair* $(x, \operatorname{dom} x)$ *consisting of a hybrid time domain* $\operatorname{dom} x$ *and a function* x *defined on* $\operatorname{dom} x$ *that is locally absolutely continuous in* t *on* $\operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{j\})$ *for each* $j \in \mathbb{N}$.

We will not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid arc x comes a hybrid time domain $\operatorname{dom} x$. In this way, hybrid arcs x are parameterized by $(t, j) \in \operatorname{dom} x$, with $x(t, j)$ being the value of x at the "hybrid instant" given by (t, j). A hybrid arc ξ is said to be *nontrivial* if $\operatorname{dom} \xi$ contains at least one point different from $(0, 0)$, *complete* if $\operatorname{dom} \xi$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} \xi$ onto $\mathbb{R}_{\geq 0}$ is bounded.

Example 1. Consider a hybrid system on $\mathbb{R}^2$ given by $D = (0, 1) \times \{0\}$, $C = \mathbb{R}^2 \setminus D$, $f(x_1, x_2) = (x_2, -x_1)$, $g(x) = x/2$. For any point ξ^0 with $0 < |\xi^0| < 1$, $\xi^0 \notin D$, a "classical" solution from ξ^0 (the solution is unique!) rotates clockwise until it hits D, then via a jump has its magnitude divided by 2, then rotates again for time 2π until it jumps again, etc; see Figure 1(a). In the presence of arbitrarily small noise, a "classical" solution may jump almost immediately after the first jump. That is, if τ is the time of the first jump, a solution to $\dot{x} = f(x+e)$, $x \in C$ while $x^+ = g(x^- + e^-)$, $x^- \in D$ will jump at τ and then again when $x_2 = -\varepsilon$, if one considers the noise $e(t) = (0, 0)$ if $t \leq \tau$, $e(t) = (0, \varepsilon)$ for $t > \tau$. In this fashion, one can in fact construct a "classical" solution and arbitrarily small noise so that the solution jumps arbitrarily many times (even infinitely many) in arbitrary short time (so it may be a Zeno solution). One can then ask what the limit of such solutions is (with the noise size decreasing to 0), and it would be reasonable to expect that the limit is a solution that jumps infinitely many times at time τ. Figure 1(b) shows this on hybrid time domains. Of course, such solution is not a "classical" solution, in fact, it can not be represented using regular time. However, it is a hybrid arc (in the sense of Definition 2) defined on a hybrid time domain.



(a) One possible "classical" solution.

(b) Convergence of solutions when the noise approaches zero. The value of the noise e at the i-th jump is given by $(0, \varepsilon_i)$ where $\varepsilon_i \to 0$ as $i \to \infty$.

Fig. 1. Solutions and their convergence under the presence of measurement noise for the system in Example 1.

2.2 Generalized solutions a la Krasovskii

The regularization of the hybrid system $\mathcal{H}$ is defined below. We remind the reader that for a set $C \subset O$, its closure relative to O is equal to the closure of C intersected with O, and is the smallest relatively closed subset of O that contains C.

Definition 3 (regularized hybrid system $\overline{\mathcal{H}}$). *Given a hybrid system $\mathcal{H} = (f, g, C, D, O)$, its regularization (a la Krasovskii) is denoted by $\overline{\mathcal{H}} = (\overline{f}, \overline{g}, \overline{C}, \overline{D}, O)$ where, for every $x \in O$,*

$$\overline{f}(x) := \bigcap_{\delta>0} \overline{\text{co}} f((x+\delta\mathbb{B}) \cap C), \quad \overline{g}(x) := \bigcap_{\delta>0} \overline{g((x+\delta\mathbb{B}) \cap D)} \tag{2}$$

and $\overline{C}, \overline{D}$ are the relative closures of the sets C, D with respect to the state space O, respectively.

