

# A Landmark-Based Controller for Global Asymptotic Stabilization on $SE(3)$

Pedro Casau, Ricardo G. Sanfelice, Rita Cunha, Carlos Silvestre

**Abstract**—In this paper, we address the problem of designing a landmark-based control law that robustly globally asymptotically stabilizes a rigid body at a desired equilibrium point on the  $SE(3)$  manifold. Synergistic potential functions are combined within a hybrid systems framework to generate such a hybrid control law. The proposed control law is solely a function of vector measurements characterizing the position of some given landmarks. We provide sufficient conditions on the geometry of the landmarks to solve the given problem. Finally, the proposed solution is simulated and compared with an almost global continuous feedback control law.

## I. INTRODUCTION

In the last decades, the problem of global stabilization of a rigid body received plenty of attention from the scientific community due to the increasing number of applications where such control strategies can be employed. These applications include the control of several vehicles, namely, satellites [5], aerial vehicles [8], and underwater vehicles [12]. The trajectories of such rigid body vehicles evolve on the Special Euclidean group of order three,  $SE(3) = \mathbb{R}^3 \times SO(3)$ , which includes the position and the orientation of the rigid body. It is well known that the attitude of a rigid body cannot be globally stabilized by means of a continuous control law [15]. The work reported in [9] demonstrates that Morse functions on  $SO(3)$  have at least four critical points which define the equilibrium points for the rotation subsystem of a rigid body. Due to topological obstacles towards global stabilization on  $SO(3)$ , solutions based of discontinuous and hybrid feedback have been suggested. Control solutions based on discontinuous feedback law are, in general, not robust to noise. On the other hand, hybrid feedback strategies with hysteretic switching between controllers have been proved to be robust to measurement noise [12].

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The most popular solutions to rigid body stabilization are based on state feedback control laws which use some kind of global parametrization of  $SO(3)$  [16]. Among these solutions, the global stabilization by quaternion feedback is specially popular [10], [12]. Nevertheless, all global parametrizations of  $SO(3)$  exhibit some kind of ambiguity because they are, most often, a many-to-one covering map of  $SO(3)$  [1]. These ambiguities require the angular velocity of the rigid body to be known, meaning that increasing the complexity of the system. Most recently, a novel strategy based on *synergistic potential functions* that achieves global stabilization using rotation matrices has been proposed in [13]. In this setting, the attitude of the rigid body is reconstructed without any ambiguity.

In this paper we extend this strategy employing the principles of synergistic potential functions and propose a control law based on the position of known landmarks that globally stabilizes not only the attitude but also the position of a rigid-body. Moreover, using the proposed control law, the state does not need to be reconstructed from the landmarks' positions, thus the control law can be computed directly from the outputs. Sufficient conditions on the geometry of the landmarks enabling the desired goal to be met are provided. The results presented in this work have direct application to rigid body vehicles that use cameras, laser sensors, and other devices that allow the position of given landmarks to be measured. Moreover, the control law proposed in this paper is robust to measurement noise as opposed to standard almost globally stabilizing control laws. More general results that those here and complete proofs are available in [4].

The remainder of this paper is organized as follows. In Section II, we present some of the notational conventions which are used throughout the paper. Section III describes problem setup which is addressed in the subsequent sections IV and V. Section IV highlights previous results on the stabilization of the rigid body and sets the stage for the novel strategy presented in Section V. Simulation results are provided in Section VI so as to demonstrate the controller's performance. Finally, some concluding remarks are given in Section VII.

## II. NOTATION & PRELIMINARIES

In this paper, we make use of recent developments on hybrid systems theory which are described in [7]. Under this framework, a hybrid system  $\mathcal{H}$  is defined as

$$\mathcal{H} = \begin{cases} \dot{\xi} \in F(\xi) & \xi \in C \\ \xi^+ \in G(\xi) & \xi \in D \end{cases},$$

where the data  $(F, C, G, D)$  is defined as follows: the set-valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is the *flow map* and governs the

continuous dynamics (also known as flows) of the hybrid system; the set  $C \subset \mathbb{R}^n$  is the *flow set* and defines the set of points where the system is allowed to flow; the set-valued map  $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is the *jump map* and defines the behavior of the system during jumps; the set  $D \subset \mathbb{R}^n$  is the *jump set* and defines the set of points where the system is allowed to jump.

We present the definition of global pre-asymptotic stability of a closed set  $\mathcal{A} \subset \mathbb{R}^n$  for a hybrid system  $\mathcal{H}$ . Let the norm  $|\cdot|_{\mathcal{A}} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  provide the shortest distance from a point  $\mathbf{x} \in \mathbb{R}^n$  to the set  $\mathcal{A} \subset \mathbb{R}^n$ , i.e.,  $|\mathbf{x}|_{\mathcal{A}} = \inf_{\mathbf{y} \in \mathcal{A}} |\mathbf{x} - \mathbf{y}|$ .

*Definition 1 (Global pre-Asymptotic Stability [7]):*

Consider a hybrid system  $\mathcal{H} = (F, C, G, D)$  defined in  $\mathbb{R}^n$ . Let  $\mathcal{A} \subset \mathbb{R}^n$  be closed. The set  $\mathcal{A}$  is said to be:

- *Globally stable* for  $\mathcal{H}$  if there exists a function  $\alpha \in \mathcal{K}_{\infty}$  such that for any solution  $\phi$  to  $\mathcal{H}$ ,  $|\phi(t, j)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0)|_{\mathcal{A}})$  for all  $(t, j) \in \text{dom } \phi$ ;
- *Globally pre-attractive* for  $\mathcal{H}$  if every solution  $\phi$  is bounded and, if it is complete<sup>1</sup>, it verifies  $\lim_{t+j \rightarrow \infty} |\phi(t, j)|_{\mathcal{A}} = 0$ ;
- *Globally pre-asymptotically stable* if it is both globally stable and globally pre-attractive.

