Tracking control of mechanical systems with a unilateral position constraint inducing dissipative impacts

J.J.B. Biemond, N. van de Wouw, W.P.M.H. Heemels, R.G. Sanfelice, and H. Nijmeijer

***Abstract*— In this paper, the tracking control problem is considered for mechanical systems with unilateral constraints with dissipative impacts. In these systems, impacts are triggered at the exact moment when the constraint becomes active. Typically, a small time mismatch is introduced between the impacts of the plant and the reference, even if this trajectory is arbitrarily close to the reference. Consequently, the Euclidean tracking error cannot behave stable in the sense of Lyapunov, such that standard tracking control approaches are unfeasible. However, desirable tracking behaviour does not imply that the Euclidean error vanishes asymptotically over time. We design continuous-time controllers that can handle the impact time mismatch and achieve accurate tracking of reference trajectories containing dissipative impacts for mechanical systems with a unilateral constraint. The behaviour of the resulting closed-loop dynamical system is illustrated with an exemplary bouncing ball system.**

I. Introduction

Many mechanical systems contain unilateral position constraints and experience impacts when a unilateral constraint becomes active. In this paper, we will design tracking controllers for mechanical systems with dissipative impacts (a restitution coefficient smaller than one), and model these as hybrid systems, which are characterised by the combination of continuous-time dynamics and jumps, cf. [1]–[3].

For mechanical systems with impacts, and more generally, for hybrid systems with state-triggered jumps, tracking controllers will encounter a "peaking" of the Euclidean tracking error, as observed in [4]–[11]: the plant and reference trajectory will generically have jumps with a small time mismatch, and during this time period, the Euclidean tracking error will be large, even though the tracking behaviour is as desired. Even for arbitrarily close initial conditions, a jump time mismatch is expected and during this time interval, the Euclidean error will be approximately equal to the norm of the jump in velocity at the impact. Hence, the Euclidean tracking error behaves unstable in the sense of Lyapunov and, consequently, standard tracking control designs do not apply.

For hybrid systems where jumps can be triggered by the controller, the "peaking" of the tracking error can be avoided by forcing the jumps of the plant to coincide with those of the reference trajectory. Such controllers are designed in [4], [7], [12], [13], and observers where jumps of the observer coincide with those of the plant are designed in [8]. However, for hybrid systems with state-triggered jumps, including mechanical systems with impacts, jumps of the plant cannot be forced to coincide with jumps of the reference trajectory. Therefore, different approaches are needed.

In the literature, several approaches are presented that allow to compare two hybrid trajectories with non-matching jump times. As suggested in [5], one could require stability in the sense of Zhukovsky, cf. [14]. In this stability concept, a rescaling of the time is allowed for the plant trajectory in order to match the jumps of the plant with the jumps of the reference trajectory. Alternatively, a Hausdorff-type metric between the graphs of the reference and plant trajectory is suggested in [15]. Both approaches rely on complete knowledge of the trajectories, and, consequently, it is not clear how these concepts can be used to formulate and solve the design problem of tracking controllers.

In [16], [17], tracking controllers are designed for mechanical systems with impacts where reference trajectories are required to be weakly stable. The weak stability property formulated in [16] implies that tracking errors are small away from the impact times. In [16], [17] reference trajectories are considered where impacts, if they occur, show accumulation points (i.e. Zeno behaviour), followed by a time interval where the constraint is active.

In the present paper, we focus on reference trajectories that experience impacts without accumulation. In that case, next to convergence of the Euclidean tracking error away from the impacts, the impact time mismatch between the reference and plant trajectories should converge to zero to obtain intuitively correct behaviour. In [5], [18], [19], tracking controllers are designed that achieve this behaviour for a class of hybrid systems with linear flow and jump dynamics, which include mechanical systems with impacts as special cases. In [5], [18], [19], periodic reference trajectories are considered, and the Euclidean tracking error is required to converge only away from the jump times. In addition, convergence of the jump times is ensured by evaluating the closed-loop dynamics using a return map.

In [10], a tracking problem is formulated by requiring convergence of a non-Euclidean tracking error measure that

J.J.B. Biemond, N. van de Wouw, W.P.M.H. Heemels and H. Nijmeijer are with the Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600MB Eindhoven, the Netherlands. Email: {j.j.b.biemond,n.v.d.wouw,m.heemels,h.nijmeijer}@tue.nl

R. G. Sanfelice is with the Department of Aerospace and Mechanical Engineering, University of Arizona, 1130 N. Mountain Ave, AZ 85721, USA. Email: sricardo@u.arizona.edu

This work is partially supported by the Netherlands organisation for scientific research (NWO), the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement no. 257462 HYCON2 Network of Excellence, the National Science Foundation under CAREER Grant no. ECS-1150306 and the Air Force Office of Scientific Research under Grant no. FA9550-12-1-0366.

