A Robust Finite-time Convergent Hybrid Observer for Linear Systems

Yuchun Li and Ricardo G. Sanfelice

***Abstract*— Motivated by the design of observers with good performance and robustness to measurement noise, the problem of estimating the state of a linear time-invariant system in finite time and with robustness to a class of measurement noise is considered. Using a hybrid systems framework, a hybrid observer producing an estimate that converges to the plant state in finite time, even under unknown constant (e.g., bias) and piecewise constant noise is presented. The stability and robustness properties of the observer are shown analytically and validated numerically.**

I. INTRODUCTION

For a linear time-invariant system defined as

$$\dot{x} = Ax, \qquad y = Hx + m, \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ denotes measurement noise, a Luenberger observer is given by

$$\dot{\hat{x}}_0 = A\hat{x}_0 - L_0(\hat{y}_0 - y), \qquad \hat{y}_0 = H\hat{x}_0. \tag{2}$$

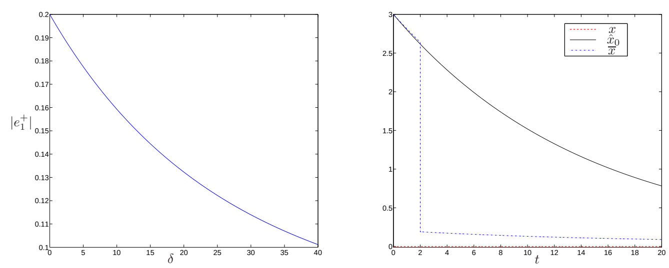
When the plant (1) is observable, the rate of convergence can be chosen arbitrarily fast by placing "large" gain L_0; however, large gain may amplify the effect of the measurement noise m. In fact, the design of Luenberger observers involves a tradeoff between the rate of convergence and robustness to measurement noise [1], [2]. Recent efforts in observer design aim at relaxing such a tradeoff. These include the adaptive gain approach in [3], the switched gain approach in [4], and the interconnected observers approach in [5].

For scenarios where fast rate of convergence is of main importance, several observer architectures that guarantee finite time convergence of the estimates without measurement noise are available in the literature. These includes those using the properties of the solutions from multiple observers, see, e.g., [6], [7], [8], [9], [10], and those utilizing the homogeneity property, see, e.g., [11], [12], [13], [14]. In particular, the finite-time convergent observer proposed in [8] without noise ($m \equiv 0$, $y = Hx$) is defined as[1]

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i - L_i(\hat{y}_i - y) && \forall t \neq k\delta, k \in \mathbb{N}, \\ \hat{x}_i^+ &= K_1\hat{x}_1 + K_2\hat{x}_2 && \forall t = k\delta, k \in \mathbb{N}, \end{aligned} \tag{3}$$

for each $i \in \{1, 2\}$, where $K_2 = (I - \exp((A - L_2H)\delta)\exp(-(A - L_1H)\delta))^{-1}$ and $K_1 = I - K_2$ if $k = 1$; $K_1 = I, K_2 = 0$ if $k > 1$ and $i = 1$; $K_1 = 0, K_2 = I$ if $k > 1$ and $i = 2$. The parameter δ is predetermined such that K_2 is well defined, and is the time that $e_i^+(\delta) = 0$ for $i \in \{1, 2\}$, where the e_i's define the estimation error, i.e., $e_i := \hat{x}_i - x$.

A tradeoff between rate of convergence and robustness to measurement noise is also present for the finite-time convergent observer in (3). Figure 1(a) illustrates such a tradeoff when (3) is used in a scalar plant. The x-axis denotes the time when the estimate is reset and the y-axis denotes the corresponding error after reset when noise m is a constant; in particular, it shows $|e_1^+|$ at $t = \delta$. It can be seen that the faster the observer jumps (δ small), the larger the effect of measurement noise after the reset would be.



(a) Trade off between the rate of convergence and the effect of measurement noise (constant) for a scalar plant.

(b) Performance of the finite-time convergent observer ($\bar{x}$) compared to a Luenberger observer ($\hat{x}_0$) for the zero solution to (1).

Fig. 1. Effect of noise m on the finite-time convergent observer in (3).

In this paper, a hybrid observer producing an estimate that converges to the plant state in finite time, even under unknown constant (e.g., bias) and piecewise constant noise, is presented. In particular, first, we study the stability and robustness properties of our hybrid observer without noise compensation (in such a case, our observer coincides with the one in (3)). Section IV-A introduces the finite-time convergent hybrid observer and establishes its properties. Then, with the goal of rejecting measurement noise and motivated by the example in Section III, we proposed hybrid observers that preserve the finite-time convergence property. Section IV-D presents the hybrid observer for constant noise compensation and Section IV-E presents the hybrid observer for piecewise constant noise compensation. In Section V, numerical results confirm the properties of the hybrid observers. Complete proofs will be published elsewhere.

II. PRELIMINARIES

A. Notation

Given a set $S \in \mathbb{R}^n$, the closure of S is the intersection of all closed sets containing S, denoted by $\overline{S}$. The operator $\overline{\text{con}}(S)$ defines the convex hull of S. Given vectors $\nu \in \mathbb{R}^n$,

Y. Li and R. G. Sanfelice are with the Department of Aerospace and Mechanical Engineering, University of Arizona, 1130 N. Mountain Ave, AZ 85721, USA. Email: yuchunli,sricardo@u.arizona.edu
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[1] $\hat{x}_i^+(t_0) := \lim_{t \to t_0^+} \hat{x}_i(t)$ for all $i \in \{1, 2\}$, $t_0 \in \mathbb{R}_{\geq 0}$.

