On the robustness to measurement noise and unmodeled dynamics of stability in hybrid systems*

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Abstract— **Results on robustness to measurement noise and unmodeled dynamics of stability in hybrid systems are presented. We show that arbitrarily small measurement noise can lead to lack of existence of solutions in hybrid systems. One solution to this problem is to pass the measurements through a filter. Robustness to measurement noise using this filtering is shown explicitly. We also study the effect of unmodeled sensor/actuator dynamics in the closed loop and we demonstrate that stability is robust to a class of singular perturbations. The results are illustrated for the inverted pendulum on a cart system when attempting to globally asymptotically stabilize the inverted position of the pendulum and the neutral cart position.**

I. Introduction

Over the last fifteen years, researchers have begun to recognize the extra capabilities of hybrid control systems compared to classical continuous-time control systems. For example, it is now well-known that hysteresis switching control can stabilize large classes of nonholonomic systems even though stabilization is impossible using time-invariant continuous state feedback, and robust stabilization is impossible using time-invariant locally bounded feedback. See, for example, [9], [15]. Also, sample and hold control (a special type of hybrid feedback) can be used to achieve stabilization that is robust to measurement noise and fast sensor/actuator dynamics, even if such robustness is impossible using purely continuous-time feedback. See, for example, [17], [5], [10].

Despite these specific studies, a general investigation of the robustness of hybrid controllers to measurement noise and fast sensor/actuator dynamics is absent from the literature. Noise in the measurement of the state arises in every implemented closed-loop system and it is desired that the hybrid controller provide certain degree of robustness to it. It is also common, in the design of a controller, to consider a simplified model of the system, exhibiting only the most important dynamics. By doing this, the controller may be easier to design but sensors/actuators dynamics remain unmodeled. Suppose that for the nonlinear system

$$\dot{x} = f(x, u), \tag{1}$$

where f does not include any sensor/actuator dynamics, there exists a hybrid controller, denoted $\mathcal{H}_c$, that renders the origin of the closed loop globally asymptotically stable. A follow-up question is whether the closed loop with the addition of the unmodeled sensor/actuator dynamics and measurement noise preserves the properties of the nominal closed loop; see Figure 1. Singular perturbation arguments show that this is the case when continuous-time controllers are used (see [11], [19]), but the answer is unknown for hybrid controllers. Moreover, when noise enters the conditions where the jumps

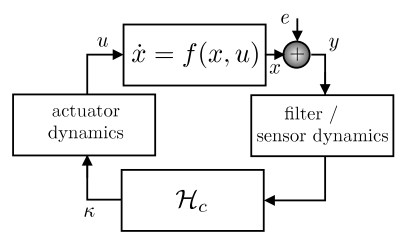

Fig. 1. Hybrid control of a continuous-time system with filter/sensor/actuator dynamics.

and flows for a hybrid system are enabled, solutions may even fail to exist.

In this paper, we give a general discussion of the issues just highlighted. If the hybrid controller $\mathcal{H}_c$ renders the origin of the nominal closed loop globally asymptotically stable, we show that each of the following scenarios:

- hybrid controller $\mathcal{H}_c$ connected to the nonlinear system (1) with filtered measurements;
- hybrid controller $\mathcal{H}_c$ connected to the nonlinear system (1) with sensor dynamics and actuator dynamics;
- hybrid controller $\mathcal{H}_c$ connected to the nonlinear system (1) with sensor dynamics and actuator dynamics that smooth the control;

all yield semiglobally practically asymptotically stable closed loops with respect to the same attractor. Moreover, we describe a particular construction of a hybrid controller $\mathcal{H}_c$, which borrows ideas from control with patchy vector fields in [1] and the hybrid approach in [16], and apply it to the problem of swinging up a pendulum on a cart with measurement error and control smoothing. In our study, we rely on the notion of a solution to a hybrid systems used in [7], [6], [8], but with disturbances explicitly included. Moreover, we exploit the robustness of stability under set perturbations established in [8].

II. Background

Throughout the paper, $\mathbb{B}$ is the closed unit ball, $\mathbb{R}_{\geq 0} := [0, +\infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$, and $|e|_\infty = \sup_{t \in \mathbb{R}_{\geq 0}} |e(t)|$.