Regarding the function f, the regularization corresponds to the one proposed by Krasovskii in [26] for discontinuous differential equations. (An equivalent description of $\overline{f}(x)$ would say that it is the smallest closed convex set containing all limits of $f(x_i)$ as $x_i \to x$, $x_i \in C$.) We note that the regularization of f as proposed by Filippov in [18] ignores the behavior of f on sets of measure zero, and thus proves to be unsuitable for hybrid systems (and even for constrained differential equations). Indeed, for example, a set C with zero measure leads to an "empty" regularization. Regarding g, the regularization is the one used in [24]; due to the nature of discrete time, the convexification is not needed.

Following the compact form for hybrid systems $\mathcal{H} = (f, g, C, D, O)$ given in (1), we can write its regularized version $\overline{\mathcal{H}} = (\overline{f}, \overline{g}, \overline{C}, \overline{D}, O)$ as

$$\overline{\mathcal{H}} \begin{cases} \dot{x} \in \overline{f}(x) & x \in \overline{C} \\ x^+ \in \overline{g}(x) & x \in \overline{D}. \end{cases} \tag{3}$$

Note that the differential and difference equations in $\mathcal{H}$ are replaced by differential and difference inclusions, since $\overline{f} : O \rightrightarrows \mathbb{R}^n$, $\overline{g} : O \rightrightarrows O$, by their very definitions, are in general set-valued mappings and not functions. A formal definition of Krasovskii solutions follows.

Definition 4 (hybrid Krasovskii solution to $\mathcal{H}$). *A hybrid arc $\psi : \operatorname{dom}\psi \to O$ is a hybrid Krasovskii solution to the hybrid system $\mathcal{H} = (f, g, C, D, O)$ with regularization given by $\overline{\mathcal{H}} = (\overline{f}, \overline{g}, \overline{C}, \overline{D}, O)$ if $\psi(0,0) \in \overline{C} \cup \overline{D}$ and:*

(K1) for all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in \operatorname{dom}\psi$,

$$\psi(t,j) \in \overline{C}, \quad \dot{\psi}(t,j) \in \overline{f}(\psi(t,j)); \tag{4}$$

(K2) for all $(t,j) \in \operatorname{dom}\psi$ such that $(t,j+1) \in \operatorname{dom}\psi$,

$$\psi(t,j) \in \overline{D}, \quad \psi(t,j+1) \in \overline{g}(\psi(t,j)). \tag{5}$$

Under minor assumptions on f and g, the system $\overline{\mathcal{H}} = (\overline{f}, \overline{g}, \overline{C}, \overline{D}, O)$ has the regularity properties (stated below, in Theorem 1) that were imposed on the hybrid systems by the authors et al. in [19] and in [20] and led to results on sequential compactness of the sets of solutions to hybrid systems. In particular, such properties guarantee that an appropriately understood limit of a sequence of solutions to a hybrid system is itself a solution.

A function $\phi : O \to \mathbb{R}^n$ (or a set-valued mapping $\phi : O \rightrightarrows \mathbb{R}^n$) is locally bounded on O if for each compact set $K \subset O$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $\phi(K) \subset K'$. It is locally bounded with respect to O on O if we request that $K' \subset O$.

Assumption 1 *The function f is locally bounded on O. The function g is locally bounded with respect to O on O.*

A set valued mapping $\phi : O \rightrightarrows \mathbb{R}^n$ (or $\phi : O \rightrightarrows O$) is outer semicontinuous if for any sequence $\{x_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} x_i = x \in O$ and any sequence $\{y_i\}_{i=1}^{\infty}$ with $y_i \in \phi(x_i)$ and $\lim_{i\to\infty} y_i = y$ we have $y \in \phi(x)$.

Theorem 1 (basic properties of $\overline{\mathcal{H}}$). *Under Assumption 1, the regularized hybrid system $\overline{\mathcal{H}} = (\overline{f}, \overline{g}, \overline{C}, \overline{D}, O)$ satisfies*

(A0) $O \subset \mathbb{R}^n$ is an open set.
(A1) $\overline{C}$ and $\overline{D}$ are relatively closed sets in O.
(A2) $\overline{f} : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\overline{f}(x)$ is nonempty and convex for all $x \in \overline{C}$.
(A3) $\overline{g} : O \rightrightarrows O$ is outer semicontinuous and $\overline{g}(x)$ is nonempty for all $x \in \overline{D}$.