If every solution to the hybrid system is complete, then the prefix *pre* can be removed.

In order to prove pre-asymptotically stability of a set for a hybrid system  $\mathcal{H}$ , we use Theorem 1 given below, which requires the hybrid system  $\mathcal{H} = (F, C, G, D)$  to meet the Basic Assumptions [4].

*Definition 2:* Given the hybrid system  $\mathcal{H} = (F, C, G, D)$  and the compact set  $\mathcal{A} \subset \mathbb{R}^n$ , the function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a Lyapunov function candidate for  $(\mathcal{H}, \mathcal{A})$  if: 1)  $V$  is continuous and nonnegative on  $(C \cup D) \setminus \mathcal{A} \subset \text{dom } V$ , 2)  $V$  is continuously differentiable on an open set  $\mathcal{O}$  satisfying  $C \setminus \mathcal{A} \subset \mathcal{O} \subset \text{dom } V$ , and 3)

$$\lim_{x \rightarrow \mathcal{A}, x \in \text{dom } V \cap (C \cup D)} V(x) = 0.$$

*Theorem 1 ([7, Theorem 20]):* Consider the hybrid system  $\mathcal{H} = (C, F, D, G)$  satisfying the Basic Assumptions and the compact set  $\mathcal{A} \subset \mathbb{R}^n$  satisfying  $G(\mathcal{A} \cap D) \subset \mathcal{A}$ . If there exists a Lyapunov function candidate  $V$  for  $(\mathcal{H}, \mathcal{A})$  such that

$$\begin{aligned} \langle \nabla V(\mathbf{x}), f \rangle &< 0 \quad \text{for all } \mathbf{x} \in C \setminus \mathcal{A}, f \in F(\mathbf{x}) \\ V(g) - V(x) &< 0 \quad \text{for all } \mathbf{x} \in D \setminus \mathcal{A}, g \in G(\mathbf{x}) \setminus \mathcal{A}, \end{aligned}$$

then the set  $\mathcal{A}$  is pre-asymptotically stable and the basin of pre-attraction contains every forward invariant, compact set.

For the application in this paper we are solely interested on solutions with components evolving on  $SE(3)$ . This motivates the following definitions.

*Definition 3:* The Special Euclidean Group of order 3 is denoted by  $SE(3)$  and it is given by the Cartesian product  $SE(3) = \mathbb{R}^3 \times SO(3)$ , where  $SO(3)$  denotes the Special Orthogonal Group of order 3, and it is given by

$$SO(3) = \{\mathcal{R} \in \mathbb{R}^{3 \times 3} : \mathcal{R}^T \mathcal{R} = I_3, \det(\mathcal{R}) = 1\}. \quad \square$$

<sup>1</sup>A solution  $\phi$  to a hybrid system  $\mathcal{H}$  is *complete* if its domain is unbounded [7, p. 41].

*Definition 4:* The Lie Algebra of the  $SO(3)$  group is denoted by  $\mathfrak{so}(3)$  and it is given by

$$\mathfrak{so}(3) = \{M \in \mathbb{R}^{3 \times 3} : M = -M^T\}. \quad \square$$

Let  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$  denote the  $n$ -dimensional sphere, defined by  $\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}^T \mathbf{x} = 1\}$  and let  $S : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$  denote the bijection between  $\mathbb{R}^3$  and  $\mathfrak{so}(3)$  (with inverse  $S^{-1} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ ), such that  $S(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}$ . A particular parametrization of the  $SO(3)$  group used in the sequel is the *angle-axis* parametrization, given by

$$\mathcal{R}(\theta, \mathbf{u}) = I_3 + \sin(\theta)S(\mathbf{u}) + (1 - \cos \theta)S(\mathbf{u})^2, \quad (1)$$

where  $\mathbf{u} \in \mathbb{S}^2$  denotes the axis of rotation and  $\theta \in [0, \pi]$  denotes the rotation angle [16]. It is easy to see that  $\mathcal{R}(\pi, \mathbf{u}) = \mathcal{R}(\pi, -\mathbf{u})$ , therefore the covering map (1) is many-to-one for all  $\theta = \pi$  and  $\mathbf{u} \in \mathbb{S}^2$ . For more information on the topological issues related to the  $SO(3)$  manifold, see [2].

We will also make use of synergistic potential functions defined in  $SO(3)$ , whose definition is given below.

