5. Intersample behavior is captured by hybrid feedback
6. Hybrid control improves performance

## On truly hybrid models:

Hybrid systems are not
- piecewise continuous / affine systems
- linear systems with nonsmooth inputs
- systems with discontinuous right-hand side

$$\dot{x} = -\operatorname{sgn}(x)$$

- mixed logical dynamical systems $x^+ = g(x, u)$

$x = (x_c, x_d)$ $\quad$ $x_c$ is the part of the state that is continuous valued

$x_d$ is " " " " " discrete valued

## A uniting local and global example:

A mechanical disk drive has a reading head that is controlled to perform the following task:

"global" – The reading head is steered to a location nearby where the data to read is. ← SPECIFICATION IS RAPID CONVERGENCE (nearby)

"local" – Once close there, there is another strategy that regulates the position of the head to read the actual data at the given location. ← SPECIFICATION IS ACCURACY ON POSITIONING.

A simple model of the position and velocity of the head in a given track is of the form $\dot{p} = v, \dot{v} = u$

where $p \in \mathbb{R}$ is position and $v$ is velocity, with $u$ the "force" control input. The desired reference to accomplish a reach task can be taken to be $p^*$ [redacted] $\in \mathbb{R}$, $v^* = 0$

$$z = \begin{bmatrix} p \\ v \end{bmatrix} \quad \text{plant state}$$

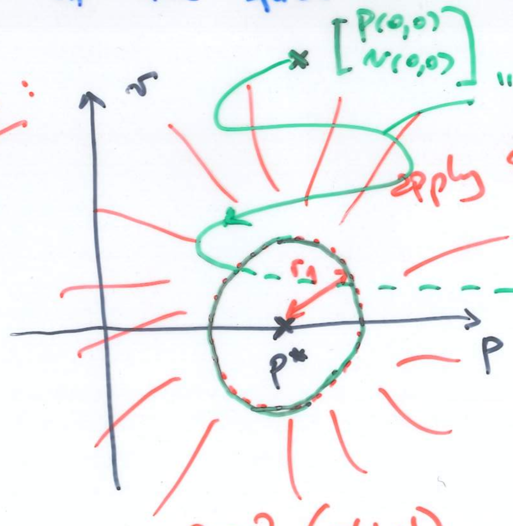
$$\dot{z} = \begin{bmatrix} v \\ u \end{bmatrix} =: f_p(z, u) \qquad (z,u) \in \mathbb{R}^2 \times \mathbb{R}$$

w/o jumps.

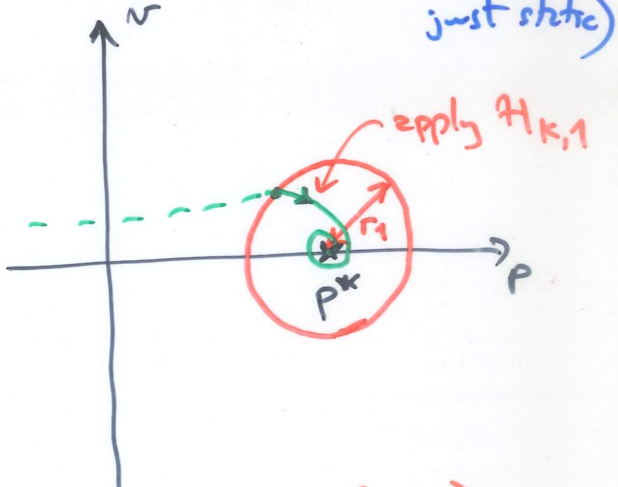
Suppose that the local controller is proportional control of the form $\mathcal{H}_{K,1}$ : $y = K_1 \begin{bmatrix} p - p^* \\ v \end{bmatrix}$ (no flows, no jumps, just static)

and that the global controller is also proportional control of the form $\mathcal{H}_{K,2}$ : $y = K_2 \begin{bmatrix} p - p^* \\ v \end{bmatrix}$ (no flows, no jumps, just static)

Try 1:




$q = 2$ (global)




$q = 1$ (local)