One of the benefits of these properties is that, for systems that possess them, very general conditions for existence of solutions can be given, and maximal solutions behave as expected: that is, they are either complete or "blow up" in finite hybrid time (a solution is complete if its domain is unbounded). More specifically, under Assumption 1, and hence in presence of the properties listed in Theorem 1, the following is true: if $\psi^0 \in D$ or the following condition holds:

(VC) $\psi^0 \in C$ and for some neighborhood U of ψ^0, for all $\psi' \in U \cap C$, $T_C(\psi') \cap F(\psi') \neq \emptyset$,

then there exists a nontrivial Krasovskii solution ψ to $\mathcal{H}$ with $\psi(0,0) = \psi^0$. If (VC) holds for all $\psi^0 \in C \setminus D$, then for any maximal solution ψ with $\psi(0,0) = \psi^0$ (a Krasovskii solution ψ is said to be *maximal* if there does not exist another Krasovskii solution ψ' such that ψ is a truncation of ψ' to some proper subset of $\operatorname{dom} \psi'$) at least one of the following statements is true:

(i) ψ is complete;

(ii) ψ eventually leaves every compact subset of O: for any compact $K \subset O$, there exists $(T, J) \in \operatorname{dom} \psi$ such that for all $(t, j) \in \operatorname{dom} \psi$ with $(T, J) \prec (t, j)$, $\psi(t, j) \notin K$;

(iii) for some $(T, J) \in \operatorname{dom} \psi$, $(T, J) \neq (0, 0)$, we have $\psi(T, J) \notin C \cup D$.

If additionally

(VD) for all $\psi^0 \in D$, $G(\psi^0) \subset C \cup D$,

then case (iii) above does not occur. For details, see [21, Proposition 2.5].

Note that the viability condition (VC) for the continuous evolution is automatically satisfied at each point ψ^0 in the interior of C. Therefore, when $C \cup D = O$ (a condition that is common in many models for hybrid systems, see the Introduction), (VC) holds for all $\psi^0 \in C \setminus D$ since $C \setminus D = O \setminus D$ and the latter set is open. Consequently, if $C \cup D = O$, for all $\psi^0 \in O$ there exists a nontrivial solution ψ with $\psi(0,0) = \psi^0$.

Example 2. Consider the system from Example 1. Since the set D is thin, arbitrarily small noise can cause "classical" solutions, or solutions understood as hybrid arcs satisfying (1), starting from initial points ξ^0 with $0 < |\xi^0| < 1$, $\xi \notin D$, to miss D and never jump. On the other hand, arbitrarily small noise can cause solutions from ξ^0 with $|\xi^0| = 1$ to jump (to a point near $(0.5, 0)$) when the solution is near $(1, 0)$. Finally, once a solution ξ is such that $0 < |\xi(t)| < 1$, arbitrarily small noise can cause it to miss D and rotate, jump several times in arbitrarily short time, or display any combination of these behaviors. (So in particular, when limits of such solutions under vanishing noise are considered, uniqueness – present for "classical" solutions – is lost.)

Such potential effects of noise on the system are captured by its Krasovskii regularization. Here, we get $\overline{C} = \mathbb{R}^2$, $\overline{D} = [0, 1] \times \{0\}$, while $\overline{f} = f$, $\overline{g} = g$. The fact that $\overline{C} = \mathbb{R}^2$ results in Krasovskii solutions that only flow, or rotate around the origin an arbitrary number of times in between jumps. The point $(1, 0)$ being in $\overline{D}$ leads to solutions starting with $|\xi(0)| = 1$ that jumps at some time. These features, and the generality of hybrid time domains, capture the behavior of the original system under (arbitrarily small) noise.