Following [7], [8] and also [6] (cf. [12], [3], and [13]), a solution to a hybrid system is a function defined on a hybrid

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time domain satisfying certain conditions. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. The set S is a *hybrid time domain* if for all $(T, J) \in S$,

$$S \cap ([0, T] \times \{0, 1, \ldots J\})$$

is a compact hybrid domain. By a *hybrid arc* or *hybrid trajectory* we understand a pair consisting of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(t, j)$ is locally absolutely continuous in t for a fixed j and $(t, j) \in \operatorname{dom} x$. We will not mention $\operatorname{dom} x$ explicitly, but always assume that given a hybrid arc x, the set $\operatorname{dom} x$ is exactly the set on which x is defined. A hybrid arc is called *complete* if $\operatorname{dom} x$ is unbounded and *maximal* if it is not a truncation of another arc x' to some proper subset of $\operatorname{dom} x'$.

A hybrid system $\mathcal{H}$ will be given on a state space O by set-valued mappings F and G describing, respectively, the continuous and the discrete dynamics, and sets C and D where these dynamics may occur. A hybrid arc $x : \operatorname{dom} x \to O$ is a *solution to* $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and:

(S1) for all $j \in \mathbb{N}$ and a.a. t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j));$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j+1) \in G(x(t, j)).$$

Some mild assumptions on the data of $\mathcal{H}$ are needed to guarantee that, among other properties, the sets of solutions to $\mathcal{H}$ have good sequential compactness properties. We refer the reader to [8] (see also [7]) for details on and consequences of these assumptions.

Assumption 2.1: State space O is open; sets C and D are relatively closed in O; mappings F and G are outer semi-continuous and locally bounded[1] on O; $F(x)$ is nonempty and convex for all $x \in C$; $G(x)$ is nonempty for all $x \in D$.

It was shown in [8] that if $C \cup D = O$, then solutions exist for every initial point in O and maximal solutions are either complete or "blow up".

III. Hybrid control of nonlinear systems

A. Nominal Hybrid Controller

Consider the nonlinear system (1) where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and a hybrid controller, denoted $\mathcal{H}_c$, with continuous state $x_c \in \mathbb{R}^p$, discrete state $q \in Q \subset \mathbb{N}$, continuous dynamics

$$\left.\begin{array}{ll} \dot{x}_c & = f_c(x, x_c, q) \\ \dot{q} & = 0 \end{array}\right\} \text{when } (x, x_c, q) \in C_c,$$

and discrete dynamics

$$\left.\begin{array}{ll} x_c^+ & = g_c(x, x_c, q) \\ q^+ & \in Q_c(x, x_c, q) \end{array}\right\} \text{when } (x, x_c, q) \in D_c,$$

[1] A set-valued mapping G defined on an open set O is *outer semicontinuous* if for each sequence $x_i \in O$ converging to a point $x \in O$ and each sequence $y_i \in G(x_i)$ converging to a point y, it holds that $y \in G(x)$. It is *locally bounded* if, for each compact set $K \subset O$ there exists $\mu > 0$ such that $G(K) := \cup_{x \in K} G(x) \subset \mu\mathbb{B}$.

that renders the set $\{0\} \times \{0\} \times Q$ of the closed-loop system, denoted $\mathcal{H}_{cl}$, asymptotically stable with basin of attraction $\mathbb{R}^n \times \mathbb{R}^p \times Q$ by using the feedback

$$u = \kappa_c(x, x_c, q), \tag{2}$$

where $\kappa_c : \mathbb{R}^n \times \mathbb{R}^p \times Q \to \mathbb{R}^m$ is the output of the controller. We will assume that the closed-loop $\mathcal{H}_{cl}$, which is a hybrid system, satisfies the following assumption.