$K_2$ should be designed to steer $z(t,j)$ to $\mathcal{A}_1$

Find $r_1 > 0$ small enough defining

$$\mathcal{A}_1 = \{ z \in \mathbb{R}^2 : \underbrace{\left| z - \begin{bmatrix} p^* \\ 0 \end{bmatrix} \right|}_{\sqrt{(p-p^*)^2 + v^2}} \le r_1 \}$$

The resulting feedback is:

$$K(z) := \begin{cases} K_2 \begin{bmatrix} p - p^* \\ v \end{bmatrix} & \text{if } [q = 2] \equiv z \notin \mathcal{A}_1 \; (z \in \mathbb{R}^2 \setminus \mathcal{A}_1) \\ K_1 \begin{bmatrix} p - p^* \\ v \end{bmatrix} & \text{if } [q = 1] \equiv z \in \mathcal{A}_1 \end{cases}$$

discontinuous, not hybrid!

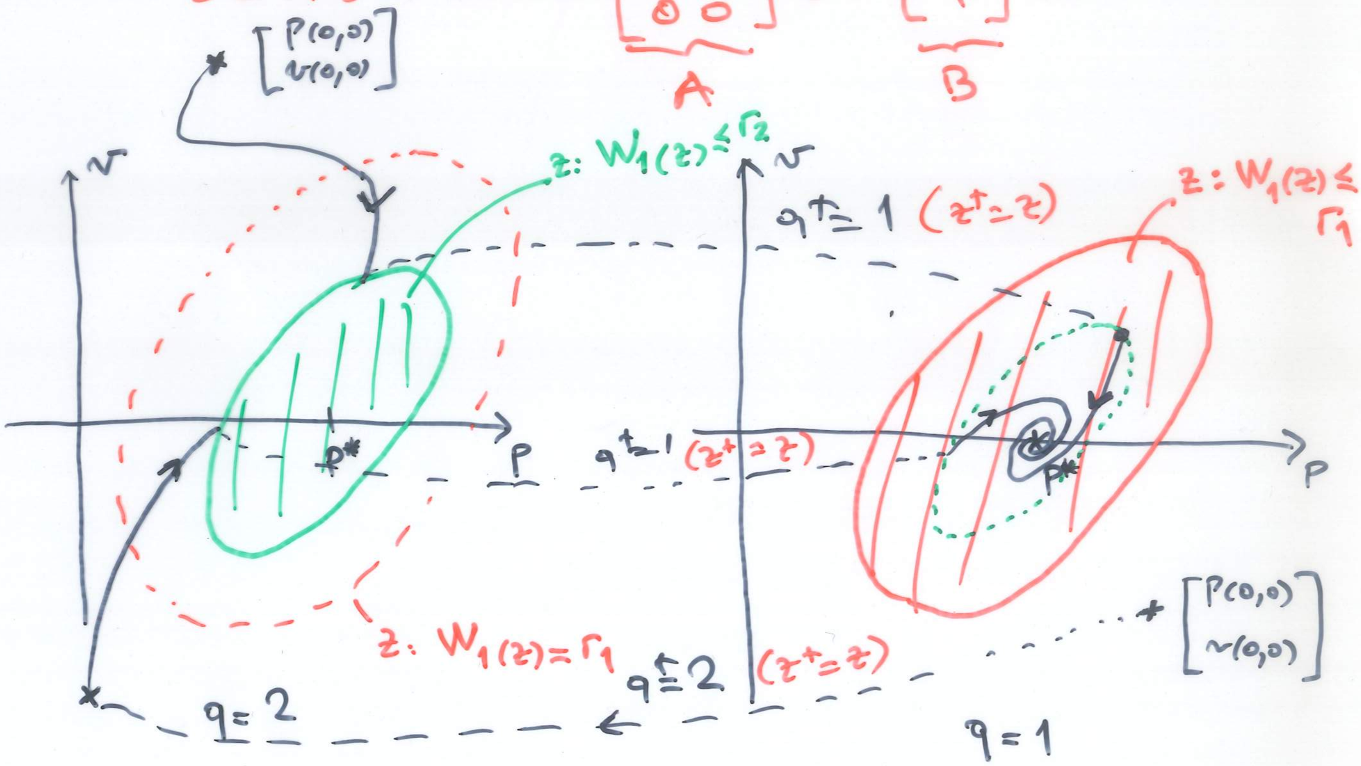
$K_1$ should be designed so that solutions from $\mathcal{A}_1$ converge to $\begin{bmatrix} p^* \\ 0 \end{bmatrix}$