$w \in \mathbb{R}^m$, $|\nu|$ defines the Euclidean vector norm $|\nu| = \sqrt{\nu^\top \nu}$, and $[\nu^\top \ w^\top]^\top$ is equivalent to (ν, w). Given a matrix $A \in \mathbb{R}^{n\times n}$, $\text{eig}(A)$ denotes the set that contains all eigenvalues of A; $\overline{\mu}(A) := \max\{Re(\lambda)/2 : \ \lambda \in \text{eig}(A + A^\top)\}$; $\underline{\mu}(A) := \min\{Re(\lambda)/2 : \ \lambda \in \text{eig}(A + A^\top)\}$; $||A|| := \max\{|\lambda|^{\frac{1}{2}} : \ \lambda \in \text{eig}(A^\top A)\}$; $\kappa(A) := \min\{||X||||X^{-1}|| : A = XJX^{-1}\}$; $\overline{\alpha}(A) := \max\{Re(\lambda) : \ \lambda \in \text{eig}(A)\}$; $\underline{\alpha}(A) := \min\{Re(\lambda) : \ \lambda \in \text{eig}(A)\}$; with $p \in \mathbb{N}$, $\text{diag}_p(A)$ defines a diagonal square matrix which has p sub-matrices A on the diagonal. Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t\in\mathbb{R}_{\geq 0}} |m(t)|$ and $|m|_\delta := \sup_{t\in[0,\delta]} |m(t)|$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, $\text{dom } f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$; the right limit of function f is defined as $f^+(x) := \lim_{t\to 0^+} f(x + t)$ if it exists. Given a set $B \subset \mathbb{R}^n$, the indicator function $\chi_B : \mathbb{R}^n \to \{0, 1\}$ is defined as $\chi_B(x) := 0$ if $x \notin B$, and $\chi_B(x) := 1$ if $x \in B$. Given a point $y \in \mathbb{R}^n$ and a closed set $A \subset \mathbb{R}^n$, $|y|_A := \inf_{x\in A} |x-y|$. $\{v_i\}_{i=1}^n$ defines an orthonormal basis for $\mathbb{R}^n$, each column vector v_i contains the only nonzero element at the i-th entry. Given two matrices A, B with proper dimensions, $\text{He}(A, B) := A^\top B + BA$.

B. Preliminaries on hybrid systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, f, D, g) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &= g(z) \quad z \in D, \end{aligned} \tag{4}$$

where $z \in \mathbb{R}^n$ is the state, f defines the flow map which captures the continuous dynamics and C defines the flow set on which f is effective. The map g defines the jump map and models the discrete behavior, while D defines the jump set, from which discrete dynamics are allowed. A solution to $\mathcal{H}$ is parametrized by (t, j), where t denotes ordinary time and j denotes the jump time. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version from another solution, and it is complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t-direction. Furthermore, a hybrid system $\mathcal{H}$ is called forward complete at a point z_0 if every maximal solution to $\mathcal{H}$ from z_0 is complete. A hybrid system is forward complete if it is forward complete for every point in $\mathbb{R}^n$. See [15], [16] for more details on this hybrid systems framework.

III. Motivational example

Consider the following scalar plant

$$\dot{x} = ax, \qquad y = x + m, \tag{5}$$

where the measurement noise m is assumed to be constant, i.e., a bias. Consider the finite-time convergent observer in (3). Because of the presence of noise, at the reset time δ, the estimate $\hat{x}_1$, $\hat{x}_2$ will not be reset to x by the matrix K_δ. Instead, defining $e_i = \hat{x}_i - x$ for $i \in \{0, 1\}$, the error after the jump is given by

$$\begin{aligned} e_1^+(\delta) = e_2^+(\delta) &= \frac{m}{\exp(\tilde{a}_2\delta) - \exp(\tilde{a}_1\delta)} \\ &\times \left(\frac{L_1}{\tilde{a}_1} (\exp(\tilde{a}_1\delta)\exp(\tilde{a}_2\delta) - \exp(\tilde{a}_2\delta)) \right. \\ &\left. - \frac{L_2}{\tilde{a}_2} (\exp(\tilde{a}_1\delta)\exp(\tilde{a}_2\delta) - \exp(\tilde{a}_1\delta)) \right), \end{aligned} \tag{6}$$

where $\tilde{a}_1 = a - L_1$ and $\tilde{a}_2 = a - L_2$. Let $m = 0.2$, $a = -0.05$, $L_1 = 0.01$ and $L_2 = 0.02$. Evaluating equation (6), we obtain the plot in Figure 1(a), which indicates the relationship between the jump time parameter δ and the error after the jump. Specifically, the sooner the jump occurs (δ small), the larger the error would be. On the other hand, if the parameter δ is large, the estimate would take longer to converge. Figure 1(b) illustrates this tradeoff by comparing a trajectory of (3) with the one of the Luenberger observer in (2) with gain $L_0 = 0.02$. The red dash line denotes the evolution of the scalar plant, the black line denotes the trajectory of the Luenberger observer, while the blue dash-dot line denotes the estimation from observer (3), namely, the average $\overline{x} = \frac{1}{2}(\hat{x}_1 + \hat{x}_2)$ with $\delta = 2$. Due to the jump, the estimation from observer (3) approaches zero much faster than that of the Luenberger observer.