*Definition 5 ([11, Definition 1]):* A continuously differentiable function  $V : SO(3) \rightarrow \mathbb{R}_{\geq 0}$  is a *potential function* on  $SO(3)$  (with respect to  $I_3$ ) if  $V(\mathcal{R}) > 0$  for all  $\mathcal{R} \in SO(3) \setminus \{I_3\}$  and  $V(I_3) = 0$ . The class of potential functions on  $SO(3)$  is denoted by  $\mathcal{P}$ .  $\square$

*Definition 6 ([11, Definition 2]):* Let  $Q \subset \mathbb{Z}$  be a finite index set with cardinality  $N$  and define  $\mu : \mathcal{P}^N \rightarrow \mathbb{R}_{\geq 0}$ , such that, for each family of potential functions  $\mathcal{V} = \{V_q\}_{q \in Q} \in \mathcal{P}^N$ ,

$$\mu(\mathcal{V}) = \min_{\mathcal{R} \in \text{Crit } V_q \setminus \{I_3\}} \max_{q \in Q} V_q(\mathcal{R}) - V_p(\mathcal{R}).$$

The family  $\mathcal{V} \in \mathcal{P}^N$  is *synergistic* if there exists  $\delta > 0$  such that  $\mu(\mathcal{V}) > \delta$ , where we say that  $\mathcal{V}$  is synergistic with gap exceeding  $\delta$ .  $\square$

In the special case of a function  $V : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  we use the notation  $[\nabla V(X)]_{ij} = \partial V(X) / \partial X_{ij}$ .

Other notational conventions include: coordinate frames consist of a right handed triad of orthonormal vectors and are denoted by an uppercase letter enclosed in brackets; vectors are represented by boldface lowercase letters; matrices are represented by uppercase letters; the symbol  $\times$  denotes the cross product operator; the symbol  $I_n \in \mathbb{R}^{n \times n}$  denotes the identity matrix of size  $n \times n$ ; the trace operator is denoted by  $\text{trace}$ ; given a continuously differentiable scalar field over  $SO(3)$ ,  $V : SO(3) \rightarrow \mathbb{R}$ , its set of critical points is given by

$$\text{Crit } V = \{\mathcal{R} \in SO(3) : \varphi(\mathcal{R}^T \nabla V) = 0\};$$

the set of all possible pairs of eigenvalues/eigenvectors of a matrix  $A \in \mathbb{R}^{n \times n}$  is given by

$$\mathcal{E}(A) = \{(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n : Av = \lambda v, |v| = 1\}; \quad (2)$$

the projection of the set (2) over  $\mathbb{C}^n$  is the *eigenspace* of  $A \in \mathbb{R}^{n \times n}$  and it is given by

$$\mathcal{E}_v(A) = \{v \in \mathbb{C}^n : \exists (\lambda, v) \in \mathcal{E}(A)\};$$

given a vector  $\mathbf{x} \in \mathbb{R}^n$ ,  $\text{diag}(\mathbf{x}) \in \mathbb{R}^{n \times n}$  denotes the matrix whose diagonal entries are the ordered elements of  $\mathbf{x}$ .

### III. PROBLEM SETUP

Consider a fully-actuated rigid body whose kinematics are given by

$$\dot{\mathbf{p}} = \mathbf{v} - S(\boldsymbol{\omega})\mathbf{p}, \quad \dot{\mathcal{R}} = \mathcal{R}S(\boldsymbol{\omega}), \quad (3)$$

where  $(\mathbf{p}, \mathcal{R}) \in SE(3)$  describes the configuration of the body-fixed orthonormal coordinate frame  $\{B\}$  with respect to an inertial orthonormal coordinate frame  $\{I\}$ ,  $\mathbf{v} \in \mathbb{R}^3$  denotes the linear velocity of  $\{B\}$  with respect to  $\{I\}$  expressed in  $\{B\}$ ,  $\boldsymbol{\omega} \in \mathbb{R}^3$  denotes the angular velocity of  $\{B\}$  with respect to  $\{I\}$  expressed in  $\{B\}$ .

*Problem 1:* Let  $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \dots \times \mathbb{R}^3$  denote the (fixed) positions of  $n$  landmarks with respect to  $\{I\}$ . Given a fixed configuration  $(\mathbf{p}_d, \mathcal{R}_d) \in SE(3)$  and a set  $\mathcal{X}$  of  $n$  landmarks, design a feedback control law  $(\mathbf{v}, \boldsymbol{\omega}) = \kappa(\ell_1, \ell_2, \dots, \ell_n)$  where  $\ell_i = \mathcal{R}^\top \mathbf{x}_i - \mathbf{p}$ , for  $i \in \{1, 2, \dots, n\}$ , such that  $\mathcal{A} = \{(\mathbf{p}, \mathcal{R}) \in SE(3) : (\mathbf{p}, \mathcal{R}) = (\mathbf{p}_d, \mathcal{R}_d)\}$  is *globally asymptotically stable* for system (3).  $\square$

The vector  $\ell_i \in \mathbb{R}^3$  denotes the position of the  $i$ -th landmark with respect to  $\{B\}$  expressed in  $\{B\}$ . Figure 1 illustrates the physical setup where the configuration of the body frame is shown as well as the inertial reference frame and four landmarks.

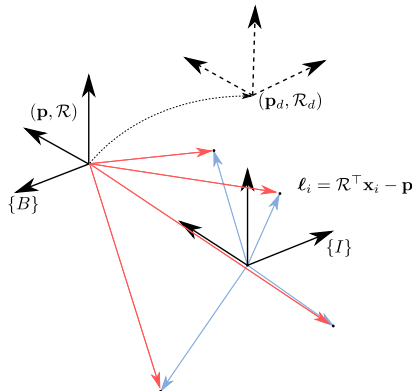


Fig. 1. System setup concept. The figure illustrates an arbitrary configuration of the body frame and the geometry of the 4 landmarks used on the simulations presented in Section VI.

