On the Synchronization of Two Impulsive Oscillators under Communication Constraints

Sean Phillips, Ricardo G. Sanfelice, and R. Scott Erwin

***Abstract*— The problem of establishing synchronization of a class of impulsive oscillators is considered. Each impulsive oscillator is modeled as a hybrid system that self resets to zero when its state reaches a threshold and is externally reset via an impulsive update law when information from other oscillators is received. At every reset to zero, the oscillators broadcast a packet and select a different communication channel. The mechanism resembles that of a firefly model in which the external resets correspond to flashes of the fireflies. Oscillators can only receive information when they are on the same channel that a packet was broadcast. This mechanism leads to the possibility of information loss. Under such a communication constraint, we show that the coupled impulsive oscillators (almost globally) synchronize their channel selections. To establish this result, we model the interconnection of oscillators as a hybrid system and apply recently developed Lyapunov stability tools. Numerical simulations are included to illustrate the results.**

I. Introduction

Impulsive oscillators are dynamical systems with states that evolve continuously until an event occurs, at which instant they exhibit an instantaneous jump; see, e.g., [1], [2], [3]. Networks of such oscillators, under the name *integrate and fire oscillators* and *pulse-coupled oscillators*, have been employed to capture the dynamics of a wide range of biological systems [4], [1], including neurons, heart muscle cells, crickets, and fireflies; as well as in the design of network communication algorithm [5], [6], [7]. Relying only on minimal information, such networks of impulsive oscillators have been shown to asymptotically synchronize their variables in ideal communication settings [1].

In this paper, we analyze the synchronization properties of pulse-coupled oscillators communicating over two bidirectional channels. Bidirectional communication channels allow for both transmission and reception of information on the same channel. These channels are available to each oscillator. In such a setting, the packets transmitted by each oscillator on the currently chosen channel generate an event in the other oscillator only if they are on same communication channel.

Our motivation for the study of this problem stems from the control of networked reconfigurable systems, in particular, cognitive radio systems in space applications, in which system parameters, such as communication channel, transmission power, and direction of transmission or reception, can be updated in real time to react to a changing environment. In space applications, two-agent communication scenarios are key as they capture the scenario consisting of a ground station establishing a link with a satellite. Cognitive radio is a form of software defined radio. It is an agile communication system capable of dynamically changing its protocols with the rapid changes of the environment, due to adversarial jammers, managing communication with a primary user, and rendezvousing with other users in a decentralized network. In this context, the impulsive oscillators represent agents or radios that, through dynamic selection of the communication channel parameters, synchronize channel access with minimal information using feedback-based protocols. This feature could be advantageous in preventing adversaries from disrupting agent-to-agent communication since no pre-specified channel selection is made. In fact, traditional algorithms for establishing communication between nodes rely on a fixed channel selection sequence, such as the so-called *frequency hopping algorithm* which assigns to each agent a frequency-hopping pattern specifying the sequence (or code) of frequencies at which transmission is allowed [8]; see related work in [9], [10] and [11]. Compared with such works, a key feature of the algorithm emerging from the impulsive synchronization problem studied in this paper is that it does not require pilot tones on a pre-specified channel and that the channel selection patterns are determined in real time and based on feedback control.

The approach taken in this paper consists of modeling the pulse-coupled oscillators as a hybrid dynamical system, with continuous dynamics capturing the evolution of the oscillator's state in between impulses and discrete dynamics modeling self- and externally-triggered impulses. The resulting hybrid system contains continuous states, which are timers corresponding to the oscillator's variables, and discrete states, which are variables denoting the channel selected by each oscillator. Synchronization is recast as a compact set stabilization problem. Asymptotic stability of this set implies that the difference between the states of the oscillators and of the logic variables representing the selected channels converge to zero. Analysis is performed using the framework of hybrid systems in [12] and tools to assert asymptotic stability in [12], [13]. We construct a Lyapunov function to show synchronization for the case of two oscillators on two channels.

The remainder of this paper is organized as follows. Section II is devoted to modeling. Section III presents the main tools for analysis as well as the main result. Numerical simulations are presented in Section IV.