is tailored to the specific hybrid system and constructed such that convergence of this tracking error measure corresponds to an intuitive notion of tracking, that is, trajectories converge to each other away from the impact times, and the impact time mismatch vanishes asymptotically over time. Additionally, the tracking error measure remains constant over jumps and, hence, does not exhibit peaks in its time evolution. Since this tracking error measure incorporates information on the velocity during the time interval near jumps, the tracking problem can be formulated based on this error only. In particular, we will only require asymptotically stable behaviour of the newly defined error. This property directly implies, in addition to the convergence of the Euclidean tracking error to zero when away from impact times, also the convergence of the jump time mismatch to zero over time, as shown in [10]. This is an advantage of our approach when compared to the approach in [5], [18], where a return map argument is required to prove convergence of jump times.

In [11], the approach presented in [10] is used to design tracking controllers for mechanical systems with non-dissipative impacts. For these systems, the state vector x, containing the position and velocity with respect to the impacting surface, is mapped onto $-x$ during an impact, where, prior to the impact, the position is zero and the velocity points in the direction of the constraint. Hence, in [11], a plant trajectory x is not required to track the reference trajectory r such that $|x - r| \to 0$, but instead, convergence of $d(r, x) = \min(|x - r|, |x + r|)$ to zero is required. Independently, in [9], [20], [21], tracking and observer problems are considered for billiard systems, and controllers are designed that ensure asymptotic stability of a set containing the reference trajectory and its mirror images. The local tracking controller developed for this set is similar to the tracking controller designed in [11].

Both the tracking controller design of [11] and the design of [9], [20], [21] exploit the property that the post-impact velocity equals minus the pre-impact velocity, and study the behaviour of $x-r$ and $x+r$ along closed-loop solutions (after an impact of x or r, $x+r$ equals the difference between the plant state x and the reference trajectory r before the impact). Due to this setup, the approach of [9], [11], [20], [21] is restricted to non-dissipative impacts where the restitution coefficient is equal to one. Tracking control for mechanical systems with dissipative impacts (with restitution coefficients strictly smaller than one) for general, non-periodic reference trajectories, has not been considered so far in the literature. This is highly relevant, since in physical systems, dissipation will always appear to some extent.

In the current paper, we will address the tracking problem for impacts with restitution coefficient $\epsilon \in (0, 1]$ for mechanical systems with one degree of freedom, inspired by the usage of the non-Euclidean tracking error measure as proposed in [10], [11]. For these systems, we show that a Lyapunov function can be defined with the following behaviour. When the reference experiences an impact prior to the plant, the velocity decrease of the reference at the impact is initially ignored by a rescaling of the reference state r with a factor $\frac{1}{\epsilon}$, such that the Lyapunov function remains constant. If, subsequently, the plant jumps, then this rescaling is undone, such that the Lyapunov function is decreasing. Effectively, in the case where the reference jumps first, the dissipative effect of the impacts of both the reference and plant trajectory is taken into account only after the jump of the plant. Using this rescaling function, a switching control law is designed that enables converging closed-loop behaviour of either $x - r$ (away from the impacts), $x + \frac{1}{\epsilon} r$ (when only r experienced an impact) or $\frac{1}{\epsilon} x + r$ (when only x experienced an impact), such that the plant trajectory converges to the reference away from the jump times, and an intuitively correct notion of tracking is achieved.

This paper is organised as follows. In Section II, we introduce the class of systems under study and define the corresponding solution concept. In Section III, we recall the tracking problem definition from [10] and define a tracking error measure for mechanical systems with dissipative impacts. Controllers solving the tracking problem are designed in Section IV, and are illustrated in Section V with an example. Section VI presents a discussion of the main result.

Notation

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}$ the set of real numbers; $\mathbb{N}$ the set of natural numbers including 0. Let $\overline{\text{co}}(S)$ denote the smallest closed convex hull containing a set $S \subset \mathbb{R}^n$, and let $S^2 = S \times S$. Given vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $|x|$ denotes the Euclidean vector norm, $\text{col}(x, y)$ denotes $\left[x^\top \; y^\top\right]^\top$ and, if $m = n$, then $\langle x, y \rangle$ denotes the inner product. A function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class-$\mathcal{K}_\infty$ (denoted $\alpha \in \mathcal{K}_\infty$) if it is continuous, zero at zero, strictly increasing and unbounded. For symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \prec 0$ ($A \succ 0$) when A is negative definite (positive definite) and $A \prec B$ ($A \succ B$) when $A - B \prec 0$ ($A - B \succ 0$). Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of a symmetric matrix A, respectively. Finally, if $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and $A \succ 0$ then $|x|_A^2 = x^\top A x$.