Assumption 3.1: For the closed-loop hybrid system $\mathcal{H}_{cl}$, the set $\{0\} \times \{0\} \times Q$ is globally asymptotically stable:

1) (local stability) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every (x^0, x_c^0, q^0) satisfying $|x^0| + |x_c^0| \leq \delta$ and $q^0 \in Q$, the x and x_c components of every solution starting at (x^0, x_c^0, q^0) satisfy $|x(t, j)| + |x_c(t, j)| \leq \varepsilon$ for all (t, j) in the domain of the solution; and
2) (global convergence) for each $(x^0, x_c^0, q^0) \in \mathbb{R}^n \times \mathbb{R}^p \times Q$, every maximal solution is complete and the x and x_c components satisfy $\lim_{t+j \to \infty} (|x(t, j)| + |x_c(t, j)|) = 0$. ■

Under some regularity assumptions on $\mathcal{H}_c$, the closed-loop system $\mathcal{H}_{cl}$ can be recast in the framework of Section II, so that Assumption 2.1 and the conditions for existence of solutions in [8] are met. A particular set of such regularity assumptions is the following.

Assumption 3.2: The following conditions hold for $\mathcal{H}_c$:

1) The set Q is a finite subset of the integers.
2) The sets C_c and D_c are closed.
3) $C_c \cup D_c = \mathbb{R}^n \times \mathbb{R}^p \times Q$.
4) The functions $\kappa_c : \mathbb{R}^n \times \mathbb{R}^p \times Q \to \mathbb{R}^m$, $f_c : \mathbb{R}^n \times \mathbb{R}^p \times Q \to \mathbb{R}^p$, and $g_c : \mathbb{R}^n \times \mathbb{R}^p \times Q \to \mathbb{R}^p$ are such that, for each $q \in Q$, the functions $\kappa_c(\cdot, \cdot, q)$, $f_c(\cdot, \cdot, q)$, and $g_c(\cdot, \cdot, q)$ are continuous.
5) The set-valued mapping $Q_c : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \rightrightarrows Q$ is outer semicontinuous and for each $(x, x_c, q) \in D_c$, $Q_c(x, x_c, q)$ is nonempty. ■

B. Vulnerability to Measurement Noise

In this section we discuss the behavior of hybrid systems $\mathcal{H}$ under the influence of measurement noise. The general form of a hybrid system with measurement noise is

$$\begin{array}{lll} \dot{\xi} & \in & F(\xi, e) \quad \xi + m(e) \in C \\ \xi^+ & \in & G(\xi, e) \quad \xi + m(e) \in D\ . \end{array} \tag{3}$$

The signal e and the solution ξ have the same domains. (Note that given a hybrid time domain S and an exogenous signal $e(t)$, we can define, with some abuse of notation, $e(t, j) := e(t)$ for each $(t, j) \in S$.) The function $m : \mathbb{R}^n \to \mathbb{R}^n$ "selects" which components of ξ are corrupted with noise, e.g. state of the plant or state of the controller.

A hybrid arc ξ and a measurement noise signal e are a *solution pair* (ξ, e) *to the hybrid system* $\mathcal{H}$ if $\operatorname{dom} \xi = \operatorname{dom} e$, $\xi(0, 0) + m(e(0, 0)) \in C \cup D$, and

(S1e) For all $j \in \mathbb{N}$ and a.a. t such that $(t, j) \in \operatorname{dom} \xi$,

$$\xi(t, j) + m(e(t, j)) \in C, \quad \dot{\xi}(t, j) \in F(\xi(t, j), e(t, j)).$$

(S2e) For all $(t, j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$,

$$\xi(t, j) + m(e(t, j)) \in D, \quad \xi(t, j+1) \in G(\xi(t, j), e(t, j)).$$

However, in the presence of measurement noise there is no guarantee that solutions exist, even if no such issues arise for the nominal system. Indeed, being the state $\xi = (x, q)$, when there exists a point (x^*, q^*) and sequences x^i and x^k both approaching x^* such that $(x^i, q^*) \notin C$ and $(x^k, q^*) \notin D$ then solutions can fail to exist even for arbitrarily small measurement noise.