As shown in Fig. 1, we are able to relate the landmark positions in  $\{B\}$  to the system state  $(\mathbf{p}, \mathcal{R}) \in SE(3)$  using the landmarks positions expressed in  $\{I\}$ . If we collect the  $n$  landmarks into a matrix  $L \in \mathbb{R}^{3 \times n}$ , such that the  $i$ -th column of  $L$  is the position vector  $\ell_i \in \mathbb{R}^3$ , then  $L = [\ell_1 \ \ell_2 \ \dots \ \ell_n]$  and  $L = \mathcal{R}^\top X - \mathbf{p}\mathbf{1}^\top$ , where the matrix  $X = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n]$  collects the positions of the landmarks with respect to  $\{I\}$  and  $\mathbf{1} = [1 \ 1 \ \dots \ 1]^\top \in \mathbb{R}^n$ . The landmarks positions when the system is at the desired set-point  $(\mathbf{p}_d, \mathcal{R}_d) \in SE(3)$  are given by  $L_d = \mathcal{R}_d^\top X - \mathbf{p}_d\mathbf{1}^\top$ .

In order to achieve the goal stated in Problem 1, we impose the following conditions on  $X \in \mathbb{R}^{3 \times n}$ .

*Assumption 1:* The origin of  $\{I\}$  belongs to the relative interior of the landmarks' convex hull,

i.e.  $\{O\} \in \mathbf{relint} \ \mathbf{conv}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ .<sup>2</sup>  $\square$

Since the landmarks are fixed with respect to  $\{I\}$ , Assumption 1 does not pose any constraints on the landmarks locations because the origin of the inertial frame can be set arbitrarily. Moreover, the assertion in Lemma 1 provides an equivalence relation for Assumption 1.

*Lemma 1* ([6, Proposition 3]): Assumption 1 is satisfied if and only if there exists a vector  $\mathbf{a} = [a_1 \ a_2 \ \dots \ a_n]^\top$  such that  $X\mathbf{a} = \mathbf{0}$ ,  $\mathbf{1}^\top \mathbf{a} = 1$ , and  $a_j > 0$  for all  $j \in \{1, 2, \dots, n\}$ .  $\square$

Another important assumption on the landmarks geometry relates to their relative positioning. There are certain landmark configurations that hinder the controller's stability and must therefore be prevented. This motivates the following assumption.

*Assumption 2:* Given  $\mathbf{a} \in \mathbb{R}^n$  satisfying the conditions of Lemma 1,  $XD_aX^\top$  is positive definite with distinct eigenvalues, where  $D_a = \text{diag}(\mathbf{a})$ .  $\square$

In order to draw some intuition out of Assumptions 1 and 2 one may compare the properties of the landmark setup to the properties of a system of point mass particles. Consider that the particle located at  $x_j \in \mathbb{R}^3$  has mass  $a_j$ , then it is easy to check that Assumption 1 requires the inertial reference frame to be located at the center of mass. It is also possible to verify that the tensor of inertia of this system of particles is given by  $P := \text{trace}(XD_aX^\top)I_3 - XD_aX^\top$ . Assumption 2 implies that the eigenvalues of  $P$  are distinct and therefore the system of particles is *anisotropic* (the details about the inertia properties of a system of point mass particles can be found in [14]).

According to Problem 1, the main goal of the controller is to stabilize the rigid-body at a given configuration  $(\mathbf{p}_d, \mathcal{R}_d) \in SE(3)$ . This objective is equivalent to the stabilization of the error variables

$$\mathbf{e} = \mathbf{p} - \mathbf{p}_d, \quad \mathcal{R}_e = \mathcal{R}^\top \mathcal{R}_d, \quad (4)$$

to the point  $(\mathbf{e}, \mathcal{R}_e) = (0, I_3)$ . The kinematics of the error system are obtained by taking the derivative of (4) and using (3), yielding

$$\dot{\mathbf{e}} = \mathbf{v} - S(\boldsymbol{\omega})(\mathbf{e} + \mathbf{p}_d), \quad \dot{\mathcal{R}}_e = -S(\boldsymbol{\omega})\mathcal{R}_e. \quad (5)$$

The next section presents a controller that provides set-point stabilization of system (5) by means of continuous output feedback, highlighting the limitations of such strategy.

### IV. ALMOST GLOBAL STABILIZATION ON $SE(3)$ BY CONTINUOUS OUTPUT FEEDBACK

Global stabilization of a rigid-body on  $SE(3)$  is hindered by the topological obstructions that are present in stabilization problems over the  $SO(3)$  manifold. In particular, since  $SO(3)$  is a compact manifold, there does not exist a continuous control law which is able to globally stabilize a given equilibrium point. Such problems are well documented

<sup>2</sup> The operator  $\mathbf{relint}C$  denotes the relative interior of a set  $C \subset \mathbb{R}^n$ . This is useful whenever  $C$  has an affine dimension which is lower than  $n$ , such as a line segment in the plane, a square in  $\mathbb{R}^3$ , etc. The interior of these sets is the empty set, but the  $\mathbf{relint}$  operator provides the interior points relative to their affine hull. The operator  $\mathbf{conv}C$  denotes the convex hull of a set  $C \subset \mathbb{R}^n$ . Both  $\mathbf{relint}$  and  $\mathbf{conv}$  are formally defined in [3].

throughout the literature, and both [2] and [12] provide a enlightening discussion on these issues.

For this reason, there exist several applications which have resorted to an *almost* globally stabilizing control law. This kind of control laws is unable to drive the system state to the desired equilibrium point if the initial condition lies on a certain set  $\mathcal{S} \subset SO(3)$  of Lebesgue measure zero. In this section, we focus our attention on the *almost* globally stabilizing control law developed in [6] as it provides useful insight for the landmark-based controller presented in Section V.