2.3 Generalized solutions a la Hermes

To define hybrid Hermes solutions to a hybrid system, we need a concept of convergence of hybrid arcs that admits sequences of arcs with potentially different domains. Consequently, we will rely on graphical convergence. Given a hybrid arc x with domain $\operatorname{dom} x$, its graph is the set

$$\operatorname{gph} x := \{(t, j, x(t, j)) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times O \mid (t, j) \in \operatorname{dom} x\} .$$

A sequence of hybrid arcs $\{x_i\}_{i=1}^{\infty}$ *converges graphically* to a hybrid arc x if the sequence of graphs $\{\operatorname{gph} x_i\}_{i=1}^{\infty}$ converges to $\operatorname{gph} x$ in the sense of set convergence. The latter concept is well-established and often used in set-valued and nonsmooth analysis; see [32, 2]. For precise definitions of general set and graphical convergence we refer the reader to [32, Chapters 4,5]; below we state a version of [32, Exercise 5.34] relevant for our purposes. For further details on graphical convergence of hybrid arcs we recommend [20]. Finally, we add that graphical convergence is closely related to convergence in the Skorokhod topology used in [15].

Lemma 1 (graphical convergence of hybrid arcs). *Let x be a hybrid arc with compact $\operatorname{dom} x$, and let (T, J) be the supremum of $\operatorname{dom} x$. A sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid arcs with $\operatorname{dom} x_i \subset \mathbb{R}_{\geq 0} \times \{0, 1, \dots, J\}$, $i = 1, 2, \dots$, converges graphically to x if and only if for all $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that, for all $i > i_0$*

(a) for all $(t, j) \in \operatorname{dom} x$ there exists s such that $(s, j) \in \operatorname{dom} x_i$, $|t - s| < \varepsilon$, and $|x(t, j) - x_i(s, j)| < \varepsilon$,
(b) for all $(t, j) \in \operatorname{dom} x_i$ there exists s such that $(s, j) \in \operatorname{dom} x$, $|t - s| < \varepsilon$, and $|x_i(t, j) - x(s, j)| < \varepsilon$.

In particular, a sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid arcs with $\operatorname{dom} x_i \subset \operatorname{dom} x$, $i = 1, 2, \dots$, converges graphically to x if for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that, for all $i > i_0$, all $(t, j) \in \operatorname{dom} x$, we have $(t, j) \in \operatorname{dom} x_i$ and $|x(t, j) - x_i(t, j)| < \varepsilon$.

Equipped with graphical convergence, we generalize the definition of Hermes solutions discussed by Hermes in [23] and later defined by Hàjek in [22].

Definition 5 (hybrid Hermes solution to $\mathcal{H}$). *A hybrid arc $\varphi : \operatorname{dom}\varphi \to O$ is a hybrid Hermes solution to $\mathcal{H} = (f, g, C, D, O)$ if for each compact hybrid time domain $\mathcal{D} \subset \operatorname{dom}\varphi$ and the truncation $\varphi^{\mathcal{D}}$ of φ to $\mathcal{D}$, there exists a sequence of hybrid arcs $\varphi_i : \operatorname{dom}\varphi_i \to O$ and measurable functions $e_i : \operatorname{dom} e_i \to \mathbb{R}^n$, $\operatorname{dom} e_i = \operatorname{dom}\varphi_i$, that satisfy, for each i,*

(H1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom}\varphi_i$,

$$\varphi_i(t,j) + e_i(t,j) \in C, \quad \dot{\varphi}_i(t,j) = f(\varphi_i(t,j) + e_i(t,j)); \tag{6}$$

(H2) for all $(t, j) \in \operatorname{dom}\varphi_i$ such that $(t, j+1) \in \operatorname{dom}\varphi_i$,

$$\varphi_i(t,j) + e_i(t,j) \in D, \quad \varphi_i(t,j+1) = g(\varphi_i(t,j) + e_i(t,j)) \tag{7}$$

with the property that $\lim_{i\to\infty} \varphi_i(0,0) = \varphi(0,0)$, $\{\varphi_i\}_{i=0}^{\infty}$ converges graphically to $\varphi^{\mathcal{D}}$, for each i we have $\sup_{(t,j)\in\operatorname{dom} e_i} |e_i(t,j)| =: \varepsilon_i < +\infty$, and the sequence $\{\varepsilon_i\}_{i=0}^{\infty}$ converges to 0.

To illustrate what graphical convergence (vs. classical convergence notions) grants us, we give two somewhat extreme, but important, examples.