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II. Modeling Impulsive Oscillators with Channel Dependency

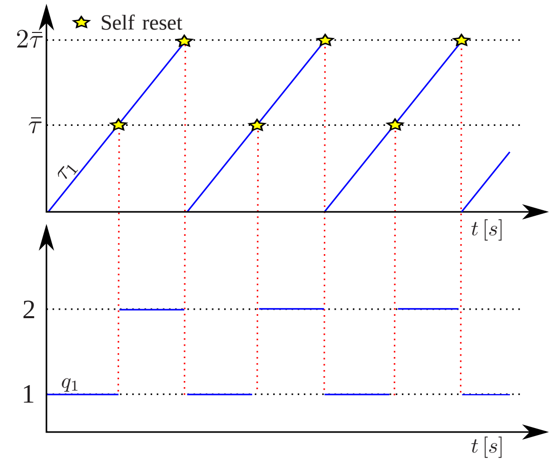
The pulse-coupled oscillator system of study consists of oscillators defining the agents with continuous states given by timers (τ_1, τ_2) and discrete states (q_1, q_2) denoting the current channel selection. These states are discretely updated when they reach a threshold and are externally reset when information is received. Information arrives to each agents from pre-defined channels. The agents can listen to one channel at a time.

Consider the case of two agents communicating over two channels via the following mechanism:

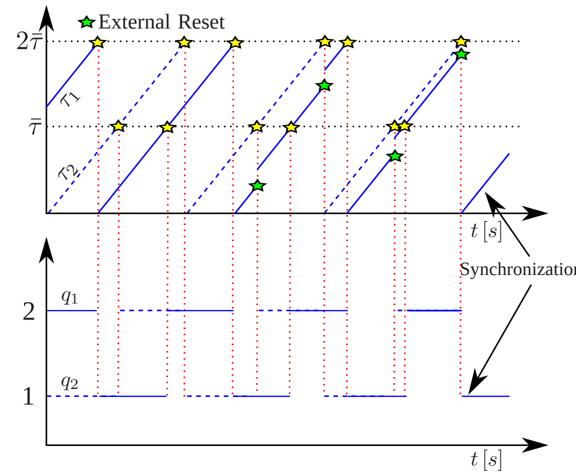
A) Each agent listens on the currently selected channel until its timer expires. Under such an event, the agent transmits a signal (or packet) on the current channel, resets its timer to zero, and switches to the other channel. Figure 1(a) shows that situation for a single pulse-coupled oscillator.

B) If an agent receives a packet while listening on the currently chosen channel, its timer is reset via an update law that reduces the listening time on that channel for the receiving agent. Figure 1(b) demonstrates the interaction between two pulse-coupled oscillators.

This mechanism can be thought as a control algorithm. It is inspired by synchronization of biological systems in [4], [1], where agents can "listen" all the time. In fact, the main difference between the mechanism above and the synchronization mechanism studied in [1] is that here there is a constraint on data reception, which depends on the channel currently chosen by the agents and does not guarantee that information sent is always received. In the case of a common channel and no information loss, the agents will synchronize as in the work of [1].



(a) A single pulse-coupled oscillator.



(b) Two pulse-coupled oscillators.

Fig. 1. (a) Trajectories (τ_1, q_1) of a single pulse-coupled oscillator over two channels. (b) Trajectories (τ_1, q_1) and (τ_2, q_2) of two pulse-coupled oscillators over two channels.

A. Hybrid Modeling

Our approach is to use a hybrid system model to capture mathematically the mechanism outlined in A) and B) above. Hybrid systems allow for states that both flow and jump, and allow for analysis of the above mechanism in which events cause timers and channel selection states jump. To this end, we follow the hybrid formalism of [12], where a hybrid system $\mathcal{H}$ is given by four objects (C, f, D, G) defining its *data*:

- *Flow map*: a single-valued map $f : \mathbb{R}^n \to \mathbb{R}^n$ defining the flows (or continuous evolution) of $\mathcal{H}$.
- *Flow set*: a set $C \subset \mathbb{R}^n$ specifying the points where flows are possible.
- *Jump map*: a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the jumps (or discrete evolution) of $\mathcal{H}$.
- *Jump set*: a set $D \subset \mathbb{R}^n$ specifying the points where jumps are possible.