II. MECHANICAL SYSTEMS WITH IMPACTS

A. Modelling

In this paper, we consider mechanical systems with one degree of freedom (1DOF) and a single unilateral position constraint with impact, as depicted in Figure 1. As shown in [11], trajectories of such systems can be modelled with

$$\dot{x} = \begin{bmatrix} x_2 \\ f(t, x) + u + \lambda(x_1, x_2) \end{bmatrix}, \qquad x \in C := [0, \infty) \times \mathbb{R} \tag{1a}$$

$$x^+ = g(x) = \begin{bmatrix} x_1 \\ -\epsilon x_2 \end{bmatrix}, \quad x \in D := \{0\} \times (-\infty, 0), \tag{1b}$$

where the constraint is positioned at $x_1 = 0$. Here, (1a) describes the flow of trajectories with control signal u, which is the controller force when the system has unit mass, λ is the contact force when the constraint is active, and $f(t, x)$ represents other possible forces. Equation (1b) models impacts in which the velocity changes sign. Energy is dissipated according to the restitution coefficient $\epsilon \in (0, 1]$. The contact force $\lambda(x_1, x_2)$ satisfies

$$\lambda(x_1, x_2) \in \begin{cases} 0, & (x_1\ x_2) \neq (0\ 0), \\ [0, \infty)\,, & (x_1\ x_2) = (0\ 0), \end{cases} \tag{2}$$

and ensures that the unilateral contact constraint $x \geq 0$ is not violated when $x = [x_1\, x_2]^T = [0\, 0]^T$.

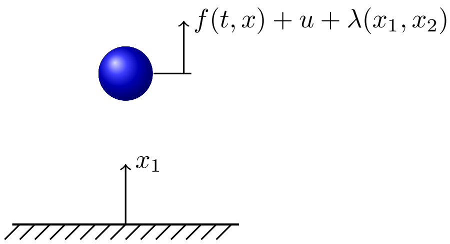

Fig. 1. Example of a mechanical system described by (1), (2).

In this paper, we present controllers that solve a local tracking problem near a reference trajectory r, that is a solution of (1) for a given feedforward signal $u = u_{\text{ff}}(t)$, where, at any time instant, r is bounded away from the origin. For all trajectories near this reference trajectory, the contact force λ vanishes, such that nearby trajectories are described by the simpler hybrid system

$$\dot{x} = F(t, x, u) := \begin{bmatrix} x_2 \\ f(t, x) + u \end{bmatrix},\ x \in C := [0, \infty) \times \mathbb{R} \tag{3a}$$

$$x^+ = g(x) := \begin{bmatrix} x_1 \\ -\epsilon x_2 \end{bmatrix},\ x \in D := \{0\} \times (-\infty, 0). \tag{3b}$$

Throughout this paper, we assume that u is bounded and f is continuous in x and locally essentially bounded in t.

In order to define solutions of the hybrid system (3), we assume that the input u satisfies $u(t) \in \mathcal{U}$ for a compact set $\mathcal{U} \subset \mathbb{R}$. Using the framework of [1], solutions φ of the hybrid system (3) are defined on a hybrid time domain $\operatorname{dom} \varphi \subset [0, \infty) \times \mathbb{N}$. A hybrid time instant is given as $(t, j) \in \operatorname{dom} \varphi$, where t denotes the continuous time lapsed, and j denotes the number of experienced jumps. The arc φ denotes a solution of (3) when, for all $(t, j) \in \operatorname{dom} \varphi$ such that $(t, j+1) \in \operatorname{dom} \varphi$, $\varphi(t, j) \in D$ and $\varphi(t, j+1) = g(\varphi(t, j))$, and, for almost all $t \in I_j := \{t |\ (t, j) \in \operatorname{dom} \varphi\}$ and all j such that I_j has non-empty interior, $\varphi(t, j) \in C$ and $\frac{d}{dt}\varphi(t, j) = F(t, \varphi(t, j), u(t, \varphi(t, j)))$. In other words, φ is a solution of (3a) during flow, and jumps satisfy (3b). In this paper, we only consider maximal solutions, i.e., solutions that cannot be continued towards a larger time domain. A solution φ is said to be non-Zeno if $\operatorname{dom} \varphi$ is unbounded in the t-direction.

The dynamics of (1)-(2) can also be represented effectively using complementarity systems or measure differential inclusions, cf. [22] and [6], [23], respectively. However, this paper builds upon results on tracking control results developed in [10] formulated in the framework of [1], which motivates the use of this framework in the present paper. Still, we foresee that the same rationale for the tracking control of jumping desired trajectories as proposed here for hybrid systems can be extended to complementarity systems and measure differential inclusions.