Suppose that for each $q \in Q := \{1, 2\}$ we design $K_1$ and $K_2$ to (globally) exponentially stabilize the origin of $z$, but $K_2$ guarantees fast convergence (e.g., small rise time) and $K_1$ " small overshoot. Let $W_q$ is the Lyapunov function associated with $K_q$; i.e., $P_q = P_q^T > 0$ s.t.

$$(A + BK_q)^T P_q + P_q (A + BK_q) < 0 \quad \forall q \in Q$$

where $W_q(z) = \left(z - \begin{bmatrix} p^* \\ 0 \end{bmatrix}\right)^T P_q \left(z - \begin{bmatrix} p^* \\ 0 \end{bmatrix}\right)$.

$$\dot{z} = Az + Bu = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$




$0 < r_2 < r_1$

jumps from $q = 2$ to $q = 1$: $D_{S,2} := \{z : W_1(z) \leq r_2\}$

flows with $q = 2$: $C_{S,2} := \{z : W_1(z) \geq r_2\}$

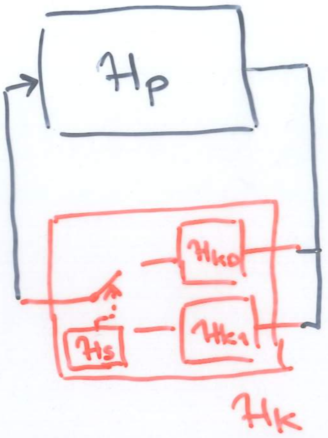
jumps from $q = 1$ to $q = 2$: $D_{S,1} := \{z : W_1(z) \geq r_1\}$

flows with $q = 1$: $C_{S,1} := \{z : W_1(z) \leq r_1\}$

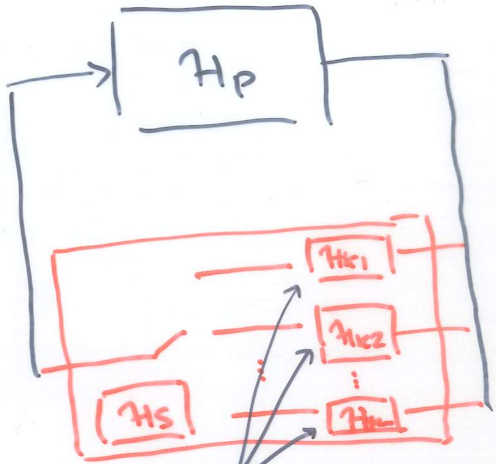
$C_{S,1} \cap C_{S,2} = \{z : r_2 \leq W_1(z) \leq r_2\}$ so feedback is not static !

## Throw-catch control

The uniting local/global controller coordinates two controllers (static state-feedback, dynamic continuous-time feedback, and individual hybrid controllers)