This is illustrated as follows. For each $q \in Q := \{-1, 1\}$, let $c_q \in \mathbb{R}_{\geq 0}$, and $C_q := \{(x, q) \in \mathbb{R}^n \times Q \;\mid q|x| \leq qc_q\}$. Define the system

$$\begin{array}{rcll} \dot{x} & = & f(x, \kappa_c(x, q)) & (x, q) \in C := \cup_{q \in Q} C_q \\ q^+ & = & -q & (x, q) \in D := \overline{(\mathbb{R}^n \times Q) \setminus C} \; . \end{array}$$

(The structure of this system is like what is used in hysteresis switching between local and global controllers: $u = \kappa_c(x, 1)$ is to be used in C_1 ($|x| \leq c_1$), while $u = \kappa_c(x, -1)$ is to be used in C_{-1} ($|x| \geq c_{-1}$). For details on uniting local and global controllers see [14].) Suppose that for a given $q \in Q$, the system is initialized at $|x| = c_q$. Let v^k and w^k be a sequences converging to zero satisfying $q|x+v^k| > qc_q$ and , $q|x+w^k| < qc_q$. Then $x+v^k \notin C$ for all k and $x+w^k \notin D$ for all k. If we let $e(0) = w_k$ and $e(t) = v_k$ for all $t > 0$ (such signal can be made arbitrarily small), the system (3), with m being an identity function, has no solutions from x. Indeed, such a solution could not jump from (x, q), as $(x(0, 0) + e(0), q(0, 0)) \notin D$. Similarly, it could not flow from x, since, for small t, $(x(t, 0) + e(t), q(t, 0)) \notin C$.

This situation can be remedied, at least for small measurement noise, if C and D always "overlap". By this we mean that for any $(x, q) \in O$, either $(x + e, q) \in C$ for all small e or $(x + e, q) \in D$ for all small e. This can be achieved in the example above by inflating the set D but with the inflation small enough so that, still, starting from the condition $|x| \leq c_{-1}$ the jump set is not reached.

In general, there always exist inflations of C and D that preserve semiglobal practical asymptotic stability, however they only guarantee existence of solutions for small measurement noise. Alternatively, solutions are guaranteed to exist locally for any locally bounded measurement noise if the measurement noise does not appear in the flow and jump sets of the hybrid system. That can be achieved by filtering the measurements, which will be described in the next section.

IV. Robustness to noise via filtered measurements

In this section, we consider the nonlinear system given by (1) with the addition of the measured output

$$y = x + e,$$

where e is an exogenous bounded signal. Our goal is to augment the controller $\mathcal{H}_c$ that nominally stabilizes the system so that the resulting closed-loop system preserves semiglobal practical stability for small measurement noise.

To overcome the existence of solutions problem, we pass y through a linear filter with matrices (A_f, B_f, L_f) and parameter ε. The filter is designed to be asymptotically stable and at jumps, its state is reset to the current value of y:

Assumption 4.1: The matrices (A_f, B_f, L_f) are such that A_f is Hurwitz and $-L_f A_f^{-1} B_f = I$.

Then, the output of the filter replaces the state x in the feedback law κ_c and in the flow and jump conditions, thereby guaranteeing local existence of solutions. Denoting the state of the filter by $\zeta \in \mathbb{R}^r$, we can write the resulting closed loop, denoted $\mathcal{H}^\varepsilon_{cl}$, as follows

$$\left.\begin{array}{rcl} \dot{x} & = & f(x, \kappa_c(L_f\zeta, x_c, q)) \\ \dot{x}_c & = & f_c(L_f\zeta, x_c, q) \\ \dot{q} & = & 0 \\ \varepsilon\dot{\zeta} & = & A_f\zeta + B_f(x+e) \end{array}\right\} \quad \begin{array}{l} \text{when} \\ (L_f\zeta, x_c, q) \in C_c \end{array}$$

$$\left.\begin{array}{rcl} x^+ & = & x \\ x_c^+ & = & g_c(L_f\zeta, x_c, q) \\ q^+ & \in & Q_c(L_f\zeta, x_c, q) \\ \zeta^+ & = & -A_f^{-1}B_f(x+e) \end{array}\right\} \quad \begin{array}{l} \text{when} \\ (L_f\zeta, x_c, q) \in D_c \; . \end{array}$$

It can be shown that for every compact set of initial conditions and positive number ν, the solutions to the family of hybrid systems $\mathcal{H}^\varepsilon_{cl}$ with small enough parameter ε satisfy a $\mathcal{KLL}$ bound with an offset given by ν.

