GRADUATE COURSE ON ROBUST HYBRID CONTROL SYSTEMS – Homework #3

Problem 1 *(20 points)* For the bouncing ball system given by the hybrid system with

$$f(x) = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix}, \qquad C := \left\{ x \in \mathbb{R}^2 \ : \ x_1 \geq 0 \ \right\}$$

$$g(x) = \begin{bmatrix} x_1 \\ -e x_2 \end{bmatrix}, \qquad D := \left\{ x \in \mathbb{R}^2 \ : \ x_1 = 0, x_2 \leq 0 \ \right\}$$

where $e \in [0, 1)$ and $\gamma > 0$:

1. Show that the origin $(0, 0)$ is asymptotically stable.
2. In the plane, plot: a) the flow and jump sets, b) level sets of the Lyapunov functions, and c) a solution starting from $(1, 0)$ and a solution starting from $(0, -1)$. Show graphically that the motion of the solutions is such that they go from larger to smaller level sets of the Lyapunov function.

Problem 2 *(20 points)* Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$\begin{aligned} C &:= \{x : x_1 \geq 0\}, & f(x) &:= \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} x \qquad \forall x \in C, \\ D &:= \{x : x_1 = 0, x_2 \leq 0\}, & g(x) &:= -\gamma x \qquad \forall x \in D, \end{aligned}$$

where $\gamma > 0$, $\omega > 0$, and $\alpha \in \mathbb{R}$ are the system parameters.

1. Using the sufficient conditions for Lyapunov stability, find conditions on the system parameters for which the origin of the hybrid system is *uniformly globally pre-asymptotically stable*. Show your work in detail.
2. Confirm your answer to item 1 via simulations.
3. Is the origin *uniformly globally asymptotically stable*? Justify your answer.

Problem 3 *(20 points)* Consider the hybrid system with state $x \in \mathbb{R}^n$, flow set $C \subset \mathbb{R}^n$, jump set $D \subset \mathbb{R}^n$, and

$$f(x) := Ax \qquad \forall x \in C, \qquad\qquad g(x) := Ex \qquad \forall x \in D,$$

where $A, E \in \mathbb{R}^{n \times n}$. Take the quadratic function

$$V(x) = x^\top P x,$$

where P is a positive definite matrix.

1. What are the conditions on A and E so that $\mathcal{A} := \{x \in \mathbb{R}^n \ : \ x = 0\ \}$ is uniformly globally pre-asymptotically stable?

2. What additional conditions on the set of solutions to the hybrid system should be imposed so that $\mathcal{A} := \{x \in \mathbb{R}^n \ : \ x = 0\ \}$ is uniformly globally asymptotically stable?

Problem 4 *(20 points)* The mechanical system in Figure 1 consists of a point mass impacting with a vertical wall. Assuming unitary mass for sake of simplicity, the system is described by the following equations:

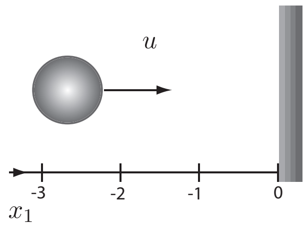

Figure 1: Point mass interacting with the environment.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 x_1 - k_2 x_2 - f_c(x), \end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}$ denotes the horizontal position, $x_2 \in \mathbb{R}$ denotes the horizontal velocity, $k_1, k_2 \in \mathbb{R}$, f_c is the contact force given by

$$f_c(x) = \begin{cases} k_c x_1 + b_c x_2 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$$

in which $k_c > 0$ and $b_c > 0$ are, respectively, the elastic and damping coefficients of the compliant contact model. When a collision with the surface located at $x_1 = 0$ occurs with a velocity of the mass greater or equal than $\bar{x}_2$, the impact is assumed to be impulsive and, accordingly, the rigid body instantaneously rebounds or jumps. The new value of the state variables after the impact is described by the reset law

$$\begin{aligned} x_1^+ &= x_1 \\ x_2^+ &= -\varrho x_2, \end{aligned}$$

where ϱ represents the restitution coefficient. For the hybrid system model $\mathcal{H}$ capturing the dynamics of the mechanical system in Figure 1:

1. Compute the corresponding regularized hybrid system model $\widehat{\mathcal{H}}$.

2. Indicate if there are solutions to $\widehat{\mathcal{H}}$ that are not solutions to $\mathcal{H}$.

3. Do Hermes and Krasovskii solutions to $\mathcal{H}$ coincide or not?

Problem 5 *(20 points)* Consider the plant

$$\dot{x} = u \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x,$$

where $x \in \mathbb{R}^2$ is the state satisfying $|x| = 1$, $\omega > 0$ is a constant, and $u \in \mathbb{R}$ is the control input.

1. Propose a control law (static or dynamic, hybrid or not) to locally stabilize the point $x^* = (1, 0)$.
2. Indicate if the resulting closed-loop system is robust to small perturbations.
3. Extend your control law so that it globally stabilizes the point $x^* = (1, 0)$ and indicate if the resulting closed-loop system is robust to small perturbations.