Assumptions 1 and 2 play an important role in the derivation of the controller presented in [6]. Assumption 1 allows the decoupling between the rotation and the position terms, providing  $\mathbf{e} = (L_d - L)\mathbf{a}$ ,  $\mathcal{R}^\top X = L(I_n - \mathbf{a}\mathbf{1}^\top)$  and  $\mathcal{R}_d^\top X = L(I_n - \mathbf{a}\mathbf{1}^\top)$ . In particular, these relations allow the so called modified trace function  $P_M(\mathcal{R}_e) = \text{trace}((I_3 - \mathcal{R})M)$  to be written as a function of the landmarks as follows

$$P_M(\mathcal{R}_e) = P_M(L) = \frac{1}{2} \text{trace}((L - L_d)(I_n - \mathbf{a}\mathbf{1}^\top)D_a \times \\ \times (I_n - \mathbf{1}\mathbf{a}^\top)(L^\top - L_d^\top)),$$

where  $M = \mathcal{R}_d^\top X D_a X^\top \mathcal{R}_d$ . Notice that there is a slight abuse of notation when we write  $P_M(\mathcal{R}_e) = P_M(L)$ , but we just intend to emphasize that the modified trace function may be written interchangeably as a function of the state or as a function of the landmarks. The set of critical points of the modified trace function is given by

$$\text{Crit } P_M = \{I_3\} \cup \mathcal{R}(\pi, \mathcal{E}_v(M)),$$

and it is the union of four different points, provided that the matrix  $M \in \mathbb{R}^{3 \times 3}$  is symmetric, positive semi-definite with distinct eigenvalues (c.f. [6, Lemma 7]). Using the modified trace function as a Lyapunov function provides the standard control result stated in the following lemma.

*Lemma 2:* For any positive semi-definite symmetric matrix  $M \in \mathbb{R}^{3 \times 3}$  with distinct eigenvalues, and for any  $k_\omega, k_e > 0$ , the closed-loop system resulting from the feedback interconnection between (5) and

$$\boldsymbol{\omega} = k_\omega \varphi(\mathcal{R}_e M), \quad \mathbf{v} = -k_e \mathbf{e} + S(\boldsymbol{\omega})(\mathbf{e} + \mathbf{p}_d), \quad (6)$$

renders the equilibrium point  $(\mathbf{e}, \mathcal{R}_e) = (0, I_3)$  *almost globally asymptotically stable*. The region of attraction  $\mathcal{R}_A \subset SE(3)$  is given by  $\mathcal{R}_A = SE(3) \setminus \mathcal{S}$  where  $\mathcal{S} = \{(\mathbf{e}, \mathcal{R}_e) \in SE(3) : \mathbf{e} = 0, \mathcal{R}_e = \mathcal{R}_e(\pi, \mathbf{u}) \text{ for some } \mathbf{u} \in \mathbb{S}^2\}$ .  $\square$

Since we consider  $M = \mathcal{R}^\top X D_a X^\top \mathcal{R}_d$ , Assumption 2 ensures that  $M$  has distinct eigenvalues and enables the control law (6) to be *almost globally asymptotically stabilizing* for  $(\mathbf{e}, \mathcal{R}_e) = (0, I_3)$ .

Additionally, the control law (6) can be rewritten as a function of the landmarks as follows

$$\boldsymbol{\omega} = k_w \varphi(L(I_n - \mathbf{a}\mathbf{1}^\top)D_a(I_n - \mathbf{1}\mathbf{a}^\top)L_d^\top) \\ \mathbf{v} = -k_e(L_d - L)\mathbf{a} + S(\boldsymbol{\omega})((L_d - L)\mathbf{a} + \mathbf{p}_d), \quad (7)$$

thus providing an output feedback almost globally stabilizing controller. However, this solution does not solve Problem 1 and, exhibits some problems when  $\mathcal{R}_e$  is in a neighborhood of the set  $\mathcal{S}$ , (similarly to the issues discussed in [12]): a) the presence of arbitrarily small noise may prevent the system from ever leaving the neighborhood of  $\text{Crit } P_M \cap \mathcal{S}$ , and

b) even in the absence of noise, the system may take a very long time to reach the desired equilibrium point. Both disadvantages are solved through the hybrid feedback control law, presented in the next section.

## V. GLOBAL STABILIZATION ON $SE(3)$ BY HYBRID OUTPUT FEEDBACK

In this section we apply the ideas of synergistic potential functions for attitude control of a rigid body described in [11] to Problem 1. The synergistic potential functions introduced in Definition 6 allow us to derive a class of hybrid controllers which is suitable for the landmark-based control of a rigid body. Although we follow very closely the solution for global stabilization in  $SO(3)$  presented in [11], there are some differences: i) we extend the global stabilization in  $SO(3)$  problem to the global stabilization in  $SE(3)$ ; ii) we consider directly  $(\mathbf{v}, \boldsymbol{\omega})$  as control inputs, instead of the torque; iii) the control law we present is a function of the outputs and not a function of the system state.

We extend the state-space model (5) of the system in order to include a logic variable  $q \in Q$ , such that  $\dot{q} = 0$ . Let  $\mathcal{V} = \{V_q\}_{q \in Q} \in \mathcal{P}^N$  be a family of synergistic potential functions with synergy gap  $\delta > 0$  and let  $\rho : SO(3) \rightarrow \mathbb{R}_{\geq 0}$  be a function given by  $\rho(\mathcal{R}) = \min_{q \in Q} V_q(\mathcal{R})$ .