To overcome the negative effect of measurement noise in the finite-time convergent observer (3), we propose a hybrid observer that, for constant or piecewise constant noise m, estimates the state of the plant in finite time with zero error. In particular, for the constant noise case, the estimation error converges to zero after the first jump, and the proposed observer for (5) is given by the hybrid system

$$\begin{aligned} \dot{\zeta} &= f_o(\zeta, y) \quad \zeta \in C_o, \\ \zeta^+ &= g_o(\zeta, y) \quad \zeta \in D_o, \end{aligned} \tag{7}$$

with state $\zeta = (\hat{x}_1, \hat{x}_2, \xi_1, \xi_2, \hat{m}, \tau, q) \in \mathcal{X}_o := \mathbb{R}^5 \times [0, \delta] \times \{0, 1\}$. The parameter $\hat{m}$ is the estimation of the bias m. The bias is estimated using ξ_i's, which are auxiliary variables, and information of y. The timer τ and logic variable q are used to trigger a jump when $\tau = \delta$ and $q = 0$. The flow map f_o and jump map g_o are given by

$$f_o(\zeta, y) = \begin{bmatrix} a\hat{x}_1 + L_1(y - \hat{x}_1) - L_1\hat{m} \\ a\hat{x}_2 + L_2(y - \hat{x}_2) - L_2\hat{m} \\ (a - L_1)\xi_1 + L_1 \\ (a - L_2)\xi_2 + L_2 \\ 0 \\ 1 - q \\ 0 \end{bmatrix}, \tag{8}$$

$$g_o(\zeta, y) = \begin{bmatrix} R(\hat{x}_1, \hat{x}_2) - T(\xi_1, \xi_2)\frac{R(\hat{x}_1,\hat{x}_2)-(y-\hat{m})}{T(\xi_1,\xi_2)-1} \\ R(\hat{x}_1, \hat{x}_2) - T(\xi_1, \xi_2)\frac{R(\hat{x}_1,\hat{x}_2)-(y-\hat{m})}{T(\xi_1,\xi_2)-1} \\ 0 \\ 0 \\ \hat{m} + \frac{R(\hat{x}_1,\hat{x}_2)-(y-\hat{m})}{T(\xi_1,\xi_2)-1} \\ 0 \\ 1 - q \end{bmatrix}, \tag{9}$$

where $R(\hat{x}_1, \hat{x}_2) = K_1\hat{x}_1 + K_2\hat{x}_2$ and $T(\xi_1, \xi_2) = K_1\xi_1 + K_2\xi_2$. The flow set is defined by $C_o := \mathcal{X}_o$ and the jump set is given by $D_o := \{\zeta \in \mathcal{X}_o : \tau = \delta, q = 0\}$. The gains K_1 and K_2 are given by $K_1 = I - K_2$, $K_2 = (I - \exp(\tilde{a}_2\delta)\exp(-\tilde{a}_1\delta))^{-1}$.

For the hybrid system resulting from interconnecting (5) and (7) with constant m, for any initial condition $(x(0,0), \zeta(0,0)) \in S_o := \{(x,\zeta) \in \mathbb{R} \times \mathcal{X}_o : \hat{x}_1 = \hat{x}_2, \xi_1 = \xi_2 = 0, \tau = 0, q = 0\}$, if L_1, L_2, and δ are chosen such that $L_2 > L_1 > 0$, $\delta > 0$, and $T(\xi_1(\delta,0), \xi_2(\delta,0)) \neq 1$, then, the states $\hat{x}_1$ and $\hat{x}_2$ converge to x in finite time δ and one jump, i.e., $\hat{x}_1(t,j) = \hat{x}_2(t,j) = x(t,j)$ for all $(t,j) \in \operatorname{dom}\zeta, t+j \geq \delta+1$. In fact, for any initial condition $(x(0,0), \zeta(0,0)) \in S_o$, and all $(t,j) \in \operatorname{dom}\zeta$, it follows that for $0 \leq t \leq \delta$,

$$\begin{aligned} e_i(t,0) &= \exp((a-L_i)t)e_i(0,0) \\ &\quad + \int_0^t \exp((a-L_i)(t-\tau))L_i(m-\hat{m})d\tau, \end{aligned}$$

and $\xi_i(t,0) = \int_0^t \exp((a-L_i)(t-\tau))L_i d\tau$. Using the definitions of K_1 and K_2, and the fact that $\hat{x}_i = e_i + x$ and that m and $\hat{m}$ are constant over $[0\ \delta]$, we get

$$\begin{aligned} &R(\hat{x}_1(\delta,0), \hat{x}_2(\delta,0)) = K_1\hat{x}_1(\delta,0) + K_2\hat{x}_2(\delta,0) \\ &= x(\delta,0) + \left[K_1 \int_0^\delta \exp((a-L_1)(\delta-\tau))L_1 d\tau \right. \\ &\quad \left. + K_2 \int_0^\delta \exp((a-L_2)(\delta-\tau))L_2 d\tau \right](m - \hat{m}(\delta,0)), \end{aligned}$$

and

$$\begin{aligned} &T(\xi_1(\delta,0), \xi_2(\delta,0)) = K_1\xi_1(\delta,0) + K_2\xi_2(\delta,0) \\ &= K_1 \int_0^\delta \exp((a-L_1)(\delta-\tau))L_1 d\tau \\ &\quad + K_2 \int_0^\delta \exp((a-L_2)(\delta-\tau))L_2 d\tau. \end{aligned}$$