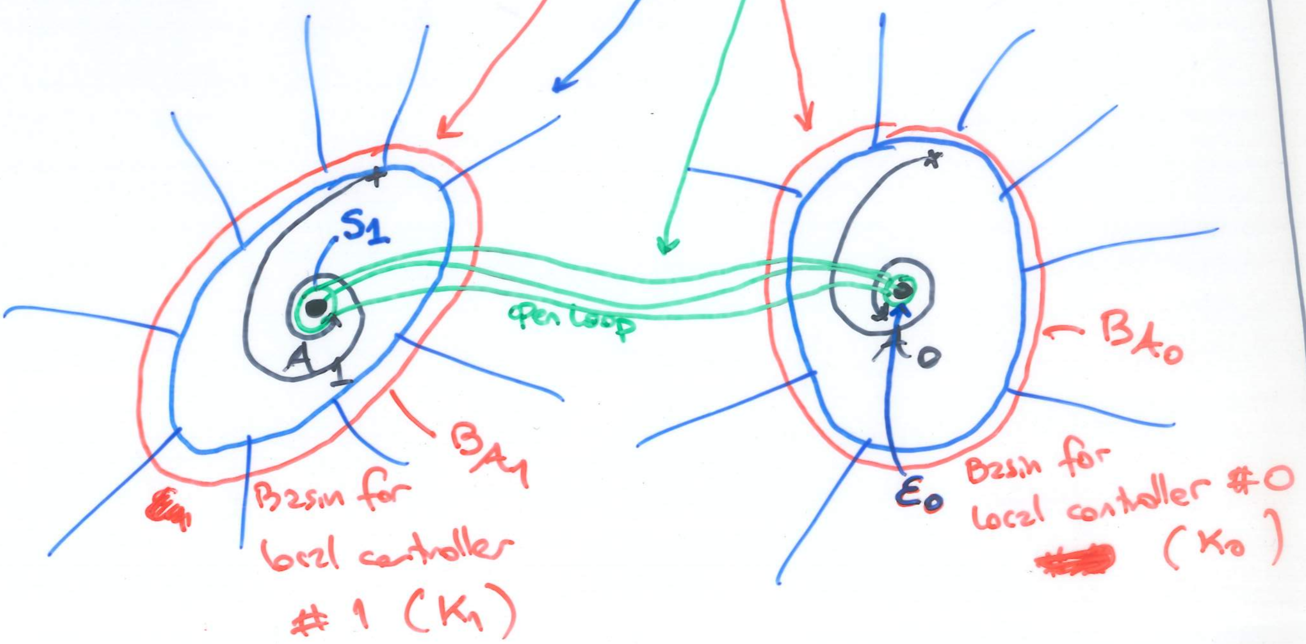



Uniting local/global




- local static state feedback
- bootstrap feedback controller
- open-loop controller

Consider the plant $\mathcal{H}_p$ $\dot{z} = F_p(z, u)$

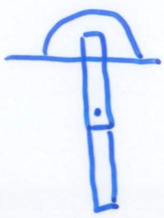



If the goal is to render $x_0$ globally asymptotically stable for the plant, then we need more than two local controllers.

In general we may need multiple local stabilizers that work nearby multiple points $A_i$, and also have the need to steer between $A_i$'s and $A_j$'s using open-loop controllers.

**Example:** Double link pendulum swing up.

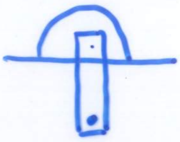
$\dot{z} = F_p(z, u)$ with $z = \begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \end{bmatrix}$ angles of links ($\phi_1$, $\phi_2$), angular velocities ($\omega_1$, $\omega_2$)