III. Tracking control problem

A. Tracking problem formulation

Tracking controllers for hybrid systems with state-triggered jumps, such as mechanical systems with impacts, will generically show the following "peaking" in the Euclidean tracking error, cf. [4]–[10], which we illustrate in Figure 3a) for an example. If jumps are state-triggered, i.e., they occur when the state reaches a certain surface in the state space, then, generically, a reference and plant trajectory that are initially close will not reach this surface exactly at the same time, but shortly after each other. Hence, in the intermediate time period, the Euclidean distance between the plant and reference trajectory will be approximately equal to the Euclidean norm of the jump. Consequently, if the Euclidean distance between plant and reference trajectory is considered as a tracking error, then, for a small time interval near the impacts, the error will be large, even when the initial error was arbitrarily small. Hence, this error behaves unstable in the sense of Lyapunov. Due to the "peaks" in the Euclidean error induced by the jump time mismatch, a tracking problem formulation that requires asymptotic stability of the Euclidean tracking error is not feasible for hybrid systems with state-triggered jumps, such as mechanical systems with impacts.

To resolve this problem, in [11], the present authors designed a local tracking controller for the case of ideal, non-dissipative impacts, i.e., $\epsilon = 1$ in (1b), by requiring asymptotic convergence of the function $d_{\text{ideal}}(r, x) = \min(|x - r|, |x + r|)$ along closed-loop trajectories, where d_{ideal} is considered as the tracking error measure. In the present paper, we formulate a tracking problem by requiring convergence of a tracking error definition d_ϵ tailored to the dissipative impact law (1b), which depends explicitly on the restitution coefficient $\epsilon \in (0, 1]$.

We will construct the function d_ϵ such that this tracking error does not change at jumps, i.e., $d_\epsilon(r, g(x)) = d_\epsilon(r, x)$ for each $x \in D$ and $d_\epsilon(g(r), x) = d_\epsilon(r, x)$ for each $r \in D$. Additionally, we design d_ϵ to be a continuous function on $(C \cup D)^2$. Consequently, when evaluated along closed-loop trajectories, the function $d_\epsilon(r, x)$ is a continuous function of t, and independent of j.

To construct a tracking error function d_ϵ with the properties given above, we will adapt d_{ideal} using the following coordinate transformation:

$$M_\epsilon(x) := \begin{bmatrix} x_1 \\ \alpha(x) x_2 \end{bmatrix}, \quad \text{with } \alpha(x) := \begin{cases} \frac{1}{\epsilon} & x_2 > 0 \\ 1 & x_2 \leq 0. \end{cases} \tag{4}$$

We can use d_{ideal} in this new coordinate system, which yields a distance function $d_\epsilon(r, x)$ given as

$$d_\epsilon(r, x) := \min(|M_\epsilon(x) - M_\epsilon(r)|, |M_\epsilon(x) + M_\epsilon(r)|). \tag{5}$$

This tracking error measure d_ϵ fulfills the requirements just mentioned as it is continuous and $d_\epsilon(r,x)$ remains constant when either x or r experiences a jump.

According to Theorem 1 of [10], convergence of this tracking error to zero ensures that for all $\delta > 0$, after a sufficiently long time, $|x(t) - r(t)| < \delta$ whenever the reference position r_1 satisfies $r_1(t) > \delta$. For the considered class of reference trajectories, if we take $\delta > 0$ sufficiently small, then the length of the individual time intervals where $r_1(t) < \delta$ holds (where 'peaking' can occur since $|x(t) - r(t)| < \delta$ may be violated) can be made arbitrarily small. Consequently, if $d_\varepsilon(r,x)$ converges to zero along closed-loop trajectories, then the duration of possible 'peaks' in the Euclidean tracking error converges to zero over time: the jump times of the plant trajectory converge to the jump times of the reference trajectory.

As already mentioned, analogous to the common approach in tracking control for ODEs, we consider reference trajectories r that are solutions to (3) for a given feedforward signal $u = u_{\text{ff}}(t)$. Now, the objective is to design a state- and time-dependent control law $u = u_d(t,r,x)$ such that $d_\epsilon(r,x)$ converges asymptotically stable to zero along the closed-loop trajectories. To investigate the evolution of $d_\epsilon(r,x)$ along trajectories of the closed-loop system, we combine the dynamics of the reference trajectory with the dynamics of the plant. For this purpose, we create an extended hybrid system with state $q = \text{col}(r,x)$. The dynamics of this hybrid system is then given by

$$\dot{q} = F_e(t,q), \qquad q \in C^2 \tag{6a}$$

$$q^+ = \text{col}(q_1, -\epsilon q_2, q_3, q_4), \qquad q \in D \times (C \cup D) \tag{6b}$$

$$q^+ = \text{col}(q_1, q_2, q_3, -\epsilon q_4), \qquad q \in (C \cup D) \times D, \tag{6c}$$

where

$$F_e(t,q) := \begin{bmatrix} q_2 \\ f(t, \text{col}(q_1,q_2)) + u_{\text{ff}}(t) \\ q_4 \\ f(t, \text{col}(q_3,q_4)) + u_d(t, \text{col}(q_1,q_2), \text{col}(q_3,q_4)) \end{bmatrix}. \tag{7}$$

We define $\bar{r}(t,j) := \text{col}(q_1,q_2)(t,j)$ and $\bar{x}(t,j) = \text{col}(q_3,q_4)(t,j)$, such that $\bar{r}, \bar{x} : \text{dom } q \to C \cup D$ are reparameterisations of $r : \text{dom } r \to C \cup D$ and $x : \text{dom } x \to C \cup D$ on the combined hybrid time domain $\text{dom } q$.