Lemma 3 employs the Lyapunov function candidate  $V : SE(3) \times Q \rightarrow \mathbb{R}_{\geq 0}$ ,  $V(\mathbf{e}, \mathcal{R}_e, q) = V_q(\mathcal{R}_e) + \frac{1}{2} \mathbf{e}^\top \mathbf{e}$ , where  $V_q \in \mathcal{V}$ .

The results presented in [11] are built upon the rotation kinematic model  $\dot{\mathcal{R}}_e = \mathcal{R}_e S(\boldsymbol{\omega})$ , which is different than the one we consider. Therefore, in order to use results from the aforementioned paper we apply a linear and invertible transformation to the control input  $\boldsymbol{\omega}$ , defined as  $\boldsymbol{\omega}' := -\mathcal{R}_e^\top \boldsymbol{\omega}$ , which in turn changes the kinematic model (5) to

$$\dot{\mathbf{e}} = \mathbf{v} + S(\mathcal{R}_e \boldsymbol{\omega}')(\mathbf{e} + \mathbf{p}_d) \quad \dot{\mathcal{R}}_e = \mathcal{R}_e S(\boldsymbol{\omega}').$$

Under these considerations we use the control law

$$\boldsymbol{\omega}' = -k_\omega \varphi(\mathcal{R}_e^\top \nabla V_q(\mathcal{R}_e)), \\ \mathbf{v} = -k_e \mathbf{e} - S(\mathcal{R}_e \boldsymbol{\omega}')(\mathbf{e} + \mathbf{p}_d), \quad (8)$$

which, together with the additional logic variable  $q$ , generates the closed-loop hybrid system  $\mathcal{H} = (F, C, G, D)$ , given by:

**State:**

$$x = (\mathbf{e}, \mathcal{R}_e, q) \quad (9a)$$

**Flow Map:**

$$F(x) = (\mathbf{v} + S(\mathcal{R}_e \boldsymbol{\omega}')(\mathbf{e} + \mathbf{p}_d), \mathcal{R}_e S(\boldsymbol{\omega}'), 0) \quad (9b)$$

**Flow Set:**

$$C = \{(\mathbf{e}, \mathcal{R}_e, q) \in SE(3) \times Q : V_q(\mathcal{R}_e) - \rho(\mathcal{R}_e) \leq \delta\} \quad (9c)$$

**Jump Map:**

$$G(x) = \{(\mathbf{e}, \mathcal{R}_e, q) \in SE(3) \times Q : V_q(\mathcal{R}_e) = \rho(\mathcal{R}_e)\} \quad (9d)$$

**Jump Set:**

$$D = \{(\mathbf{e}, \mathcal{R}_e, q) \in SE(3) \times Q : V_q(\mathcal{R}_e) - \rho(\mathcal{R}_e) \geq \delta\}, \quad (9e)$$

similarly to the construction in [13] (except that we have expanded the system state space to include the position error). The global stabilization of the rigid body at  $(\mathbf{e}, \mathcal{R}_e) = \{0\} \times \{I_3\}$  is possible, under the hybrid systems framework, as proved in the following lemma.

*Lemma 3:* Given a family  $\mathcal{V} = \{V_q\}_{q \in Q} \in \mathcal{P}^N$  of potential functions on  $SO(3)$  that is synergistic with gap exceeding  $\delta > 0$ , for any  $k_w > 0$  and  $k_e > 0$ , the compact set  $\mathcal{A}_Q = \{0\} \times \{I_3\} \times Q$  is globally asymptotically stable for the closed-loop hybrid system  $\mathcal{H} = (F, C, G, D)$  resulting from replacing the control law (8) into the flow map (9b).

Since the closed-loop hybrid system verifies the Basic Assumptions and the set  $\mathcal{A}$  is compact, we may conclude that it is robust to small measurement noise [7]. Also, since  $\mathcal{A} = \text{proj}_{SE(3)}(\mathcal{A}_Q)$ , we conclude that the control law (8) globally stabilizes  $\mathcal{A} \subset SE(3)$ . However, we still need to find a family of synergistic potential functions.

It has been shown in [11] that two modified trace functions become synergistic by *angular warping*; that is, for

$$\begin{aligned} V_q(\mathcal{R}_e) &= \text{trace}((I_3 - \mathcal{T}_q(\mathcal{R}_e))M), \\ \mathcal{T}_q(\mathcal{R}_e) &= \exp(k_q P_M(\mathcal{R}_e)S(\mathbf{u}_q))\mathcal{R}_e, \end{aligned} \quad (10)$$

where  $\mathcal{T}_q : SO(3) \rightarrow SO(3)$  is a diffeomorphism, there exist at least two functions  $V_1(\mathcal{R}_e)$  and  $V_2(\mathcal{R}_e)$  with synergy gap  $\delta > 0$  (c.f [13]). The function  $\mathcal{T}$  corresponds to a rotation of  $\mathcal{R}_e \in SO(3)$  by an amount  $k_q P_M(\mathcal{R}_e) \in \mathbb{R}$  around the axis  $\mathbf{u}_q \in \mathbb{R}^3$ , as long as  $\sqrt{2}k_q \max \|\nabla P_M(\mathcal{R}_e)\|_F < 1$ .