both down

$\phi_1 = \phi_2 = \pi$

$\omega_1 = \omega_2 = 0$



one down / one up

$\phi_1 = \pi$

$\phi_2 = 0$

$\omega_1 = \omega_2 = 0$

one up / one down

$\phi_1 = 0$

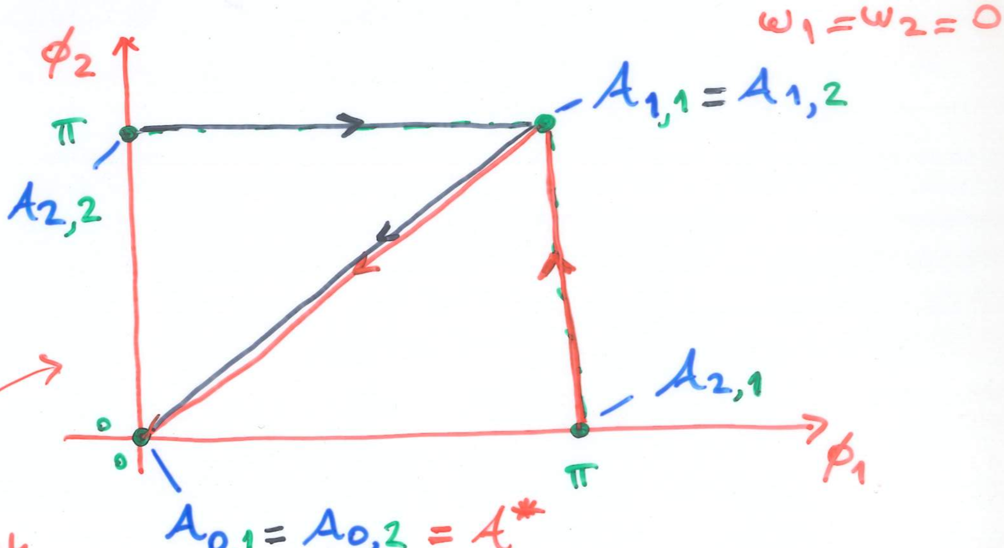
$\phi_2 = \pi$

$\omega_1 = \omega_2 = 0$

both up

$\phi_1 = \phi_2 = 0$

$\omega_1 = \omega_2 = 0$

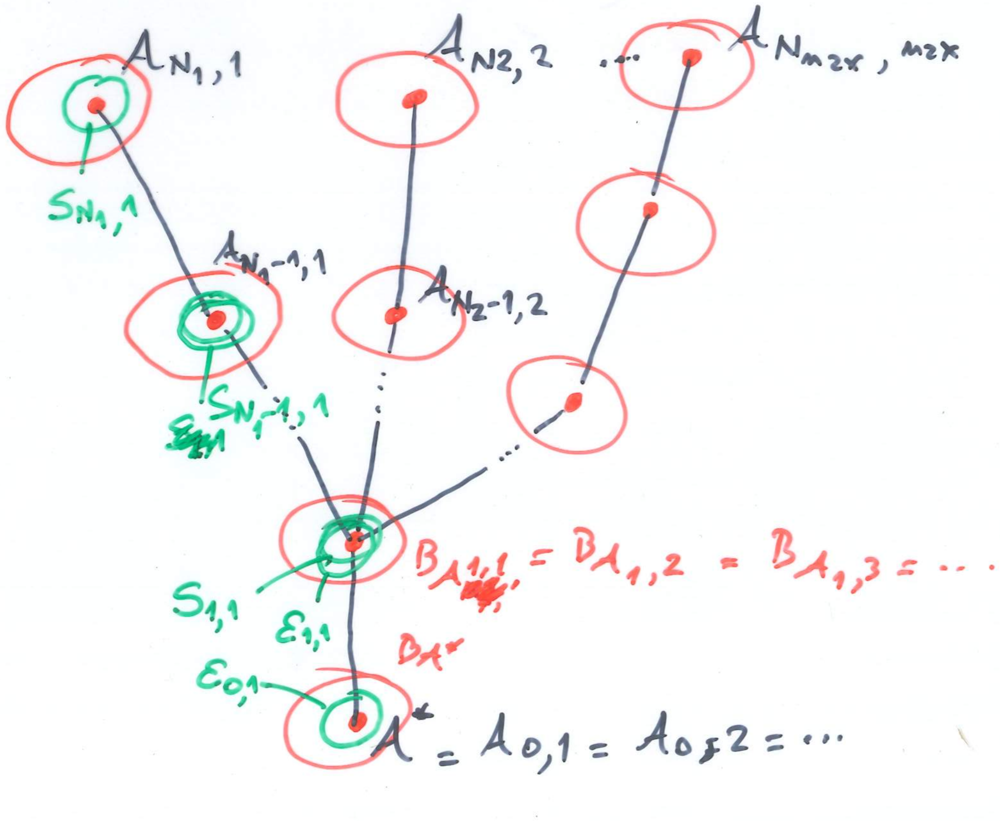



graph with two paths to the goal

Throw-and-catch problem: Given $A^*$ find local feedback controllers, open-loop controllers, and bootstrap controller such that the basin of attraction is the entire state space.

A solution:

- Define a graph with paths ending at the goal $A^*$
- Design each local controller $K_{i,s}$ with basin of attraction $B_{A\,i,s}$
- Design open-loop controllers $K_{(i,s)\to(i-1,s)}$ capable of steering solutions to $\mathcal{H}_p$ from $S_{i,s}$ to a set $\mathcal{E}_{i-1,s}$
- Design a bootstrap controller $K_b$ capable of steering solutions in a finite amount of time to where the other controllers operate.