Then, a hybrid system $\mathcal{H} := (C, f, D, G)$ can be written in the compact form

$$\mathcal{H}: \qquad x \in \mathbb{R}^n \qquad \left\{ \begin{array}{lcll} \dot{x} & = & f(x) & x \in C \\ x^+ & \in & G(x) & x \in D \end{array} \right. , \quad (1)$$

where the state x can contain both continuous and discrete states. We refer the reader to [12] for more details about this modeling framework.

From the outline in A) and B) above a two agent/two channel mechanism can be modeled as a hybrid system in (1). We denote it as $\mathcal{H}_{2,2}$. For each $i \in \{1, 2\}$, the i-th agent has a timer state τ_i and a channel state $q_i \in Q := \{1, 2\}$. The timer state takes value in the set $[0, 2\bar{\tau}]$, where $\bar{\tau} > 0$ is a parameter defining the threshold for jumps. Then, the state of the two agent/two channel system is given by

$$x := \begin{bmatrix} \tau_1 \\ q_1 \\ \tau_2 \\ q_2 \end{bmatrix} \in P := [0, 2\bar{\tau}] \times Q \times [0, 2\bar{\tau}] \times Q.$$

The flow and jump sets are defined to constrain the evolution of the timers (τ_1, τ_2) and the channel state (q_1, q_2). For example, when agent 1 is listening to channel one, that is, $q_1 = 1$, the timer τ_1 takes value in the set $[0, \bar{\tau}]$, while when agent 1 is listening to channel two, $q_1 = 2$, and τ_1 takes value in $[\bar{\tau}, 2\bar{\tau}]$. Then, flows are allowed when each of the agent's timers are within the range corresponding to the current channel on which they are listening. This is captured via the flow set

$$C := \{x \in P : (\tau_1, q_1) \in C_1, (\tau_2, q_2) \in C_2\}, \quad (2)$$

where, for each $i \in Q$,

$$C_i := \{(\tau_i, q_i) \in [0, 2\bar{\tau}] \times Q : (q_i - 1)\bar{\tau} \leq \tau_i \leq q_i\bar{\tau}\}.$$

During flows, the timers count ordinary time and the channel state remains constant, i.e.,

$$f(x) := \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^\top \qquad \forall\, x \in C. \quad (3)$$

The discrete events described in A) and B) above are modeled by a jump set D and a jump map G. The events or jumps are triggered when a timer expires, i.e., the jump set D captures timer resets and packet reception events. These events correspond to either timer reaching its threshold. More precisely:

$$D := \{x \in C : (\tau_1, q_1) \in D_1\} \cup \{x \in C : (\tau_2, q_2) \in D_2\}, \quad (4)$$

where $D_i := \{(\tau_i, q_i) \in [0, 2\bar{\tau}] \times Q : \tau_i = q_i\bar{\tau}\}$. In such a case, the agent whose timer expired transmits a packet on its current channel and changes channel by updating its channel state and timer state appropriately. If the other agent is in the same channel, then its timer is incremented by a timer advance constant, $\varepsilon \in (0, 2\bar{\tau}]$, as to reduce the listening time on that channel. This is captured via the jump map

$$G(x) = \begin{cases} g_1(x) & \text{if} \quad \tau_1 = q_1\bar{\tau},\, \tau_2 < q_2\bar{\tau},\, q_1 = q_2 \\ g_2(x) & \text{if} \quad \tau_1 < q_1\bar{\tau},\, \tau_2 = q_2\bar{\tau},\, q_1 = q_2 \\ g_3(x) & \text{if} \quad \tau_1 = q_1\bar{\tau},\, \tau_2 < q_2\bar{\tau},\, q_1 \neq q_2 \\ g_4(x) & \text{if} \quad \tau_1 < q_1\bar{\tau},\, \tau_2 = q_2\bar{\tau},\, q_1 \neq q_2 \\ g_5(x) & \text{if} \quad \tau_1 = q_1\bar{\tau},\, \tau_2 = q_2\bar{\tau},\, q_1 = q_2 \\ g_6(x) & \text{if} \quad \tau_1 = q_1\bar{\tau},\, \tau_2 = q_2\bar{\tau},\, q_1 \neq q_2 \end{cases} \quad (5)$$

$$\forall\, x \in D.$$

The first case (g_1) of G corresponds to the case when agent 1's timer reaches a threshold, which means that it is about to transmit a packet, agent 2's timer is not at a threshold, which means that it is listening, while both agents are on the same channel. In this way, g_1 is defined as

