Problem 1 (30 points) Consider a simplified model of the Newton’s cradle consisting of a pair of pendulums with mass $m_L$, $m_R$ as shown in Figure 1.

![Figure 1: Simplified Newton’s cradle.](image)

1.1 Determine the state of the system.

1.2 Define each element of a hybrid system model $(C, f, D, g)$ assuming the following:

- Viscous friction for circular motion.
- Conservation of momentum at impacts.
- Dissipation of energy at impacts.

1.3 Perform the following simulations of the resulting hybrid system for parameters $m_L = 1, m_R = 2$, unitary viscous friction, and a restitution law with parameters such that there is dissipation at impacts:

   (a) Plot trajectories as a function of $t$ for pendulums starting at $x_1 = -\pi/4$ and $x_3 = \pi/4$ with zero velocity.

   (b) Plot trajectories as a function of $t$ for pendulums starting at $x_1 = 0$ and $x_3 = \pi/4$ with zero velocity.

   (c) Plot trajectories as a function of $t$ for pendulums starting at $x_1 = \pi/4$ and $x_3 = \pi/2$ with zero velocity.
Problem 2 (30 points) Consider the implementation of a static controller
\[ \kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m \]
for the continuous-time plant
\[ \dot{\xi} = \tilde{f}(\xi, u) \]
in a digital device, e.g. computer, microcontroller, digital signal processor, etc. This is depicted in Figure 2, where the controller is interfaced with sample-and-hold devices. The sample-and-hold device that samples the state \( \xi \) of the plant is referred to as sampling device (or analog-to-digital (A/D) converter), while the sample-and-hold device that stores the output of the controller in between computations is referred to as hold device (or digital-to-analog (D/A) converter) which is assumed to be of zero-order type, that is, a zero-order hold (ZOH).

Figure 2: Sample-and-hold control of a nonlinear system.

2.1 Define each element of a hybrid system model \((C, f, D, g)\) and its state assuming the following:

- The computation of the static feedback law takes no time, i.e., is instantaneous.
- The positive constants \( T_s \) and \( T_c \) are not necessarily equal.

2.2 Explain how the model would change if the computation of the control law takes \( \delta > 0 \) units of time.
Problem 3 (30 points) For the hybrid system $\mathcal{H}$ with data

$$
C := \{ x \in \mathbb{R}^2 : |x| < 1 \}, \quad f(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \\
D := \{ x \in \mathbb{R}^2 : x_1 = 0, x_2 \in (-\frac{1}{2}, 0] \}, \quad g(x) := \begin{bmatrix} 0 \\ x_1 \end{bmatrix}
$$

with state $x \in \mathbb{R}^2$,

1. Classify all of its solutions following the definitions in Definition 2.5 and Definition 2.7.

2. Indicate the points in $C \cup D$ from which nontrivial solutions exist.

3. Indicate the properties of maximal solutions to $\mathcal{H}$.

You may want to apply Proposition 2.10 in answering these questions.

Problem 4 (10 points) For the hybrid system $\mathcal{H}$ that you proposed in Homework 1, answer items 1-3 in Problem 3 above.