Theorem 4.2: Under Assumptions 3.1, 3.2, and 4.1, there exists $\beta \in \mathcal{KLL}$ and for each $\mu > 0$ and $\nu > 0$, there exist $\varepsilon^* > 0$ and $\delta > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and $|e|_\infty \leq \delta$, the solutions to $\mathcal{H}^\varepsilon_{cl}$ satisfy

$$|x(t, j)| + |x_c(t, j)| \leq \beta(|x^0| + |x_c^0|, t, j) + \nu$$

for all initial conditions $(x^0, x_c^0, q^0, \zeta^0) \in \mathbb{R}^n \times \mathbb{R}^m \times Q \times \mathbb{R}^r$ with $|x^0| + |x_c^0| \leq \mu$ and $|\zeta^0| \leq \mu$.

V. Robustness to certain singular perturbations

A. Fast actuator and sensor dynamics

We now analyze the robustness of the closed-loop $\mathcal{H}_{cl}$ defined in Section III-A to unmodeled sensor/actuator dynamics. Figure 1 shows the closed-loop $\mathcal{H}_{cl}$ with two additional blocks: a model for the sensor and a model for the actuator. Such blocks can be modeled as stable filters with parameters that depend on the characteristics of the sensors and actuators used in the loop. To simplify the controller design procedure, these dynamics are usually not included in the model of the nonlinear system (1) for which a hybrid controller $\mathcal{H}_c$ is to be designed.

We denote the state of the filter that models the sensor dynamics by $\zeta_s \in \mathbb{R}^{r_s}$ with matrices (A_s, B_s, L_s) and parameter ε, and the state of the filter that models the actuator dynamics by $\zeta_a \in \mathbb{R}^{r_a}$ with matrices (A_a, B_a, L_a) and parameter ε. For the case of study, it is reasonable to assume that the added sensors and actuators are stable with unity DC gain:

Assumption 5.1: The matrices (A_s, B_s, L_s) and (A_a, B_a, L_a) are such that A_s and A_a are Hurwitz, $-L_s A_s^{-1} B_s = I$ and $-L_a A_a^{-1} B_a = I$.

Since the filters are not internal components of the hybrid controller, their state cannot be reset at jumps (cf. the filter in Section IV). We employ temporal regularization with parameter $\overline{\tau}$ to eliminate Zeno solutions. Augmenting $\mathcal{H}_{cl}$ by adding filters and by the temporal regularization leads to a family $\mathcal{H}^\varepsilon_{cl}$ given as follows

$$\left.\begin{array}{rcl} \dot{x} & = & f(x, L_a\zeta_a) \\ \dot{x}_c & = & f_c(L_s\zeta_s, x_c, q) \\ \dot{q} & = & 0 \\ \dot{\tau} & = & -\tau + \tau^* \\ \varepsilon\dot{\zeta}_s & = & A_s\zeta_s + B_s(x+e) \\ \varepsilon\dot{\zeta}_a & = & A_a\zeta_a + B_a\kappa_c(L_s\zeta_s, x_c, q) \end{array}\right\} \begin{array}{l} \text{when} \\ (L_s\zeta_s, x_c, q) \in C_c \\ \text{or } \tau \leq \overline{\tau} \end{array}$$

$$\left.\begin{array}{rcl} x^+ &=& x \\ x_c^+ &=& g_c(L_s\zeta_s, x_c, q) \\ q^+ &\in& Q_c(L_s\zeta_s, x_c, q) \\ \zeta_s^+ &=& \zeta_s \\ \zeta_a^+ &=& \zeta_a \\ \tau^+ &=& 0 \end{array}\right\} \begin{array}{l}\text{when} \\ (L_s\zeta_s, x_c, q) \in D_c \\ \text{and } \tau \geq \overline{\tau}\end{array}$$

where τ^* is a constant satisfying $\tau^* > \overline{\tau}$. The following result states that for fast enough sensors and actuators and small enough temporal regularization parameter, the closed loop has the semiglobal practical asymptotic stability property.