$$g_1(x) = \begin{bmatrix} (2 - q_1)\bar{\tau} \\ 3 - q_1 \\ n_2(x) \end{bmatrix},$$

$$n_2(x) = \begin{cases} \begin{bmatrix} \tau_2 + \varepsilon \\ q_2 \end{bmatrix} & \text{if} \quad \tau_2 + \varepsilon < q_2\bar{\tau} \\ \begin{bmatrix} (2 - q_2)\bar{\tau} \\ 3 - q_2 \end{bmatrix} & \text{if} \quad \tau_2 + \varepsilon > q_2\bar{\tau} \\ \left\{ \begin{bmatrix} \tau_2 + \varepsilon \\ q_2 \end{bmatrix}, \begin{bmatrix} (2 - q_2)\bar{\tau} \\ 3 - q_2 \end{bmatrix} \right\} & \text{if} \quad \tau_2 + \varepsilon = q_2\bar{\tau} \end{cases}.$$

The function n_2 describes how agent 2 reacts to incoming information, allowing for timer resetting and channel switching if the jump pushes the timer past the threshold. More precisely, if the new value of the timer τ_2, which after the reset is $\tau_2 + \varepsilon$, is below the current threshold, then update τ_2 to $\tau_2 + \varepsilon$, but if it is above the current threshold, then reset it as if it expired and switch channels. When $\tau_2 + \varepsilon = q_2\bar{\tau}$ then the jump set is within a set of values and will do either.

The definition of the function g_2 in G is as g_1, but with reverse roles. More precisely, it is given as

$$g_2(x) = \begin{bmatrix} n_1(x) \\ (2 - q_2)\bar{\tau} \\ 3 - q_2 \end{bmatrix},$$

$$n_1(x) = \begin{cases} \begin{bmatrix} \tau_1 + \varepsilon \\ q_1 \end{bmatrix} & \text{if} \quad \tau_1 + \varepsilon < q_1\bar{\tau} \\ \begin{bmatrix} (2 - q_1)\bar{\tau} \\ 3 - q_1 \end{bmatrix} & \text{if} \quad \tau_1 + \varepsilon > q_1\bar{\tau} \\ \left\{ \begin{bmatrix} \tau_1 + \varepsilon \\ q_1 \end{bmatrix}, \begin{bmatrix} (2 - q_1)\bar{\tau} \\ 3 - q_1 \end{bmatrix} \right\} & \text{if} \quad \tau_1 + \varepsilon = q_1\bar{\tau} \end{cases}.$$

Functions g_3 and g_4 capture the cases when the agents are in different channels. They are given by

$$g_3(x) = \begin{bmatrix} (2 - q_1)\bar{\tau} \\ 3 - q_1 \\ \tau_2 \\ q_2 \end{bmatrix}, \quad g_4(x) = \begin{bmatrix} \tau_1 \\ q_1 \\ (2 - q_2)\bar{\tau} \\ 3 - q_2 \end{bmatrix}.$$

When both agents reach their threshold at the same time neither are listening, so they both change channels and reset their timers. Functions g_5 and g_6 correspond to such a case, where g_5 corresponds to the case that the agents are in the same channel while g_6 to the case when they are not. Then

$$g_5(x) = g_6(x) = \begin{bmatrix} (2 - q_1)\bar{\tau} \\ 3 - q_1 \\ (2 - q_2)\bar{\tau} \\ 3 - q_2 \end{bmatrix}.$$

B. Basic Properties of $\mathcal{H}_{2,2}$

To apply analysis tools for hybrid systems, which will be presented in Section III, the data of the hybrid system $\mathcal{H}_{2,2}$ must meet certain mild conditions [12]. These conditions, referred to as *Basic Assumptions*, are as follows:

A1) C and D are closed sets in $\mathbb{R}^n$.

A2) $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on C.

A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on D, and such that $G(x)$ is nonempty for each $x \in D$.

A set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graph $\{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$ is closed. In terms of set convergence, G is outer semicontinuous if and only if, for each $x \in \mathbb{R}^n$ and each sequence $x_i \to x$, the outer limit $\limsup_{i\to\infty} G(x_i)$ is contained in $G(x)$. The mapping G is *locally bounded* on a set D if, for each compact set $K \subset D$, $G(K)$ is bounded.