## A simple vehicle control example:

$u, z \in \mathbb{R}^2$

Given a vehicle model of the form $\dot{z} = u$ design a feedback controller that, globally, steers $z$ to the target location $\mathcal{A}^*$ under the following constraints:

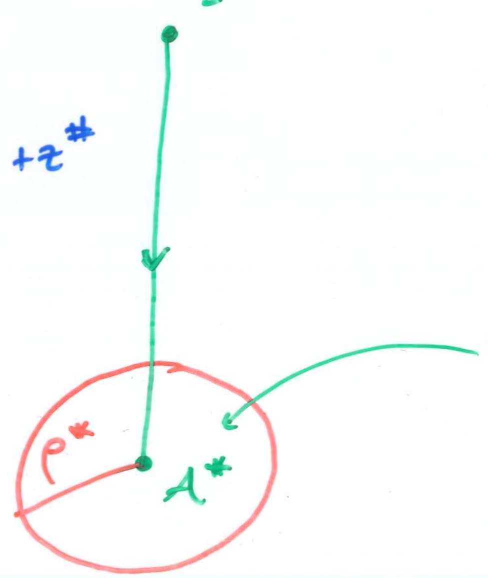

- Relative position information (to $\mathcal{A}^*$) only available nearby $\mathcal{A}^*$ -- denoted as $\rho^* > 0$
- Relative position information to an intermediate location $z^\#$ is always available
- Information on how to reach a neighborhood of $\mathcal{A}^*$ (from where rel. pos. info to $\mathcal{A}^*$ is available) is only available nearby intermediate location.

$\dot{z} = u \quad u = k(z) = -z$

$z = 0$ GAS (GES)

Define graph associated w/ throw-catch strategy

Outline the controllers to use $\begin{cases} \text{local state feedback} \\ \text{open loop controller} \\ \text{bootstrap} \end{cases}$

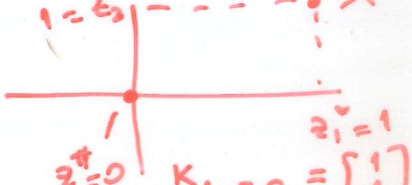



$\kappa_b(z) = -z + z^\#$

$z^* = \begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$

$\mathcal{A}^* = \{z^*\}$

$\kappa_0(z) = -z + z^*$




$\kappa_{1 \to 0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

# Event triggered control

Warm up example:

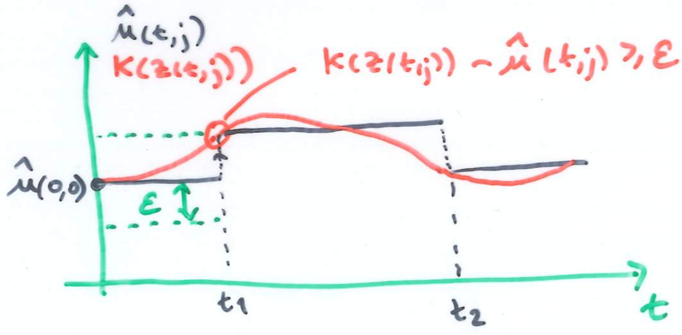
Define an event triggering condition as a jump set for

$$\dot{z} = F_p(z, u)$$
$$u = K(z)$$

so that the input to the plant is only updated to the one defined by $K(z)$ only [when the input error is $\geq \varepsilon > 0$]:

setting: $\dot{z} = F_p(z, \hat{u})$, $\dot{\hat{u}} = 0$ ; $z^+ = z$, $\hat{u}^+ = K(z)$ when

Find $D$ & $C$.