From [10], [11], we adopt the following stability definition and tracking problem formulation.

Definition 1 (Stability with respect to distance d_ϵ) *Let d_ϵ be given in* (5)*. A reference trajectory $r(t,j)$ of system* (3) *is called*

- stable with respect to d_ϵ *if for all $t_0, j_0 \geq 0$ and $\delta_1 > 0$ there exists a $\delta_2(t_0, j_0, \delta_1) > 0$ such that $\forall t \geq t_0, \forall j \geq j_0$*

$$d_\epsilon(\bar{r}(t_0,j_0), \bar{x}(t_0,j_0)) < \delta_2(t_0,j_0,\delta_1) \Rightarrow d_\epsilon(\bar{r}(t,j), \bar{x}(t,j)) < \delta_1; \tag{8}$$

- locally asymptotically stable with respect to d_ϵ *if it is stable with respect to d_ϵ and for any $t_0, j_0 \geq 0$ there exists a $\delta_3(t_0,j_0) > 0$ such that*

$$d_\epsilon(\bar{r}(t_0,j_0), \bar{x}(t_0,j_0)) < \delta_3(t_0,j_0) \Rightarrow \lim_{t+j\to\infty} d_\epsilon(\bar{r}(t,j), \bar{x}(t,j)) \to 0. \tag{9}$$

Using this definition, the tracking problem is formalised as follows.

Definition 2 (Tracking problem) *Given a hybrid system* (3) *with reference trajectory r, corresponding to a feedforward signal u_{ff}, design a control law $u_d(t,r,x)$ such that the trajectory r is locally asymptotically stable with respect to d_ϵ given in* (5)*.*

B. Sufficient conditions for stability

In order to guarantee that trajectories of (6) have hybrid time domains that are unbounded in $t-$direction, we require that r is non-Zeno, unique and bounded, as formalised in the following assumption.

Assumption 1 *The reference trajectory $r = \text{col}(r_1, r_2)$ is non-Zeno, bounded, satisfies $\inf_{(t,j)\in\text{dom } r} |r(t,j)| > 0$ and is the unique solution of* (3) *with a bounded feedforward signal u_{ff} and initial condition $r(0,0)$.* △

Sufficient conditions for the uniqueness of solutions to hybrid systems are given in [1, Proposition 2.11]. In our case, the required uniqueness of the reference trajectory implies that $\text{col}(q_1(t,j), q_2(t,j))$ represents a reparameterisation of the reference trajectory $r(t,j)$ when the initial condition $\text{col}(q_1,q_2)(0,0) = r(0,0)$ is chosen.

The following theorem of [11] provides sufficient conditions for the (local) asymptotic stability of a reference trajectory r using a Lyapunov function V.

Theorem 1 [11] *Consider a hybrid system* (3)*, distance d_ϵ given in* (5)*, reference trajectory r and feedforward signal u_{ff} satisfying Assumption 1. Let the control law $u_d(t,r,x)$ be given, and let $F_e(t, \text{col}(r,x))$ be defined in* (7)*. If there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuously differentiable function $V(r,x)$ and scalars $c, \delta_1 > 0$ such that*

$$\alpha_1(d_\epsilon(r(t,j),x)) \leq V(r(t,j),x) \leq \alpha_2(d_\epsilon(r(t,j),x)) \tag{10a}$$

holds for all $x \in C \cup D$, $(t,j) \in \text{dom } r$, and

$$V(g(r(t,j)),x) \leq V(r(t,j),x), \quad \text{for } r(t,j) \in D \tag{10b}$$

$$V(r(t,j),g(x)) \leq V(r(t,j),x), \quad \text{for } x \in D \tag{10c}$$

$$\langle \nabla_{col(r,x)} V, F_e(t, col(r(t,j),x)) \rangle \leq -cV(r(t,j),x), \quad \text{for } x, r(t,j) \in C \tag{10d}$$

hold for all $(t,j) \in \text{dom } r$ and all $x \in C \cup D$ such that $d_\epsilon(r(t,j),x) < \delta_1$, then the reference trajectory r is asymptotically stable with respect to d_ϵ for the system (6)*.*

IV. Controller design

In this section, we design a state feedback $u = u_d(t,r,x)$ for mechanical systems (1) with restitution coefficient $\epsilon \in (0,1]$. The rationale behind the controller design is, loosely speaking, that the controller makes $M_\epsilon(x) - M_\epsilon(r)$ converge to zero away from the jump instants, and ensures convergence of $M_\epsilon(x) + M_\epsilon(r)$ to zero near the jump instants.