Example 3. Consider the system from Example 1, and a sequence of points on the line $x_1 = x_2$ converging to $(0,0)$. From each such point, one can find noise e_i and a "classical" solution ξ_i so that ξ_i rotates to D, and then jumps infinitely many times, with jumps separated by less than $1/i$ amount of time. (We argued that this is possible in Example 1.) The resulting sequence of hybrid arcs ξ_i converges graphically to a hybrid arc ξ with $\operatorname{dom}\xi = \{0\} \times \mathbb{N}$ (that is, ξ never flows) and for all $j \in \mathbb{N}$, $\xi(0,j) = (0,0)$. Such ξ is a Hermes solution. It is also a Krasovskii solution, since $(0,0) \in \overline{D}$ and $\overline{g}(0,0) = (0,0)$. (Recall though that $(0,0) \notin D$!)

Example 4. Consider a hybrid system on $\mathbb{R}^2$ given by $D = \mathbb{R}^2$; $C = [0, +\infty) \times \{0\}$; $f(x_1, x_2) = (1,1)$ for every point (x_1, x_2) where x_1 is rational, otherwise $f(x_1, x_2) = (1, -1)$; and $g(x) = 0$. For any point $\xi^0 \in C$ every classical solution cannot flow since it would be pushed away from the set C. On the other hand, in the presence of arbitrarily small noise, a "classical" solution can flow along the C set towards $+\infty$. Note that such a solution is also a Krasovskii solution since the regularization of f is given by $\overline{f}(x_1, x_2) = (1, [-1, 1])$.

In many control applications, the state of the system cannot be measured exactly since it is corrupted by noise. The measurement noise can appear in some but not every component of the state (e.g. when state feedback is implemented, noise appears only on states measured with specific sensors). To account for such cases, we consider functions f and g given as

$$\forall x \in C \quad f(x) := f'(x, \kappa_c(x)), \quad \forall x \in D \quad g(x) := g'(x, \kappa_d(x)) \tag{8}$$

where $f' : O \times U \to \mathbb{R}^n$ and $g' : O \times U \to O$, $\kappa_c : C \to U$, and $\kappa_d : D \to U$, $U \subset O$. We allow for κ_c, κ_d to be discontinuous.

The notion of Hermes solution in Definition 5 changes for a hybrid system $\mathcal{H} = (f, g, C, D, O)$ with f and g given by (8) since the noise is affecting the differential and difference equations only through the function κ_c and κ_d.

Definition 6 (hybrid control-Hermes solution to $\mathcal{H}$). *A hybrid arc $\varphi : \operatorname{dom}\varphi \to O$ is a hybrid control-Hermes solution to $\mathcal{H} = (f, g, C, D, O)$ with f and g given in (8) if for each compact hybrid time domain $\mathcal{D} \subset \operatorname{dom}\varphi$ and the truncation $\varphi^{\mathcal{D}}$ of φ to $\mathcal{D}$, there exists a sequence of hybrid arcs $\varphi_i : \operatorname{dom}\varphi_i \to O$ and measurable functions $e_i : \operatorname{dom} e_i \to \mathbb{R}^n$, $\operatorname{dom} e_i = \operatorname{dom}\varphi_i$, that satisfy, for each i,*

(cH1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom}\varphi_i$,

$$\varphi_i(t,j) + e_i(t,j) \in C, \quad \dot{\varphi}_i(t,j) = f'(\varphi_i(t,j), \kappa_c(\varphi_i(t,j) + e_i(t,j))); \quad (9)$$

(cH2) for all $(t, j) \in \operatorname{dom}\varphi_i$ such that $(t, j+1) \in \operatorname{dom}\varphi_i$,

$$\varphi_i(t,j) + e_i(t,j) \in D, \quad \varphi_i(t,j+1) = g'(\varphi_i(t,j), \kappa_d(\varphi_i(t,j) + e_i(t,j))) \quad (10)$$

with the property that $\lim_{i\to\infty} \varphi_i(0,0) = \varphi(0,0)$, $\{\varphi_i\}_{i=0}^{\infty}$ converges graphically to $\varphi^{\mathcal{D}}$, for each i we have $\sup_{(t,j)\in\operatorname{dom} e_i} |e_i(t,j)| =: \varepsilon_i < +\infty$, and the sequence $\{\varepsilon_i\}_{i=0}^{\infty}$ converges to 0.