From [11, Theorem 6] we obtain the relation  $\varphi(\mathcal{R}_e^\top \nabla V_q) = \Theta(\mathcal{R}_e, q)^\top \varphi(\mathcal{T}(\mathcal{R}_e)^\top \nabla V_q(\mathcal{T}(\mathcal{R}_e)))$  with  $\Theta(\mathcal{R}_e, q) = I_3 + k_q \mathcal{R}_e^\top \mathbf{u}_q \varphi(\mathcal{R}_e M)^\top \mathcal{R}_e$ .

Replacing the aforementioned relations into (8), we obtain a hybrid control law which achieves global stabilization of the set  $\mathcal{A}_Q$  for the particular family of synergistic potential functions (10). The feedback control law is given by

$$\begin{aligned} \boldsymbol{\omega}' &= k_\omega \Theta(\mathcal{R}_e, q)^\top \varphi(\mathcal{T}(\mathcal{R}_e)^\top M), \\ \mathbf{v} &= -k_e \mathbf{e} - S(\mathcal{R}_e \boldsymbol{\omega}')(\mathbf{e} + \mathbf{p}_d). \end{aligned}$$

In order to solve Problem 1, we need to rewrite the controller as a function of the landmarks. Analyzing the hybrid system (9), one may check that this task amounts to rewriting the definitions of the flow set (9c), the jump map (9d), the jump set (9e) and the control law (11) as functions of the landmarks  $L$ . It is possible to rewrite the modified trace function  $P_M(\mathcal{R}_e)$  as a function of the landmarks, using  $M = \mathcal{R}_d^\top X D_a X^\top \mathcal{R}_d$ , as shown in Section IV. Similarly, it is also possible to show that  $V_q(\mathcal{R}_e)$  can be written as a function of the landmarks as follows

$$\begin{aligned} V_q(\mathcal{R}_e) &= \frac{1}{2} \text{trace}((L - L_q^*)(I_n - \mathbf{a}\mathbf{1}^\top)D_a \cdot \\ &\quad \cdot (I_n - \mathbf{1}\mathbf{a}^\top)(L - L_q^*)^\top), \end{aligned}$$

where  $L_q^*$  is given by

$$L_q^* := \exp(-k_q P_M(L)S(\mathbf{u}_q))\mathcal{R}_d^\top X - \mathbf{p}_d \mathbf{1}^\top.$$

Again, since the function  $V_q(\mathcal{R}_e)$  can be written as a function of the landmarks, we use the notation  $V_q(L)$  in the sequel. Naturally, it is straightforward to verify that  $\rho(\mathcal{R}_e) := \min_{q \in Q} V_q(\mathcal{R}_e)$  can be written as a function of the landmarks as well, thus we use the notation  $\rho(L)$  when referring to this function. Lemma 4 shows that there is also a hybrid feedback law as a function of the landmarks that matches (11) for all  $(\mathbf{e}, \mathcal{R}_e, q) \in SE(3) \times Q$ .

*Lemma 4:* Suppose that there exist  $k_q > 0$  and  $\mathbf{u}_q \in \mathbb{R}^3$  for  $q \in Q := \{1, 2\}$  such that  $\mathcal{V} = \{V_q\}_{q \in Q}$  is a family of potential functions on  $SO(3)$  that is synergistic gap exceeding  $\delta > 0$ , and let  $M = \mathcal{R}_d^\top X D_a X^\top \mathcal{R}_d \in \mathbb{R}^{3 \times 3}$ . Let Assumptions 1 and 2 be verified. Then, for any  $k_w > 0$ ,  $k_e > 0$ , the set  $\mathcal{A}_Q = \{0\} \times \{I_3\} \times Q$  is globally asymptotically stable for the closed-loop hybrid system  $\mathcal{H} = (F, C, G, D)$ , resulting from replacing the feedback control law  $(\mathbf{v}, \boldsymbol{\omega}) = \kappa_q(L)$ , given by (12) into the hybrid system's flow map (9b).

Notice that, in particular, choosing  $k_q = 0$ , for all  $q \in Q$ , yields the control law (7).

Lemma 4 constitutes the main result of this paper as it proves that a global stabilizing control law can be written as a function of the landmarks using synergistic potential functions on  $SO(3)$ . The reader should also notice that the jump and flow sets definition can be also written as a function of the landmarks, thus achieving a solution to Problem 1.

In the next section we present some simulation results that let us compare the continuous approach to the hybrid approach.

## VI. SIMULATION RESULTS

Two different simulation results are presented in this section, illustrating the advantages of the hybrid controller presented in Section V over the continuous controller presented in Section IV. The chosen landmarks for these simulations are collected in the columns of matrix  $X \in \mathbb{R}^{3 \times 4}$ , given by

$$X = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

such that  $X\mathbf{a} = 0$  with  $\mathbf{a} = 0.5\mathbf{1}$ , satisfying Assumption 1. Moreover, the eigenvalues of  $X D_a X^\top \in \mathbb{R}^{3 \times 3}$  are  $\lambda_1 = 0.125$ ,  $\lambda_2 = 0.5$  and  $\lambda_3 = 1$ , thus meeting Assumption 2. It is possible to verify that for these parameters, the functions  $V_{R_1} = P_M(\mathcal{T}_1(\mathcal{R}_e))$  and  $V_{R_2} = P_M(\mathcal{T}_2(\mathcal{R}_e))$  are synergistic for  $k_1 = 0.1$ ,  $k_2 = -0.1$  and  $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{z}/|\mathbf{z}|$  with  $\mathbf{z} = [0 \ 1 \ 1]^\top$  and synergy gap exceeding  $\delta \approx 0.0017$ .