input error is what the input is minus what the input should be:

$$\hat{u} - K(z)$$

To trigger events when $\hat{u} - K(z)$ in norm is $\geq \varepsilon$ define $D$ as: $x = \begin{bmatrix} z \\ \hat{u} \end{bmatrix}$

$$D = \{ x : |\hat{u} - K(z)| \geq \varepsilon \}$$

The flow set is the set of points where $|\hat{u} - K(z)|$ $\geq \varepsilon$ does not hold, closed: $C := \overline{(\mathbb{R}^{n_p} \times \mathbb{R}^{m_p}) \setminus D}$ $= \{ x : |\hat{u} - K(z)| \leq \varepsilon \}$

\ returns

Another event triggered control implementation of a static state-feedback law:

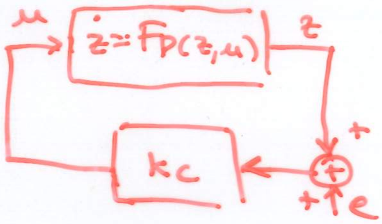
For $\dot{z} = f_p(z, u)$ consider the following assumptions:

- $\exists$ $k_c$ , $V_p$ and $\mathcal{K}_\infty$ $\alpha_1, \alpha_2, \rho_1, \rho_2$

$V_p$ is an ISS-Lyapunov function wrt $e$

$$\alpha_1(|z|) \leq V_p(z) \leq \alpha_2(|z|) \quad \forall z \in \mathbb{R}^{n_p}$$

$$\dot{V}_p = \langle \nabla V_p(z), f_p(z, k_c(z+e)) \rangle \leq -\rho_1(|z|) + \rho_2(|e|) \quad \forall z, e \in \mathbb{R}^{n_p}$$




Then an event-triggering implementation of $k_c$ is as follows:

- Use $\hat{y}$ instead of $\hat{u}$ in the ETM model
- Assume $y = z \Rightarrow \hat{y}$ is nothing but a memorized previous $z$ value.
- We can only apply $k_c(\hat{y})$ to the plant

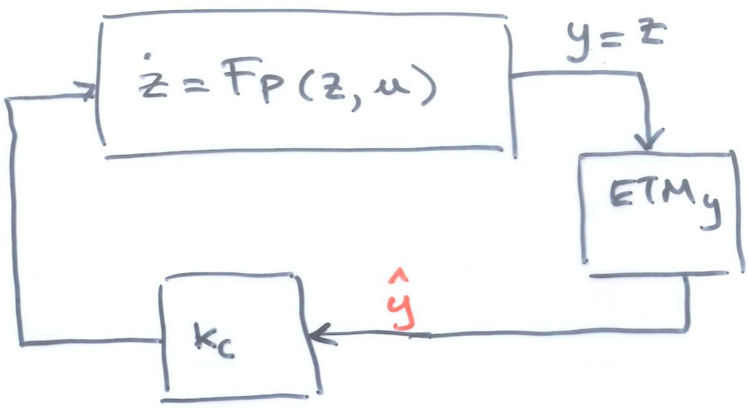
Note that $e = \hat{y} - y$ = " $\hat{z} - z$ " leads to

$$k_c(y + e) = k_c(y + \hat{y} - y) = k_c(\hat{y}) \ (= k_c(\hat{z}))$$

strategy: - Use $\hat{y}$ to store $y$ at events

- When $\rho_2(|\hat{y} - y|) \geq \sigma \rho_1(|\overset{\text{"}y\text{"}}{z}|)$

$\sigma \in (0,1)$ then trigger an event resetting $\hat{y}$ to $y$




event is triggered when $\rho_2(|\hat{y} - \overset{z}{\overset{\shortparallel}{y}}|) \geq \sigma \rho_1(|z|)$

$$e = \hat{y} - y = \hat{y} - z$$

Recall that

$$\langle \nabla V_P, F_P(z, K_c(z+e)) \rangle \leq -\rho_1(|z|) + \rho_2(|e|)$$

The closed loop has state $x = \begin{bmatrix} z \\ \hat{y} \end{bmatrix}$

$$F(x) = \begin{bmatrix} F_P(z, K_c(\hat{y})) \\ 0 \end{bmatrix} \qquad C = \{x : \rho_2(|\hat{y} - z|) \leq \sigma \rho_1(|z|)\}$$

$$G(x) = \begin{bmatrix} z \\ z \end{bmatrix} \qquad D = \{x : \rho_2(|\hat{y} - z|) \geq \sigma \rho_1(|z|)\}$$