Then, at the jump which occurs when $\tau = \delta$, we have

$$\begin{aligned} \hat{m}(\delta,1) &= \hat{m}(\delta,0) \\ &\quad + \frac{R(\hat{x}_1(\delta,0), \hat{x}_2(\delta,0)) - (y(\delta,0) - \hat{m}(\delta,0))}{T(\xi_1(\delta,0), \xi_2(\delta,0)) - 1} \\ &= m, \end{aligned}$$

which implies that

$$\begin{aligned} \hat{x}_1(\delta,1) &= \hat{x}_2(\delta,1) \\ &= R(\hat{x}_1(\delta,0), \hat{x}_2(\delta,0)) - T(\xi_1(\delta,0), \xi_2(\delta,0))(m - \hat{m}) \\ &= x(\delta,0), \end{aligned}$$

and since $x(\delta,1) = x(\delta,0)$, we have finite time convergence of e_i's to zero.

IV. Robust finite-time convergent hybrid observer

This section presents a hybrid observer for finite time convergence with robustness to constant and piecewise constant measurement noise. For simplicity, we introduce first the observer for the case of no noise and then establish its properties (In this case, our observer is equivalent to the one in (3)). Then, we introduce the observer with noise compensation.

A. Nominal observer $\mathcal{H}_n$

A finite-time convergent hybrid observer (nominal, without noise compensation) is denoted by $\mathcal{H}_n$ and is given by

$$\begin{aligned} \dot{\zeta} &= f_n(\zeta, y) \quad \zeta \in C_n, \\ \zeta^+ &= g_n(\zeta, y) \quad \zeta \in D_n, \end{aligned} \tag{10}$$

where $\zeta = (\hat{x}_1, \hat{x}_2, \tau, q) \in \mathcal{X}_n := \mathbb{R}^{2n} \times [0,\delta] \times \{0,1\}$. When the variable τ reaches the predetermined constant $\delta > 0$, the updating law in (3) is applied. The logic variable q ensures that only one jump occurs when (10) is appropriately initialized. Therefore, the flow set is defined as $C_n = \mathcal{X}_n$ and the flow map is given by

$$f_n(\zeta, y) = \begin{bmatrix} (A - L_1H)\hat{x}_1 + L_1Hx \\ (A - L_2H)\hat{x}_2 + L_2Hx \\ 1 - q \\ 0 \end{bmatrix}. \tag{11}$$

The jump condition for the system (10) is when the timer reaches δ with $q = 0$. Therefore, the jump set is $D_n = \{\zeta \in \mathcal{X}_n : \tau = \delta, q = 0\}$ and the jump map is

$$g_n(\zeta, y) = \begin{bmatrix} K_\delta[\hat{x}_1^\top\ \hat{x}_2^\top]^\top \\ K_\delta[\hat{x}_1^\top\ \hat{x}_2^\top]^\top \\ 0 \\ 1 - q \end{bmatrix}, \tag{12}$$

where K_δ is defined by $K_\delta = [K_1\ \ K_2]$, with $K_2 = (I - \exp(F_2\delta)\exp(-F_1\delta))^{-1}$, $K_1 = I - K_2$, and $F_1 := A - L_1H$, $F_2 := A - L_2H$. The estimate of x generated by $\mathcal{H}_n$ is denoted[2] by $\overline{x} := \frac{1}{2}(\hat{x}_1 + \hat{x}_2)$. Then, the estimation error is defined by $\overline{e} := \frac{1}{2}(e_1 + e_2)$, where $e_1 = \hat{x}_1 - x$ and $e_2 = \hat{x}_2 - x$. This hybrid model is different from the one introduced in [15, Example 11], which jumps recursively.

Based on [6], [8], K_δ is well defined if parameters L_1, L_2, and δ are chosen to satisfy the following conditions.

Assumption 4.1:

A1) There exist L_1 and L_2 such that F_i is Hurwitz for each $i \in \{1,2\}$.

A2) There exists $\delta > 0$ such that
$$\det(I - \exp(F_2t)\exp(-F_1t)) \neq 0 \quad \forall t \in (0,\delta].$$

Following [6], [8], a sufficient condition for A2 is stated in the following lemma.

Lemma 4.2: If there exist L_1 and L_2 such that $\overline{\alpha}(F_2) < \underline{\alpha}(F_1)$ and $\overline{\alpha}(F_1) < 0$, then, there exists $\delta > 0$ such that K_1 and K_2 are well defined and A2 in Assumption 4.1 holds.

Remark 4.3: If the pair (A, H) is observable, then, there always exist L_1 and L_2 such that $\overline{\alpha}(F_2) < \underline{\alpha}(F_1)$.

[2] In general, the estimate can be defined as $\overline{x} = a_1\hat{x}_1 + a_2\hat{x}_2$, where $a_1, a_2 \in \mathbb{R}$ such that $a_1 + a_2 = 1$, the corresponding estimation error $\overline{e} = a_1e_1 + a_2e_2$.