Lemma 2.1: The data of $\mathcal{H}_{2,2}$ satisfies the Basic Assumptions.

III. Synchronization Properties of the Hybrid System Model for Two Impulsive Oscillators with Channel Dependency

In this section, we summarize tools for stability analysis of hybrid systems and then apply them to the two agents/two channels system $\mathcal{H}_{2,2}$.

A. Tools for Stability Analysis of Hybrid Systems

Solutions to general hybrid systems $\mathcal{H}$, $\mathcal{H}_{2,2}$ in particular, can evolve continuously (or flow) and/or discretely (or jump) depending on the continuous and discrete dynamics and the sets where those dynamics apply. We treat the number of jumps as an independent variable j and we parameterize the state by (t, j). Solutions to $\mathcal{H}$ will be given by *hybrid arcs* on *hybrid time domains*.

Definition 3.1: (hybrid time domain) A subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. A subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Definition 3.2: (hybrid arc) A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is locally absolutely continuous.

Definition 3.3: (solution) A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and:

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) = F(x(t, j)) \ .$$

(S2) For all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)) \ .$$

A solution x is said to be *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *maximal* if there does not exist a solution x' such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded.

Our goal is to show that the solutions $x = (\tau_1, q_1, \tau_2, q_2)$ to $\mathcal{H}_{2,2}$ are such that

$$\tau_1(t, j) - \tau_2(t, j) \to 0 \qquad \text{and} \qquad q_1(t, j) - q_2(t, j) \to 0$$

as $t + j \to \infty$, and that if the initial conditions $\tau_1(0, 0)$, $q_1(0, 0)$ and $\tau_2(0, 0)$, $q_2(0, 0)$ are close, then the solutions stay close. In other words, our goal is to show that the compact set

$$\mathcal{A} := \{x \in C \cup D : \tau_1 = \tau_2, q_1 = q_2\} \qquad (6)$$

is asymptotically stable for the hybrid system $\mathcal{H}_{2,2}$. Due to the evolution of the timers being periodic when in $\mathcal{A}$, asymptotic stability of $\mathcal{A}$ is a synchronization property for the agent timers. A precise definition of asymptotic stability for hybrid systems $\mathcal{H}$ is given next.

Definition 3.4 (stability): A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution x with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$;
- *attractive* if there exists $\mu > 0$ such that every solution x with $|x(0, 0)|_{\mathcal{A}} \leq \mu$ is complete and satisfies $\lim_{(t,j) \in \operatorname{dom} x, t+j \to \infty} |x(t, j)|_{\mathcal{A}} = 0$;
- *asymptotically stable* if stable and attractive;

where $|x|_\Sigma$ is generally defined as $\inf_{y \in \Sigma} |x - y|$ for the set $\Sigma \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$. The set of points from where the attractivity property holds is the basin of attraction.

A Lyapunov function can be employed to show that the compact set in (6) is asymptotically stable. For a function V to be considered a Lyapunov candidate it must meet the following requirements.

Definition 3.5: (Lyapunov function candidate) Given the hybrid system $\mathcal{H}$ with data (C, f, D, G) and the compact set $\mathcal{A} \subset \mathbb{R}^n$, the function $V : \operatorname{dom} V \to \mathbb{R}$ is a *Lyapunov function candidate* for $(\mathcal{H}, \mathcal{A})$ if

i) V is continuous and nonnegative on $(C \cup D) \setminus \mathcal{A} \subset \operatorname{dom} V$,

ii) V is continuously differentiable on an open set $\mathcal{O}$ satisfying $C \setminus \mathcal{A} \subset \mathcal{O} \subset \operatorname{dom} V$, and

iii) $\lim_{\{x \to \mathcal{A},\, x \in \operatorname{dom} V \cap (C \cup D)\}} V(x) = 0$.

Conditions i) and iii) hold when $\operatorname{dom} V$ contains $\mathcal{A} \cup C \cup D$, V is continuous and nonnegative on its domain, and $V(z) = 0$ for all $x \in \mathcal{A}$. The following result from [12, Theorem 23] states the conditions on V for asymptotic stability of a compact set. Below, a level set $L_V(\mu)$ refers to the set of all points in $C \cup D$ such that $V(x) = \mu$, i.e., $L_V(\mu) := \{x \in C \cup D : V(x) = \mu\}$.