Theorem 5.2: Under Assumptions 3.1, 4.1, and 5.1 there exists $\beta \in \mathcal{KLL}$, for each $\mu > 0$ and $\nu > 0$ there exist $\overline{\tau}^* > 0$ and $\delta > 0$, and for each $\overline{\tau} \in (0, \overline{\tau}^*]$ there exist $\varepsilon^* > 0$ such that, for each $\overline{\tau} \in (0, \overline{\tau}^*]$, each $\varepsilon \in (0, \varepsilon^*(\overline{\tau})]$, and each $|e|_\infty \leq \delta$, the solutions to $\mathcal{H}_{cl}^\varepsilon$ satisfy

$$|x(t,j)| + |x_c(t,j)| \leq \beta(|x^0| + |x_c^0|, t, j) + \nu$$

for all initial conditions $(x^0, x_c^0, q^0, \zeta_s^0, \zeta_a^0, \tau^0)$ with $|x^0| + |x_c^0| \leq \mu$, $|\zeta_s^0| \leq \mu$, and $|\zeta_a^0| \leq \mu$.

B. Fast sensor dynamics and control smoothing

The control law generated by the hybrid controller $\mathcal{H}_c$ given in equation (2) can have jumps in its value when q switches. In many applications it is not possible for the actuator to switch between control laws instantaneously; moreover, especially when the control law $\kappa_c(\cdot, \cdot, q)$ is continuous for each $q \in Q$, it is desired to have a smooth transition between them when q change. We now add such operation to the nominal closed loop and we show that the robustness properties are practically and semiglobally preserved.

Figure 2 shows the final closed loop that we denote as $\mathcal{H}_{cl}^\varepsilon$. We model the smoothing control block by filtering the variable q with a linear filter with matrices (A_u, B_u, L_u) and parameter ε, and then computing (possibly by the actuator) the control law

$$\alpha(x, x_c, L_u\zeta_u) = \sum_{q \in Q} \lambda_q(L_u\zeta_u)\kappa_c(x, x_c, q)$$

where for each $q \in Q$, $\lambda_q : \mathbb{R} \to [0, 1]$ is continuous and $\lambda_q(q) = 1$. Since the filter for q may not be part of the controller, its state cannot be reset at jumps. We also include the sensor dynamics in the loop as in Section V-A.

The closed loop $\mathcal{H}_{cl}^\varepsilon$ can be written as

$$\left.\begin{array}{rcl} \dot{x} &=& f\left(x, \alpha(x, x_c, L_u\zeta_u)\right) \\ \dot{x}_c &=& f_c(L_s\zeta_s, x_c, q) \\ \dot{q} &=& 0 \\ \dot{\tau} &=& -\tau + \tau^* \\ \varepsilon\dot{\zeta}_s &=& A_s\zeta_s + B_s(x + e) \\ \varepsilon\dot{\zeta}_u &=& A_u\zeta_u + B_u q \end{array}\right\} \begin{array}{l}\text{when} \\ (L_s\zeta_s, x_c, q) \in C_c \\ \text{or } \tau \leq \overline{\tau}\end{array}$$

$$\left.\begin{array}{rcl} x^+ &=& x \\ x_c^+ &=& g_c(L_s\zeta_s, x_c, q) \\ q^+ &\in& Q_c(L_s\zeta_s, x_c, q) \\ \zeta_u^+ &=& \zeta_u \\ \zeta_s^+ &=& \zeta_s \\ \tau^+ &=& 0 \end{array}\right\} \begin{array}{l}\text{when} \\ (L_s\zeta_s, x_c, q) \in D_c \\ \text{and} \\ \tau \geq \overline{\tau}\ .\end{array}$$

The result for this system is the same as the result in Theorem 5.2. The specific statement is thus omitted. In Section VII, we add fast sensor dynamics and control smoothing to the problem of swinging up a pendulum on a cart.

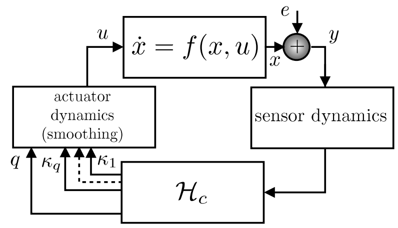

Fig. 2. Closed-loop system with fast sensor dynamics and control smoothing.