Remark 4.4: The function $t \mapsto p(t)$ given by $p(t) = \det(I - \exp(F_2 t)\exp(-F_1 t))$ is analytic. In order to determine the possible values of δ, it suffices to determine the zeros of p near the origin, so that δ is bounded by $\min\{|t| : p(t) = 0\}$. For a concrete system, these zeros can be computed numerically.

The following result will be useful in establishing robustness properties of (10).

Proposition 4.5: Suppose A2 in Assumption 4.1 holds. Then, the interconnection[3] *between $\mathcal{H}_n$ and the plant satisfies the Hybrid Basic Conditions (HBC) [16, Assumption 6.5].*

It is worth to note that Proposition 4.5 gives one set of sufficient conditions for which the HBC are implied; however, these conditions are not necessary.

B. Solution properties of the interconnection between $\mathcal{H}_n$ and plant

The following results on the solutions to the interconnection between $\mathcal{H}_n$ and the plant in (1) are established.

Proposition 4.6: Suppose the parameters L_1, L_2, and δ are such that the interconnection between $\mathcal{H}_n$ and the plant satisfies HBC. Then, the interconnection is forward complete. Moreover, every maximal solution is unique and non Zeno.

Considering that the objective is to make the state of observers converge to the state of plant, i.e., $\hat{x}_1 \to x$ and $\hat{x}_2 \to x$ as $t + j \to \infty$ (especially, $\hat{x}_1$ and $\hat{x}_2$ converge to x in finite time with measurement noise $m \equiv 0$), the set to be stabilized is defined as

$$\mathcal{A} = \{z_n \in \mathbb{R}^n \times \mathcal{X}_n : x = \hat{x}_1 = \hat{x}_2\}. \tag{13}$$

Note that set $\mathcal{A}$ is not necessarily a compact set[4].

Theorem 4.7: For the interconnection between $\mathcal{H}_n$ and plant, suppose Assumption 4.1 holds. Then, each solution to the interconnection from $z_n(0,0) \in S_n := \{z_n \in \mathbb{R}^n \times \mathcal{X}_n : \hat{x}_1 = \hat{x}_2, \tau = 0, q = 0\}$ satisfies

$$|z_n(t,j)|_{\mathcal{A}} \le (1-j)\sqrt{\frac{c_2}{c_1}}|z_n(0,0)|_{\mathcal{A}} \exp\left(-\frac{\lambda_{\min}(P)}{2c_2}t\right),$$

for all $(t,j) \in \operatorname{dom} z_n$ with $c_1 = \min\{\underline{\alpha}(P_1), \underline{\alpha}(P_2)\}$, $c_2 = \max\{\overline{\alpha}(P_1), \overline{\alpha}(P_2)\}$, for some $P_1 = P_1^\top > 0$, $P_2 = P_2^\top > 0$ such that $P = \begin{bmatrix} -He(F_1,P_1) & 0 \\ 0 & -He(F_2,P_2) \end{bmatrix} > 0$.

Remark 4.8: Note that, as only one jump is allowed in the interconnection between $\mathcal{H}_n$ and the plant, global asymptotic stability of $\mathcal{A}$ is not guaranteed. If recursive jumps are allowed, the set $\mathcal{A}$ is globally asymptotically stable as established in [15, Example 11].

[3]The hybrid observer $\mathcal{H}_n$ interconnected with the plant (1) defines a hybrid system $\mathcal{H} = (C, f, D, g)$ with state $z_n = (x, \zeta)$, $C = \mathbb{R}^n \times C_n$, $f(z_n) = (Ax, f_n(\zeta, y))$, $D = \mathbb{R}^n \times D_n$ and $g(z_n) = (x, g_n(\zeta))$.

[4]When the plant state x is constrained to a compact set, the set $\mathcal{A}$ of interest is compact.

C. $\mathcal{H}_n$ with measurement noise m

For the case where the noise m is a general function, the interconnection between $\mathcal{H}_n$ and the plant is given by

$$\begin{aligned} \dot{z}_m &= (Ax, f_n(\zeta, y)) =: f_m(z_m) \quad && z_m \in C_m, \\ z_m^+ &= (x, g(\zeta)) =: g_m(z_m) && z_m \in D_m, \end{aligned} \tag{14}$$

where $z_m = (x, \zeta) \in \mathbb{R}^n \times \mathcal{X}_n$, and $y = Hx + m$. Moreover, $C_m := \mathbb{R}^n \times C_n$ and $D_m := \mathbb{R}^n \times D_n$. Denote (14) by $\mathcal{H}_m$. To explore the robustness property of $\mathcal{H}_m$, under HBC (recall that Proposition 4.5 provides a sufficient condition), the following robustness property of the hybrid system $\mathcal{H}_m$ with respect to small measurement noise can be established.

Theorem 4.9: (Small robustness) For the hybrid system $\mathcal{H}_m$, assume HBC are satisfied. For each $z_m(0,0) \in \mathcal{K}$ with $\mathcal{K} \subset \mathbb{R}^n \times \mathcal{X}_n$ compact, each $\varepsilon > 0$, and each $(T,J) \in \mathbb{R}_{\ge 0} \times \mathbb{N}$, there exists $\sigma^\star > 0$ with the following property: for each $\sigma \in (0, \sigma^\star]$, the solution of $\mathcal{H}_m$ with $z_m(0,0)$ and $|m|_\infty \le \sigma$ and the solution of $\mathcal{H}_m$ with $z_m(0,0)$ and $m \equiv 0$ are (T,J,ε)-close.