Theorem 3.6: [12, Theorem 23] Consider a hybrid system $\mathcal{H} = (C, f, D, G)$ satisfying the Basic Assumptions and a compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfying $G(D \cap \mathcal{A}) \subset \mathcal{A}$. If there exists a Lyapunov function candidate V for $(\mathcal{H}, \mathcal{A})$ that is positive on $(C \cup D) \setminus \mathcal{A}$ and satisfies

$$\langle \nabla V(x), f(x) \rangle \leq 0 \quad \textit{for all } x \in C \setminus \mathcal{A},$$

$$V(g) - V(x) \leq 0 \quad \textit{for all } x \in D \setminus \mathcal{A}, g \in G(x) \setminus \mathcal{A}$$

then the set $\mathcal{A}$ is stable. If, furthermore, there exists a compact neighborhood K of $\mathcal{A}$ such that, for each $\mu > 0$, no complete solution to $\mathcal{H}$ remains in $L_V(\mu) \cap K$, then the set $\mathcal{A}$ is asymptotically stable. In this case, the basin of attraction contains every compact set contained in K that is forward invariant.

B. Asymptotic Stability Analysis of $\mathcal{H}_{2,2}$

The overall goal of this section is to determine the stability and attractivity properties of the set of points (6). We consider the function $V : \mathbb{R}^4 \to \mathbb{R}$ given by

$$V(x) = (1 - \rho(x))V_1(x) + \rho(x)V_2(x) \qquad \forall x \in C \cup D, \quad (7)$$

where V_1 is a piecewise function given by

$$V_1(x) = \begin{cases} \frac{1}{\varepsilon}(\tau_1 - \tau_2)^2 + \frac{\varepsilon}{4} \\ \qquad\qquad \text{if } |\tau_1 - \tau_2| \le \frac{\varepsilon}{2}, \\ \frac{1}{\varepsilon}(\tau_1 - \tau_2 - 2\bar{\tau})^2 + \frac{\varepsilon}{4} \\ \qquad\qquad \text{if } \tau_1 - \tau_2 \ge 2\bar{\tau} - \frac{\varepsilon}{2}, \\ \frac{1}{\varepsilon}(\tau_1 - \tau_2 + 2\bar{\tau})^2 + \frac{\varepsilon}{4} \\ \qquad\qquad \text{if } \tau_1 - \tau_2 \le -2\bar{\tau} + \frac{\varepsilon}{2}, \\ V_2(x) \\ \qquad\qquad \text{if } |\tau_1 - \tau_2| \in \left(\frac{\varepsilon}{2}, 2\bar{\tau} - \frac{\varepsilon}{2}\right), \end{cases},$$

here $\varepsilon \in (0, 2\bar{\tau}]$, V_2 is given by

$$V_2(x) = \min\{|\tau_1 - \tau_2|, 2\bar{\tau} - |\tau_1 - \tau_2|\},$$

and, ρ is a C^1 function satisfying

$$\rho(x) = \begin{cases} 0 & \text{if } q_1 \neq q_2, \; q_1, q_2 \in \{1, 2\} \\ 1 & \text{if } q_1 = q_2 \in \{1, 2\} \end{cases}$$

and, for all $x \in C$,

$$\nabla_{\tau_1}\rho(x) = 0 \qquad \text{and} \qquad \nabla_{\tau_2}\rho(x) = 0.$$

For points not in $C \cup D$, the Lyapunov function V is given by any positive and continuous function that is continuously differentiable (almost everywhere). Furthermore, it can be verified that the Lyapunov function satisfies the conditions in Definition 3.5.

Using $\bar{\tau} = 1$ and $\varepsilon = 0.3$, Figure 2 shows V when $q_1 = q_2$, and $q_1 \neq q_2$. Note that when $q_1 = q_2$, we have $V(x) = V_2(x)$, while when $q_1 \neq q_2$, we have $V(x) = V_1(x)$. This function was constructed in this way to eliminate points where $V(x) = 0$ outside of the compact set $\mathcal{A}$, which are points belonging to the blue lines in Figure 3. The function is not differentiable at these points and at points $\tau_2 = \tau_1 \pm \bar{\tau}$. The latter points, which are denoted by the green lines in Figure 3, will need to be removed from the basin of attraction.