$$\boldsymbol{\omega} = k_\omega \varphi(L_d(I_n - \mathbf{a}\mathbf{1}^\top)D_a(I_n - \mathbf{1}\mathbf{a}^\top)L^\top) \text{trace}(L(I_n - \mathbf{a}\mathbf{1}^\top)D_a(I_n - \mathbf{1}\mathbf{a}^\top)L_q^* k_q S(\mathbf{u}_q)) \quad (12a)$$

$$\begin{aligned} &- k_\omega \varphi(L_q^*(I_n - \mathbf{a}\mathbf{1}^\top)D_a(I_n - \mathbf{1}\mathbf{a}^\top)L^\top) \\ \mathbf{v} &= -k_e(L_d - L)\mathbf{a} - S(\boldsymbol{\omega})((L_d - L)\mathbf{a} + \mathbf{p}_d), \end{aligned} \quad (12b)$$

In the following simulations, the desired configuration of the body frame  $(\mathbf{p}_d, \mathcal{R}_d) \in SE(3)$  is set to  $\mathbf{p}_d = \mathbf{e}_3$  and  $\mathcal{R}_d = \mathcal{R}(\pi/2, \mathbf{e}_3)$ , where  $\mathbf{e}_3 = [0 \ 0 \ 1]^\top$ . The controller parameters  $k_\omega$  and  $k_e$  should be tuned to the specific application at hand. In general, increasing these parameters leads to faster response times and increased rejection of additive input disturbances, at the cost of higher actuation authority. In the simulations we chose  $k_\omega = 1$  and  $k_e = 1$ .

For the first simulation, we selected the following initial configuration of the body frame:  $\mathbf{p}_d = [1 \ 0 \ 1]^\top$  and  $\mathcal{R}_d = \mathcal{R}(-\pi/2, \mathbf{e}_3)$ , which is such that  $\mathcal{R}_e(0, 0)$  is in a neighborhood  $\text{Crit } P_M$  and the initial position is offset from the desired one. Figure 2 depicts the evolution of the distance between  $\mathcal{R}(t)$  and  $\mathcal{R}_d \in \mathbb{R}^{3 \times 3}$ . It also depicts the evolution of the distance between  $\mathbf{p}(t)$  and  $\mathbf{p}_d \in \mathbb{R}^3$ . It is possible to verify that the hybrid controller reacts immediately in order to correct its offset rotation, but the continuous controller does not react at all, being seemingly unable to correct its rotation. The position error is driven to 0 regardless of the chosen controller and their performances are indistinguishable.

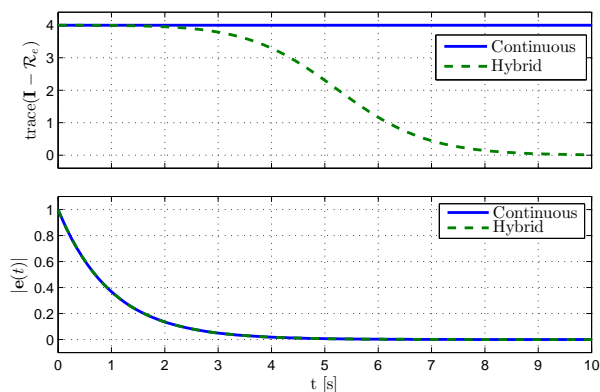


Fig. 2. Distance between the current state and the desired set-point for simulation 1.

For the second simulation we changed the initial rotation matrix to

$$\mathcal{R}(0, 0) = \begin{bmatrix} 0.0874 & 0.9923 & -0.0874 \\ -0.9962 & 0.0874 & -0.0038 \\ 0.0038 & 0.0874 & 0.9962 \end{bmatrix},$$

so as to place  $\mathcal{R}_e(0, 0)$  near a critical point of  $P_M(\mathcal{T}_1(\mathcal{R}_e))$ . Since  $q(0, 0) = 1$ , the initial condition lies in the jump set, immediately changing the selected controller. It is shown in Figure 3 that the hybrid controller still achieves better performance than the continuous controller for this particular initial condition.

## VII. CONCLUSIONS

In this work we have presented an output-feedback control law which enabled the stabilization of a rigid body vehicle at a desired set-point, using solely the measurements from the locations of given landmarks. We have employed recent developments on synergistic Lyapunov functions and proved that, under mild assumptions on the geometry of the landmarks, the problem is solved by the proposed control

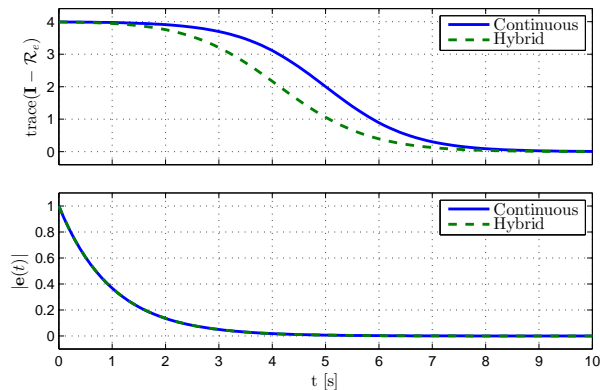


Fig. 3. Distance between the current state and the desired set-point for simulation 2.

law. We also presented simulation results which illustrate the advantages of the proposed control law over standard continuous feedback strategies.

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