To determine robustness properties of $\mathcal{H}_m$ with respect to large measurement noise m, we derive a bound on the estimation error. For $\mathcal{H}_m$ in (14), the estimation error after the jump can be calculated as[5]

$$\begin{aligned} e_1(\delta,1) = e_2(\delta,1) = K_1 &\int_0^\delta \exp\left((A - L_1 H)(\delta - \tau)\right) L_1 m(\tau) d\tau \\ &+ K_2 \int_0^\delta \exp\left((A - L_2 H)(\delta - \tau)\right) L_2 m(\tau) d\tau. \end{aligned} \tag{15}$$

The following bounds can be established.

Theorem 4.10: For the hybrid system $\mathcal{H}_m$, suppose $\overline{\alpha}(F_2) < \underline{\alpha}(F_1)$ and $\overline{\alpha}(F_1) < 0$. Moreover, if F_1 and F_2 are dissipative and $\overline{\mu}(F_2) < \underline{\mu}(F_1)$, then, (15) is bounded as

$$|e_1(\delta,1)| \le \left((1 + w_1)\frac{||L_1||}{|\overline{\mu}(F_1)|} + w_1 \frac{||L_2||}{|\overline{\mu}(F_2)|}\right)|m|_\infty,$$

where $w_1 = \frac{1}{1 - \exp\left((\overline{\mu}(F_2) - \underline{\mu}(F_1))\delta\right)}$.

Theorem 4.11: (Large robustness) For the hybrid system $\mathcal{H}_m$, suppose $\overline{\alpha}(F_2) < \underline{\alpha}(F_1)$, and $\overline{\alpha}(F_1) < 0$. Furthermore, suppose F_1 and F_2 are dissipative and $\overline{\mu}(F_2) < \underline{\mu}(F_1)$, Then, there exist a class-$\mathcal{KL}$ function β and class-$\mathcal{K}$ functions γ and γ_1 such that each solution to $\mathcal{H}_m$ from $z_m(0,0) \in S_n$ satisfies

$$\begin{aligned} |\overline{e}(t,j)| \le (1-j)\beta(|\overline{e}(0,0)|, \min\{t,\delta\}) + (1-j)\gamma_1(|m|_\delta) \\ + j\beta(\gamma_1(|m|_\delta), \max\{t-\delta, 0\}) + j\gamma(|m|_\infty) \end{aligned} \tag{16}$$

for all $(t,j) \in \operatorname{dom} \overline{e}$. In particular, the functions β, γ, γ_1

[5]With solutions on hybrid time domains, $e_1^+(\delta)$ is equivalent to $e_1(\delta,1)$.

can be chosen as

$$\beta(s,t) = s\exp(\overline{\mu}(F_1)t),$$
$$\gamma(s) = \max_{i\in\{1,2\}} \frac{||L_i||}{|\overline{\mu}(F_i)|}s,$$
$$\gamma_1(s) = \left((1+w_1)\frac{||L_1||}{|\overline{\mu}(F_1)|} + w_1\frac{||L_2||}{|\overline{\mu}(F_2)|}\right)s,$$

for all $s,t \in \mathbb{R}_{\geq 0}$.

Remark 4.12: It is worth to note that when the measurement noise is zero, (16) reduces to $|\overline{e}(t,j)| \leq (1-j)\beta(|\overline{e}(0,0)|, \min\{t,\delta\})$. When $t \in [0,\delta)$ and $j = 0$, $|\overline{e}(t,j)| \leq \beta(|\overline{e}(0,0)|, \min\{t,\delta\})$ is a standard $\mathcal{KL}$ bound, but, when $t \in [\delta,\infty)$ and $j = 1$, $|\overline{e}(t,j)| = 0$, which indicates the finite-time convergence property of (3).

D. Finite-time convergent hybrid observer with one jump under constant noise

Motivated by the fact that the estimate of $\mathcal{H}_m$ is not able to converge in finite time under any nonzero noise, we propose a finite-time convergent hybrid observer for the plant in (1) with constant noise. The observer $\mathcal{H}_c$ has state

$$\zeta_c = (\hat{x}_1, \hat{x}_2, \xi_1, \xi_2, \hat{m}, \tau, q) \in \mathcal{X}_c := \mathbb{R}^r \times [0,\delta] \times \{0,1\},$$

where $r = 2n + (2n+1)p$. Its flow map and jump map are

$$f_c(\zeta_c, y) = \begin{bmatrix} A\hat{x}_1 + L_1(y - H\hat{x}_1) - L_1\hat{m} \\ A\hat{x}_2 + L_2(y - H\hat{x}_2) - L_2\hat{m} \\ \mathrm{diag}_p(A - L_1H)\xi_1 + \tilde{L}_1 \\ \mathrm{diag}_p(A - L_2H)\xi_2 + \tilde{L}_2 \\ 0 \\ 1-q \\ 0 \end{bmatrix}, \qquad (17)$$