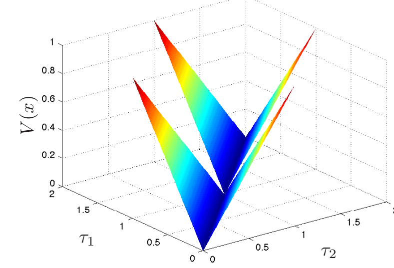
The following stability result for $\mathcal{H}_{2,2}$ can be established using the Lyapunov function in (7) and Theorem 3.6.

Theorem 3.7: (Timer synchronization with limited information) For every $\bar{\tau} > 0$ and $\varepsilon \in (0, 2\bar{\tau}]$, the hybrid system $\mathcal{H}_{2,2}$ is such that $\mathcal{A}$ is asymptotically stable with basin of attraction containing every sublevel set $L_V(\mu)$ with $\mu \in [0, \bar{\tau})$.

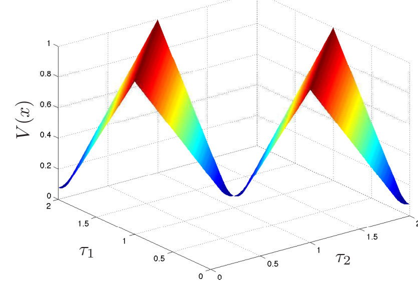
Remark 3.8: For initial conditions in $\{C \cup D \; : \; |\tau_1 - \tau_2| = \bar{\tau}, q_1 \neq q_2\}$ solutions $x(t, j)$ stay in the level set $V(\phi(t, j)) = \bar{\tau}$. Note the green lines in Figure 3. For such solutions, the state does not converge to $\mathcal{A}$ because both agents jump simultaneously but on opposite channels, and thus missing the information transmitted. This point is corroborated by a Lyapunov local maximum at these states. Solutions from all other initial conditions in $C \cup D$ approach the synchronization condition defined by $\mathcal{A}$.

IV. Numerical Analysis

Solutions to $\mathcal{H}_{2,2}$ fall into three categories: always synchronized, asymptotically synchronized, and desynchronized. The simulations below show the evolution of these solution types. The parameters used are $\bar{\tau} = 1$, $\varepsilon = 0.05$.



(a) $q_1 = q_2$



(b) $q_1 \neq q_2$

Fig. 2. A plot of the Lyapunov function V in (7) for each x in $C \cup D$

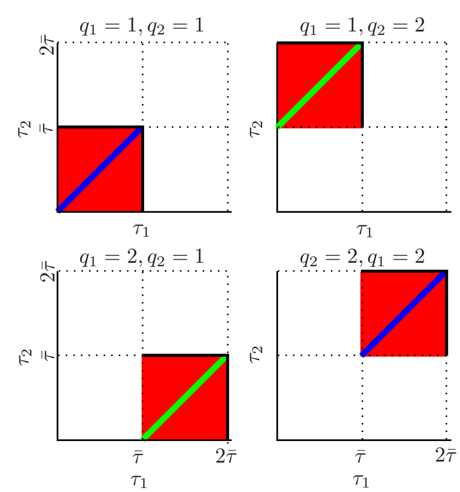
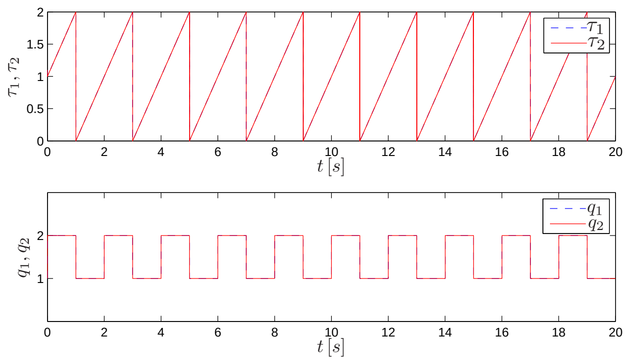