When $F_P$ is continuous, $K_c$ is continuous $\Rightarrow$ $F$ is continuous

$G$ is continuous. When $\rho_1, \rho_2$ are continuous, then $C$ & $D$ are closed. Moreover the event triggering function is

$$\gamma^y(x) := \rho_2(|\hat{y} - z|) - \sigma \rho_1(|z|)$$

Note: we can rewrite this event triggering controller as $\mathcal{H}_K$

$$\eta = \hat{y}, \quad F_K(\eta) = 0 \qquad C_K \ldots$$

$$G_K(\eta, v) = \begin{bmatrix} \phantom{v} \\ v \end{bmatrix} \qquad D_K \ldots$$

- From $\underbrace{x = \begin{bmatrix} z \\ \hat{u} \end{bmatrix} \in D}_{\Downarrow \; |\hat{u} - k(z)| \geq \varepsilon}$ , $G(x) = \begin{bmatrix} z \\ k(z) \end{bmatrix} \notin D$

after the event

$|\underbrace{k(z)}_{\hat{u} \text{ after event.}} - k(z)| = 0$

then if $F_p$ is bounded (locally) then we have flow in between every event for this particular event trigger law.

- The flow and jump set overlap due to them satisfying the hybrid basic cond$^n$. The overlap is

$$C \cap D = \{ x : \; |\hat{u} - k(z)| = \varepsilon \}$$

If flow within $C \cap D$ is not possible, then only jumps are possible and we don't have sliding solutions on $(\underbrace{\partial C}_{\text{boundary of } C}) \cap D$ . Moreover, we may have $\dot{\overbrace{\hat{u} - k(z)}} > 0 \quad \forall \; \hat{u}, z$ on $D \cap C$

Comparing $D = \{x : \rho_2(|\hat{y} - z|) \geq \underbrace{\sigma \rho_1(|z|)}\}$

and

$D = \{x : |\hat{u} - k(z)| \geq \varepsilon\}$

we see a few differences:

- the "threshold" inside the $\gamma^y$ function is essentially $\varepsilon(z)$ with $\varepsilon(z) \to 0$ as $z \to 0$
- At $x \in D$, $\rho_2(|\hat{y} - z|) \xrightarrow{\text{after jump}} \rho_2(|z - z|)$

$\sigma \rho_1(|z|) \longrightarrow \sigma \rho_1(|z|)$

when, $z = 0$ $\Rightarrow$ $G(x) \subset D$ !

$x = \begin{bmatrix} z \\ \hat{y} \end{bmatrix}$ s.t.

Zeno !

There are examples where we do not have Zeno outside the set to stabilize, but arbitrarily small perturbations can lead to Zeno (even w/ arbitrarily small size of the perturbation)

Conditions guaranteeing a (uniform) lower bound on the inter-event times: Let $\mathcal{H}$ be the c-loop system obtained from event trigger control. Pick a maximal solution $x$ to it. We would like to have $\delta_x > 0$ s.t.

$$t_{j+1} - t_j \geq \delta_x \quad \forall j : (t, j) \in \operatorname{dom} x$$

Suppose that $\mathcal{H}$ has one type of event ($\gamma^{\mu}$ events and $N_{\mu} = 1$). If $\mathcal{H}$ satisfies the hybrid basic conditions and every maximal solution to it is bounded, then for every maximal solution $x$ to $\mathcal{H}$ there exists $\delta_x^{\mu} > 0$ s.t.

$$t_{j+1} - t_j \geq \delta_x^{\mu} \quad \forall j : (t,j) \in \operatorname{dom} x$$

if and only if

$$\underbrace{D_{K,\mu}}_{\text{jump set of controller } \mathcal{H}_K \text{ implementing the ETM}^{\mu}} \cap \overbrace{G_{K,\mu}}^{\text{jump map of } \mathcal{H} \text{ implementing ETM}^{\mu}} (D_{K,\mu}) = \text{empty set}$$