$$g_c(\zeta_c, y) = \begin{bmatrix} R - T(HT-I)^{-1}(HR - (y-\hat{m})) \\ R - T(HT-I)^{-1}(HR - (y-\hat{m})) \\ 0 \\ 0 \\ \hat{m} + (HT-I)^{-1}(HR - (y-\hat{m})) \\ 0 \\ 1-q \end{bmatrix}, \quad (18)$$

where, for all $i \in \{1,2\}$[6],

$$\tilde{L}_i = \sum_{k=1}^{p} \mathcal{I}_k^\top L_i v_k, \qquad R(\hat{x}_1,\hat{x}_2) = K_1\hat{x}_1 + K_2\hat{x}_2,$$
$$T(\xi_1,\xi_2) = K_1\sum_{k=1}^{p}\mathcal{I}_k\xi_1 v_k^\top + K_2\sum_{k=1}^{p}\mathcal{I}_k\xi_2 v_k^\top,$$

where[7] $\mathcal{I}_k \in \mathbb{R}^{n\times np}$ contains p $n\times n$ sub-matrices, and the only nonzero sub-matrix among them is an identity $I_{n\times n}$ at the k-th entry. The flow set is defined by $C_c := \mathcal{X}_c$, the jump set is given by $D_c := \{\zeta_c \in \mathcal{X}_c : \tau = \delta, q = 0\}$. The gains K_1 and K_2 are given by $K_1 = I - K_2$, $K_2 = (I - \exp(F_2\delta)\exp(-F_1\delta))^{-1}$. These definitions of C_c and D_c ensure that $\mathcal{H}_c$ jumps only once.

[6] For simplicity, the arguments in T and R are dropped in (17), and (18).

[7] $\tilde{L}_i$ is a vector generated by the columns of the matrix L. $\sum_{k=1}^{p}\mathcal{I}_k\xi_1 v_k^\top$ pulls each k-th "piece" of ξ_1 and put it in the k-th column of the resulting matrix, where $\xi_i \in \mathbb{R}^{np}$.

Theorem 4.13: For the interconnection between $\mathcal{H}_c$ and the plant (1) *with state $z_c := (x,\zeta_c)$, assume the noise m is constant. Moreover, suppose Assumption 4.1 holds and that $HT(\xi_1(\delta,0),\xi_2(\delta,0)) - I$ is nonsingular. Then, for any initial condition $z_c(0,0) \in S_c := \{z_c \in \mathbb{R}^n \times \mathcal{X}_c : \hat{x}_1 = \hat{x}_2, \xi_1 = \xi_2 = 0, \tau = 0, q = 0\}$, the states $\hat{x}_1$ and $\hat{x}_2$ converge to x in finite time δ and one jump, i.e., $\hat{x}_1(t,j) = \hat{x}_2(t,j) = x(t,j)$ for all $(t,j) \in \{(t,j) \in \mathrm{dom}\ z_c : t \geq \delta, j \geq 1\}$.*

Note that the invertibility of $HT(\xi_1(\delta,0),\xi_2(\delta,0)) - I$ can be checked offline since the ξ_i trajectories have analytical expressions.

E. Finite-time convergent hybrid observer with recursive jumps under piecewise constant noise

When the noise is a piecewise constant function, it is also possible to estimate the state of the plant in finite time. However, only one jump is not enough. Therefore, recursive jumps are embedded in the observer. To this end, we propose a hybrid observer, denoted $\mathcal{H}_{pc}$, with state

$$\zeta_{pc} = (\hat{x}_1, \hat{x}_2, \xi_1, \xi_2, \hat{m}, \tau) \in \mathcal{X}_{pc} := \mathbb{R}^r \times [0,\delta].$$

The first five components of the flow map f_{pc} coincide with those of f_c and the sixth is equal to one. The first five components of jump map g_{pc} is equal to the first five components of g_c, and the sixth is equal to zero. The flow set is defined by $C_{pc} := \mathcal{X}_{pc}$ and the jump set is given by $D_{pc} := \{\zeta_{pc} \in \mathcal{X}_{pc} : \tau = \delta\}$. The gain K_1 and K_2 are given by $K_1 = I - K_2$, $K_2 = (I - \exp(F_2\delta)\exp(-F_1\delta))^{-1}$. These definitions of C_{pc} and D_{pc} ensure that the system jumps recursively.

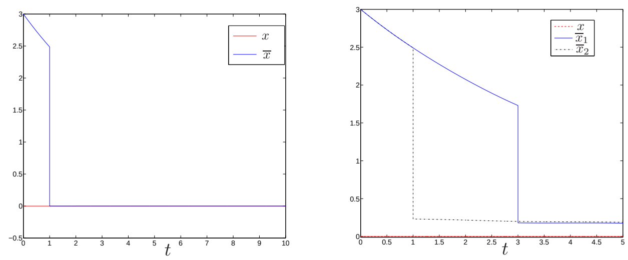
Theorem 4.14: For the interconnection between the observer $\mathcal{H}_{pc}$ and the plant (1) *with $z_{pc} := (x,\zeta_{pc})$, assume the measurement noise m is a piecewise constant function defined as $m(t) := \sum_k c_k\chi_{B_k}(t)$ for all $t \in [0,\infty)$, where $c_k \in \mathbb{R}$, $B_k := [d_{k-1}, d_k)$ with $0 \leq d_{k-1} < d_k$, d_k finite or infinite for integers $k \geq 1$, and $\cup_k^\infty B_k = [0,\infty)$. Moreover, suppose Assumption 4.1 holds, $0 < \delta < \frac{1}{2}\min_{k\in\{1,2,\dots\}}\{d_k - d_{k-1}\}$, and that $HT(\xi_1(\delta,1),\xi_2(\delta,1)) - I$ is nonsingular. Then, for each initial condition $z_{pc}(0,0) \in S_{pc} := \{z_{pc} \in \mathbb{R}^n \times \mathcal{X}_{pc} : \hat{x}_1 = \hat{x}_2, \xi_1 = \xi_2 = 0, \tau = 0\}$ and for each $I_j \times \{j\} \subset \mathrm{dom}\ z_{pc}$, there exists $\tilde{I} \subset (I_j \cup I_{j+1})$ with nonempty interior such that $\hat{x}_i(t,j) = x(t,j)$ for each $t \in \tilde{I}$ and each $i \in \{1,2\}$.*