3 Statement of main results

Following the work by Hermes [23] and Hàjek [22], we show that hybrid Krasovskii solutions to $\mathcal{H}$ are equivalent to hybrid Hermes solutions to $\mathcal{H}$.

Theorem 2 (Krasovskii solutions $\equiv$ Hermes solutions). *Under Assumption 1, a hybrid arc is a hybrid Krasovskii solution to $\mathcal{H}$ if and only if it is a hybrid Hermes solution to $\mathcal{H}$.*

The two implications are stated and proved as Corollary 4.4 and Corollary 5.2 in [34].

We note that Theorem 2 generalizes, to the hybrid framework, the result by Hàjek [22] given for differential equations. In proving the theorem, we first extend some results by Hàjek to differential equations with a constraint (and we give a proof quite different from that by Hàjek). We will also rely on results on perturbations of hybrid systems given in [20].

Assumption 2 *The functions f' is locally Lipschitz in the first argument uniformly in the second argument. The function g' is continuous in the first argument uniformly in the second argument.*

The result below is a generalization to the hybrid framework of the result given by Coron and Rosier [16] in the context of robust stabilizability of nonlinear systems with time-varying feedback laws.

Theorem 3 (Krasovskii solutions $\equiv$ control-Hermes solutions). *Under Assumptions 1 and 2, for a hybrid system $\mathcal{H}$ with f and g given in (8), a hybrid arc is a hybrid Krasovskii solution to $\mathcal{H}$ if and only if it is a hybrid control-Hermes solution to $\mathcal{H}$.*

The two implications are stated and proved as Corollary 4.7 and Proposition 5.1 in [34] (One of them naturally follows from Theorem 2.)

4 Examples

Here we discuss examples that illustrate that generalized solutions to hybrid systems play a very important role in the robust stabilization problem.

Example 5 (reset and impulsive control systems). For the problem of stabilizing dynamical systems with state feedback, controllers that have states that jump when certain conditions are satisfied have been proposed in the literature as it is the case of reset and impulsive control systems, see e.g. [14], [27], [6], [42]. A reset controller is a linear system with the property that its output is reset to zero whenever its input and output satisfy certain algebraic condition. The first reset integrator was introduced in [14] in order to improve the performance of linear systems. Several models for reset control systems and various design tools are currently available in the literature. One of the models for (closed-loop) reset control systems that has been widely used in the literature, see e.g. [6] and the references therein, assumes the form

$$\dot{x} = A_{cl}x + B_{cl}d \quad x \notin \mathcal{M} \tag{11}$$

$$x^+ = A_R x \qquad\qquad x \in \mathcal{M} \tag{12}$$

where $\mathcal{M} := \{x \in \mathbb{R}^n \mid C_{cl}x = 0, (I - A_R)x \neq 0\}$; A_{cl}, B_{cl}, C_{cl} are the closed-loop system matrices; A_R is the reset control matrix; x is the state of the system; and d is an exogenous signal. The set where resets are possible is a subset of $\{x \in \mathbb{R}^n \mid C_{cl}x = 0\}$ and is given by $D := \mathcal{M}$, while the set where the flows are active is given by $C := \mathbb{R}^n \setminus \mathcal{M}$. Note that the latter set corresponds to almost every point in the state space. It follows that for every trajectory of the system it is possible to construct an arbitrarily small measurement noise signal so that the measurement of the state never belongs to the jump set $\mathcal{M}$, so that the solution never jumps.

Therefore, in the presence of arbitrary small measurement noise, there exist solutions to the reset control system that never jump. Note that since the measurement noise can be picked arbitrarily small, a sequence of solutions converging to a solution that never jumps under the presence of measurement noise with magnitude converging to zero can be constructed, a Hermes solution to the reset control system. The limiting solution corresponds to a Krasovskii solution to the reset control system and it satisfies (K1) and (K2) in Definition 4 on the regularized sets $\overline{C} = \mathbb{R}^n$ and $\overline{D} = \mathcal{M}$, respectively.