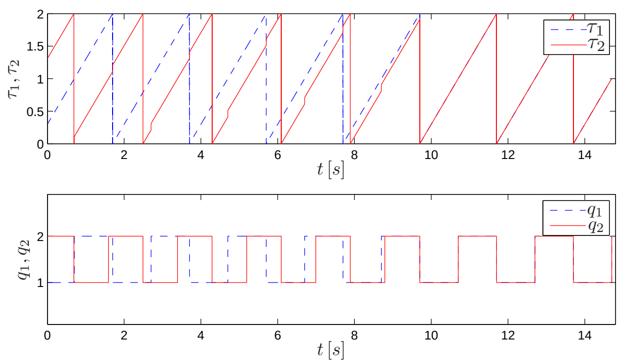
Fig. 3. The flow set (red), the jump set (solid black), and basin of attraction with μ-level sets $L_V(\bar{\tau})$ (in blue) and $L_V(0)$ (in green) for each pair $(q_1 .q_2)$

1) Always synchronized: A solution that starts in the set $\mathcal{A}$ will always stay synchronized, that is, $\mathcal{A}$ is forward invariant. Figure 4(a) shows the evolution of such a solution. The top figure shows the timer value and the bottom figure shows the channel of the agents.

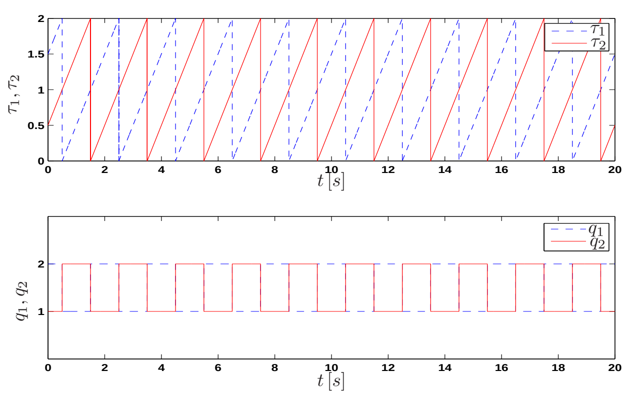
2) Asymptotically synchronized: A solution that starts close to $\mathcal{A}$ reaches synchronization rapidly. The initial condition for the simulation is such that $|\tau_1 - \tau_2| < \varepsilon$, so after one jump the two timers are the same. When the two timers start close to the set of points from where synchronization is not possible, the time needed to reach synchronization is much larger. The simulation in Figure 4(b) shows that the solution starts far from $\mathcal{A}$ but still converges. The initial conditions for these simulations are $\tau_1(0,0) = 0.3, q_1(0,0) = 1$, and $\tau_2(0,0) = 1.31$, $q_2(0,0) = 2$.



(a) A solution to $\mathcal{H}_{2,2}$ that is always synchronized.



(b) A solution to $\mathcal{H}_{2,2}$ that asymptotically synchronizes after several transitions.



(c) A solution to $\mathcal{H}_{2,2}$ that never synchronizes.

3) Desynchronized: When the agents start from an initial condition satisfying $|\tau_1(0,0) - \tau_2(0,0)| = \bar{\tau}$ and $q_1(0,0) \neq q_2(0,0)$, they stay desynchronized. The initial conditions $\tau_1(0,0) = 1.5$, $\tau_2(0,0) = .5$, $q_1(0,0) = 2$ and $q_2(0,0) = 1$ are used for the simulation in Figure 4(c). It shows that each agent has an offset leading to continually miss the other agent's transmission since they switch to opposite channels at every jump.

V. Conclusion

Synchronization of a class of two impulsive oscillators was shown through Lyapunov analysis in a hybrid framework. For almost every point in the space of the timers, the oscillators synchronize. Lost packets do not effect the asymptotic stability property, but leads to slower convergence than when there is no channel constraints. Extensions of the results to the multiple agents/multiple channels case as well as of more general update laws building from the arguments are currently under investigation.

VI. Acknowledgments

Research partially supported by the U.S. Air Force Summer Faculty Fellowship Program, American Society for Engineering Education, and the National Science Foundation under CAREER Grant no. ECS-1150306. The authors are indebted to Clayton Chu who helped implementing in the test bed the two agents/two channels hybrid system, to Tom Cleary for the initial research that led to this paper, and to David Copp for his help developing the hybrid simulator used for numerical analysis.

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