We design a tracking controller that switches based on three functions $V_a, V_b, V_c : (C \cup D)^2 \to \mathbb{R}_{\geq 0}$, given as

$$V_a(r,x) = |x-r|_P^2, \qquad V_b(r,x) = |x + \tfrac{1}{\epsilon} r|_P^2, \qquad V_c(r,x) = |\tfrac{1}{\epsilon}x + r|_P^2, \tag{11}$$

where $P \succ 0$ will be suitably chosen. We construct u_d as:

$$u_d(t,r,x) = \begin{cases} -f(t,x) + (u_{\mathrm{ff}}(t) + f(t,r)) - \begin{bmatrix} k_p & k_d \end{bmatrix}(x-r) \\ \qquad \text{if } V_a(r,x) \leq V_b(r,x) \wedge r_2 \geq 0 \text{ or } V_a(r,x) \leq V_c(r,x) \wedge r_2 < 0 \\ -f(t,x) - (u_{\mathrm{ff}}(t) + f(t,r)) - \begin{bmatrix} k_p & k_d \end{bmatrix}(x + \frac{1}{\epsilon} r) \\ \qquad \text{if } V_b(r,x) < V_a(r,x) \wedge r_2 \geq 0, \\ -f(t,x) - \epsilon(u_{\mathrm{ff}}(t) + f(t,r)) - \begin{bmatrix} k_p & k_d \end{bmatrix}(x + \epsilon r) \\ \qquad \text{if } V_c(r,x) < V_a(r,x) \wedge r_2 < 0, \end{cases} \tag{12}$$

where the controller parameters k_p, k_d satisfy $k_p, k_d > 0$.

Remark 1 *This controller structure has the controller given in [11] as a special case, since $V_b = V_c$ when $\epsilon = 1$.* △

The following theorem provides conditions on the parameters k_p, k_d and P, that guarantee that this controller solves the local tracking problem as in Definition 2.

Theorem 2 *Consider system* (1) *with $\epsilon \in (0,1]$ and reference trajectory r corresponding to the feedforward signal u_{ff} satisfying Assumption 1. If the controller parameters P and $k_p, k_d > 0$ of* (11), (12) *satisfy $P = P^\top \succ 0$ and*

$$P \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}^\top P \prec 0, \tag{13}$$

then the controller (12) *solves the tracking problem formulated in Definition 2.*

Proof: The proof of this theorem is omitted due to length constraints, and can be found in [24]. In this proof, a Lyapunov function V is constructed that, for small $d_\varepsilon(r,x)$, coincides with $\min(V_a(r,x), V_b(r,x))$ when $r_2 \geq 0$ and coincides with $\min(V_a(r,x), V_c(r,x))$ when $r_2 < 0$. ∎

Remark 2 *It can be shown that when $d_\epsilon(r,x)$ is small, then $V_a(r,x) \neq V_b(r,x)$ if $r_2 \geq 0$ and $V_a(r,x) \neq V_c(r,x)$ if $r_2 < 0$. In addition, if $d_\epsilon(r,x)$ is small and $r_2 = 0$, then one can observe that $V(r,x) = V_a(r,x)$. Hence, switches of the controller* (12) *can only be triggered by jumps of x or r. For this reason, the first case of* (12) *corresponds to the case where the reference and plant states are close to each other, the second case of* (12) *is active when the reference trajectory experienced an impact and the plant trajectory did not, and the last case of* (12) *corresponds to the case where the plant trajectory did experience an impact and the reference trajectory did not.* △

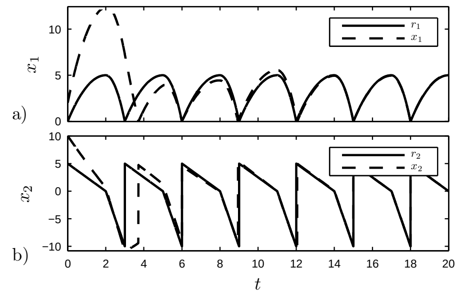

Fig. 2. Exemplary reference trajectory r given in (15) and plant trajectory x of system (1) with $\epsilon = \frac{1}{2}$ and control input (12) with $(k_p \ k_d) = (0.2 \ 0.4)$.