This lack of robustness not only arises in situations where exogenous signals are present in the system but also in numerical simulation. When the reset control system (11)-(12) is implemented in Simulink with an integrator with reset and simple function blocks, the discretization in time produced by the ODE solver may prevent the resets from being triggered and one has to appeal to special Simulink blocks with zero-crossing detection. These special blocks confer certain robustness properties to the closed loop and, in some situations, make the simulation possible while affecting the model considered in the first place.

Now consider the state-dependent impulsive dynamical system first introduced in [5] that is modeled as (see also [12] and the references therein)

$$\dot{x} = f_c(x) \qquad x \notin \mathcal{M} \tag{13}$$

$$x^+ = x + f_d(x) \qquad x \in \mathcal{M} \tag{14}$$

where the function f_c defines the continuous dynamics, the function f_d defines the discrete dynamics, and $\mathcal{M}$ is the reset set. In most applications of state-dependent impulsive dynamical systems, the reset set $\mathcal{M}$ defines a surface in $\mathbb{R}^n$ (for example, see the modeling examples in [12] or the feedback control strategies proposed in [37, 33]). In such situations, it is also the case that arbitrarily small measurement noise in the state x can prevent every solution to the closed-loop system from jumping.

Example 6 (optimal control). In many robotics applications, optimal navigation algorithms for mobile robots are designed by switching between several feedback laws when the state of the system reaches the switching surface corresponding to the current operation mode, see e.g. [1],[17], [7]. Since the switches between modes occur when the state reaches the switching surface, arbitrarily small measurement noise can prevent the switches from occurring, and consequently, can cause the navigation task to fail.

Consider the example given in [7, Section 3] where a mobile robot of the unicycle type is optimally steered from its initial location to a target (by optimality the authors mean that the vehicle reaches the target while avoiding obstacles so that it minimizes a cost function that penalizes the distance from the obstacle and the proximity to the target). In this case, a hysteresis-type switching scheme is designed around a circular obstacle by defining two circular surfaces given by $g_i(x, y, a_i) = (x_0 - x)^2 + (y_0 - y)^2 - a_i^2$, $i = 1, 2$, $a_2 > a_1$. When the surface g_1 is reached with the vector field pointing inwards, the control law switches to the one that drives the vehicle away from the obstacle while when the surface g_2 is reached with the vector field pointing outwards, the control law is switched to the one that steers the vehicle to the target. Figure 6 depicts this scenario. Even though this strategy solves the chattering problem when only one switching surface is considered, arbitrary small measurement noise can prevent the switch on the surface g_1 from happening (causing the vehicle to crash against the obstacle) or it can also preclude the switch on the other surface to occur (causing the vehicle to miss its target). Note that the nonrobustness phenomenon in this example

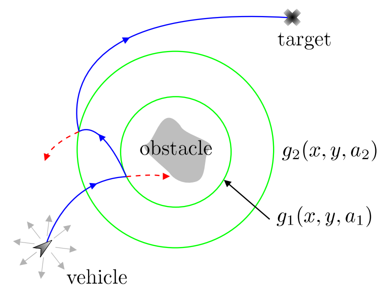


Fig. 2. Steering a vehicle to its target: the circles represent the switching surfaces for the control strategy. "Classical" (solid) and generalized (dashed) solutions to the optimal control problem in Example 6.

is not due to the existence of obstacles itself (see [39]), it is mainly related to the fact that the concept of solution and the modeling framework were not designed for asymptotic stability to be robust.

5 Conclusions

In this paper, motivated by the problem of robust stabilization of hybrid systems, we have discussed the concepts of hybrid Krasovskii, Hermes, and control-Hermes solutions. We have established that these three concepts of generalized solutions are equivalent. This equivalence implies that hybrid Krasovskii solutions can be approximated with arbitrary precision by solutions to the unregularized system with (arbitrarily small) measurement noise. By examples of theoretical and practical relevance, we have motivated the use of generalized solutions in the design of robust hybrid control systems.

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