The existence of $\tilde{I}$ guarantees that whenever noise changes, the proposed observer estimates the new value of the piecewise constant noise in finite time (within 2δ).

V. Examples

A. Constant noise and selection of δ for a scalar plant

Consider the plant in (5) with $a = -0.05$, $L_1 = 0.1$ and $L_2 = 0.2$. The solution obtained with the hybrid observer $\mathcal{H}_c$ is shown in Figure 2(a). As the figure indicates, the constant noise is compensated after one jump and the estimation error converges to zero when $\tau = \delta$. In general, the sooner the observer jumps, the larger the error after the jump would be. By evaluating the error after the jump as defined in (15), Figure 2(b) shows the effect of noise when $\delta = 1$ and $\delta = 3$. Note that the error is sensitive to the parameter δ.



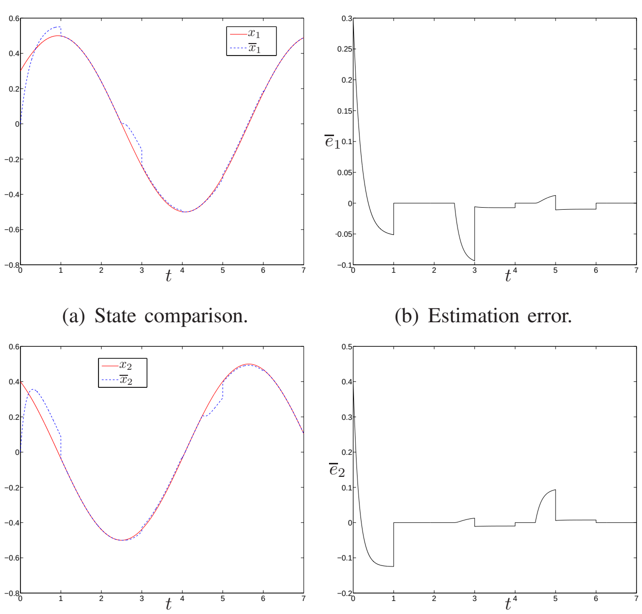
(a) The parameters are $a = -0.05$, $L_1 = 0.1$, $L_2 = 0.2$, $\delta = 1$, $x(0,0) = 0$, and $m \equiv 0.2$.

(b) Comparison when jumps are triggered by different timers for $\mathcal{H}_n$, $x(0,0) = 0$, $m(t) = 0.1\sin(t)\cos(t) + 0.2$.

Fig. 2. Noise compensation by $\mathcal{H}_c$ and effect parameter δ on $\mathcal{H}_n$.

B. Piecewise constant noise with recursive jumps

For the plant in (1) with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, and parameters $\delta=1$, $L_1 = \begin{bmatrix} -1.67 & 3.33 \\ 3.33 & -1.67 \end{bmatrix}$, $L_2 = \begin{bmatrix} -2.67 & 5.33 \\ 5.33 & -2.67 \end{bmatrix}$, a simulation is shown in Figure 3 with the hybrid observer $\mathcal{H}_{pc}$, where the noise is $m(t) = c_1\chi_{[0,2.5)} + c_2\chi_{[2.5,4.5)} + c_3\chi_{[4.5,\infty)}$ with $c_1 = (0.3, 0.2)$, $c_2 = (0.4, 0.4)$, $c_3 = (0.2, 0.3)$. The initial conditions are $x(0,0) = (0.3, 0.4)$, $\hat{x}_1(0,0) = \hat{x}_2(0,0) = 0$. As shown in Figure 3, every time the noise changes its value, the observer estimates the new noise value within two more consecutive jumps, which is when the state estimates converge to the exact value of the plant's state.



(a) State comparison. (b) Estimation error.

(c) State comparison. (d) Estimation error.

Fig. 3. Piecewise constant noise with recursive jumps. x_i's are the state components of the plant, $\bar{x}_i$'s are the corresponding estimation by using observer $\mathcal{H}_{pc}$, and $\bar{e}_i = x_i - \bar{x}_i$.

VI. Conclusion

In this work, robust finite-time hybrid observers are proposed. With the HBC satisfied, the solutions of the finite-time convergent observer with and without noise are proved to be (T, J, ε)-close. Large robustness under general noise is established based on $\mathcal{KL}$ estimates. When the noise is constant or piecewise constant, finite time convergence is guaranteed. Numerical results confirm the results concerning finite-time convergence and robustness.

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