V. Illustrative example

To illustrate our results in the exemplary bouncing ball system, we consider the system (1) with $f(t,x) = -G$, with gravitational acceleration $G = 10$ and $\epsilon = \frac{1}{2}$. To induce a reference trajectory which does not converge to zero and is non-Zeno, we design the periodic feedforward signal

$$u_{\mathrm{ff}}(t) = \begin{cases} (1-\epsilon^2)G, & \tau(t) < \frac{v}{G\epsilon^2}, \\ 0, & \tau(t) \geq \frac{v}{G\epsilon^2}, \end{cases} \tag{14}$$

with parameter $v = 5$ and $\tau(t) := t \bmod \frac{v}{\epsilon G}(1 + \frac{1}{\epsilon})$. The following reference trajectory satisfies Assumption 1:

$$r(t) = \begin{cases} \begin{bmatrix} v\tau(t) - \frac{G\epsilon^2}{2}\tau(t)^2 \\ v - G\epsilon^2 \tau(t) \end{bmatrix}, & \tau(t) < \frac{v}{G\epsilon^2} \\ \begin{bmatrix} \frac{v^2}{2G\epsilon^2} - \frac{G}{2}(\tau(t) - \frac{v}{G\epsilon^2})^2 \\ -G(\tau(t) - \frac{v}{G\epsilon^2}) \end{bmatrix}, & \tau(t) \geq \frac{v}{G\epsilon^2}. \end{cases} \tag{15}$$

In Figure 2, the reference trajectory r is shown.

Choosing $(k_p \ k_d) = (0.2 \ 0.4)$ and $P = \begin{bmatrix} 1.25 & 1.25 \\ 1.25 & 3.75 \end{bmatrix}$, Theorem 2 ensures that the controller (12) asymptotically stabilises the reference trajectory r in the sense of Definition 1, as illustrated in Figure 2 for a plant trajectory x with initial condition $x(0,0) = (2 \ 10)^\top$. The Euclidean tracking error $|x - r|$ and the distance $d_\epsilon(r,x)$ are shown in Figure 3. As shown in this figure, the Euclidean tracking error $|x-r|$ shows unstable behaviour, with peaks whose amplitude converges to the difference between the reference's pre- and post-impact velocity and whose width (i.e. the timing mismatch between impacts of the reference and plant trajectory) converges to zero. In contrast, the tracking error measure d_ϵ converges asymptotically to zero. As shown in Figure 2, this corresponds to an intuitive notion of tracking: the impact times of the plant converge to those of the reference trajectory, and away from the impact times, after a transient period, the distance between the reference and plant trajectory becomes arbitrarily small.

VI. Discussion

In this paper, tracking controllers are designed for mechanical systems with a unilateral position constraint and dissipative impact law. Although this case is of significant

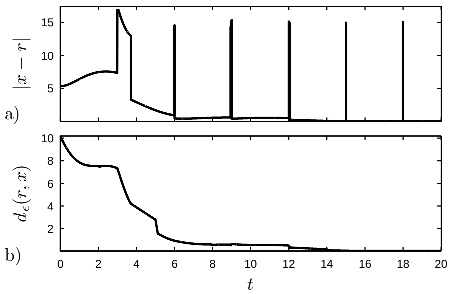

Fig. 3. a) Euclidean tracking error $|x - r|$ and b) tracking error $d_\epsilon(r, x)$ given in (5) between the reference trajectory r and plant trajectory x given in Figure 2.

practical relevance, tracking control problems with non-periodic reference trajectories for this class of systems were, up to now, not studied in the literature. The controller design ensures that, despite the "peaking" in the Euclidean tracking error, the tracking error measure introduced in this paper behaves in an asymptotically stable fashion, thereby guaranteeing an intuitively correct notion of tracking. For the design of a suitable controller, we employ a Lyapunov function that is based on distinguishing three different cases, which correspond to the situations where either the reference and plant states are close to each other, the reference trajectory jumped and the plant trajectory did not yet experience a jump, or the plant trajectory jumped and the reference trajectory did not. In the latter two cases, the Lyapunov function is based on a re-scaled version of the reference or plant trajectory, respectively. Using this Lyapunov function, a control law is designed that ensures accurate tracking. This was illustrated using the exemplary bouncing ball system.

Although the focus in this paper is on mechanical systems with dissipative impacts, we believe that the controller design, based on a Lyapunov function using the distinction between the three mentioned cases, will enable tracking controller design procedures for a larger class of hybrid systems with state-triggered jumps, which is the subject of future research. In particular, we envision that the definition of stability for time-varying trajectories with jumps could also be applied for mechanical systems with multiple degrees of freedom or impact accumulations (Zeno time) by designing an appropriate distance function.

ACKNOWLEDGMENT

The authors thank F. Forni, A.R. Teel, and L. Zaccarian for their helpful suggestions.

REFERENCES

[1] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid dynamical systems: Modeling, Stability and Robustness.* Princeton University Press, Princeton, 2012.

[2] A. J. van der Schaft and J. M. Schumacher, *An introduction to hybrid dynamical systems*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, London, 2000, vol. 251.

[3] J. Lunze and F. Lamnabhi-Lagarrigue, Eds., *Handbook of hybrid systems control.* Cambridge University Press, Cambridge, 2009.

[4] B. Brogliato, S. Niculescu, and M. Monteiro-Marques, "On tracking control of a class of complementary-slackness hybrid mechanical systems," *Systems & Control Letters*, vol. 39, no. 4, pp. 255 – 266, 2000.

[5] L. Menini and A. Tornambè, "Asymptotic tracking of periodic trajectories for a simple mechanical system subject to nonsmooth impacts," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1122–1126, 2001.

[6] R. I. Leine and N. van de Wouw, *Stability and convergence of mechanical systems with unilateral constraints*, ser. Lecture Notes in Applied and Computational Mechanics. Springer-Verlag, Berlin, 2008, vol. 36.

[7] R. G. Sanfelice, J. J. B. Biemond, N. van de Wouw, and W. P. M. H. Heemels, "Tracking control for hybrid systems via embedding of known reference trajectories," in *Proceedings of the 2011 American Control Conference*, 2011, pp. 869–874.

[8] W. P. M. H. Heemels, M. K. Camlibel, J. M. Schumacher, and B. Brogliato, "Observer-based control of linear complementarity systems," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 10, pp. 1193–1218, 2011.

[9] F. Forni, A. R. Teel, and L. Zaccarian, "Follow the bouncing ball: global results on tracking and state estimation with impacts," *submitted to IEEE Transactions Automatic Control*, 2011.

[10] J. J. B. Biemond, N. van de Wouw, W. P. M. H. Heemels, and H. Nijmeijer, "Tracking control for hybrid systems with state-triggered jumps," *submitted to IEEE Transactions on Automatic Control*, 2011.

[11] ——, "Tracking control of mechanical systems with impacts," in *Proceedings of the 2012 American Control Conference*, 2012, pp. 258–263.

[12] R. I. Leine and N. van de Wouw, "Uniform convergence of monotone measure differential inclusions: with application to the control of mechanical systems with unilateral constraints," *International Journal of Bifurcation and Chaos*, vol. 18, no. 5, pp. 1435–1457, 2008.

[13] R. G. Sanfelice, A. R. Teel, and R. Sepulchre, "A hybrid systems approach to trajectory tracking control for juggling systems," in *Proceedings of the 46th IEEE Conference on Decision and Control, New-Orleans*, 2007, pp. 5282–5287.

[14] G. A. Leonov, *Strange attractors and classical stability theory*. St. Peterburg University Press, St. Petersburg, 2008.

[15] J. J. Moreau, "Approximation en graphe d'une évolution discontinue," *RAIRO Analyse Numérique*, vol. 12, no. 1, pp. 75–84, 1978.

[16] B. Brogliato, S.-I. Niculescu, and P. Orhant, "On the control of finite-dimensional mechanical systems with unilateral constraints," *IEEE Transactions on AutomaticControl*, vol. 42, no. 2, pp. 200–215, 1997.

[17] I. C. Morărescu and B. Brogliato, "Trajectory tracking control of multiconstraint complementarity Lagrangian systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1300–1313, 2010.

[18] S. Galeani, L. Menini, A. Potini, and A. Tornambè, "Trajectory tracking for a particle in elliptical billiards," *International Journal of Control*, vol. 81, no. 2, pp. 189–213, 2008.

[19] S. Galeani, L. Menini, and A. Potini, "Robust trajectory tracking for a class of hybrid systems: an internal model principle approach," *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 344–359, 2012.

[20] F. Forni, A. R. Teel, and L. Zaccarian, "Tracking control in billiards using mirrors without smoke, Part I: Lyapunov-based local tracking in polyhedral regions," in *Proceedings of the 50th IEEE Conference on Decision and Control, Orlando*, 2011, pp. 3283–3288.

[21] ——, "Tracking control in billiards using mirrors without smoke, Part II: additional Lyapunov-based local and global results," in *Proceedings of the 50th IEEE Conference on Decision and Control, Orlando*, 2011, pp. 3289–3294.

[22] W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland, "Linear complementarity systems," *SIAM Journal on Applied Mathematics*, vol. 60, no. 4, pp. 1234–1269, 2000.

[23] Ch. Glocker, *Set-valued force laws*, ser. Lecture notes in applied mathematics, F. Pfeiffer, Ed. Springer-Verlag, Berlin, 2001, vol. 1.

[24] J. J. B. Biemond, N. van de Wouw, W. P. M. H. Heemels, R. G. Sanfelice, and H. Nijmeijer, "Tracking control of mechanical systems with a unilateral position constraint inducing dissipative impacts," Eindhoven University of Technology, Tech. Rep. DC 2012.039, 2012. [Online]. Available: http://purl.tue.nl/900578747384136