C. When there are no instantaneous Zeno solutions

We remark that if there are no instantaneous Zeno solutions to $\mathcal{H}_{cl}$ then $\mathcal{H}_{cl}$ is uniformly non-Zeno on compact sets, i.e., for each compact set K there exist $T > 0$ and a positive integer J such that $|t' - t| \leq T$ implies $|j' - j| \leq J$ for all solutions ϕ to $\mathcal{H}_{cl}$ and all sets S such that $\phi(\operatorname{dom}\phi \cap S) \subset K$, $(t, j) \in \operatorname{dom}\phi \cap S$, and $(t', j') \in \operatorname{dom}\phi \cap S$. (See [6] or [8].) In fact, if $\mathcal{H}_{cl}$ has no instantaneous Zeno solutions then $\mathcal{H}_{cl}^\varepsilon$ has no instantaneous Zeno solutions and thus is uniformly non-Zeno on compact sets. In this case, the singular perturbation arguments used above go through without the temporal regularization. The details are omitted for now. This observation will be used for the simulations in Section VII.

VI. A CLASS OF STABILIZING HYBRID CONTROLLERS

Up to this point we have assumed the existence of a hybrid controller $\mathcal{H}_c$ for system (1) that yields global stability of $\{0\} \times \{0\} \times Q$ for the closed-loop $\mathcal{H}_{cl}$. This section discusses a particular construction for $\mathcal{H}_c$, a static hybrid controller.

We start by considering two families of sets, $\{\Omega_q\}_{q \in Q}$ and $\{C_q\}_{q \in Q}$, and a family of feedback functions $\{\kappa_c(\cdot, q)\}_{q \in Q}$. These sets and the control laws are designed so that when 1) a trajectory hits the boundary of the current C_q set and may be able to flow with larger q, it does not belong to Ω_α with α smaller than the current mode; 2) trajectories that never switch converge to the origin; 3) the trajectories do not go unbounded; and 4) every control law corresponding to a q such that $0 \in C_q$ renders the origin of the closed loop stable (these ideas are similar to the ones in [16]). More precisely:

Assumption 6.1: The set Q is finite, $\cup_{q \in Q}\Omega_q = \mathbb{R}^n$ and, for each $q \in Q$,

1) Ω_q and C_q are closed and satisfy $\Omega_q \subset C_q$,
2) The map $\kappa_q : \mathbb{R}^n \to \mathbb{R}^m$, defined by $x \mapsto \kappa_c(x, q)$ is continuous.
3) The trajectories of $\dot{x} = f(x, \kappa_c(x, q))$ starting in C_q have the following properties:
   a) If $x(0) \in \Omega_q$ and $x(t) \in \partial C_q \setminus \{0\}$ for some $t \geq 0$ then $x(t) \notin \Omega_\alpha$ for any $\alpha < q$.
   b) If $x(t) \in C_q$ for all t in its domain and x is maximal, then x is complete and $\lim_{t \to \infty} x(t) = 0$.
   c) There does not exists an unbounded trajectory.
   d) For each $\varepsilon_q > 0$ there exists $\delta_q > 0$ such that $|x(0)| \leq \delta_q$ implies $|x(t)| \leq \varepsilon_q$ for all t where $x(\cdot)$ is defined. (Notice that if $0 \notin C_q$ then, since C_q is closed, there is nothing to check.)

For each $q \in Q$, we define $D_q := \overline{\mathbb{R}^n \setminus C_q}$. Then we define $D_c := \{(x, q) \in \mathbb{R}^n \times Q \mid x \in D_q\}$ and $C_c :=$

$\{(x, q) \in \mathbb{R}^n \times Q \mid x \in C_q\}$ and note that $C_c \cup D_c = \mathbb{R}^n \times Q$. Then the closed loop system $\mathcal{H}_{cl}$ obtained from connecting this particular controller $\mathcal{H}_c$ with the nonlinear system (1) is given by

$$\begin{array}{rcll} \dot{x} & = & f(x, \kappa_c(x, q)) & (x, q) \in C_c \\ q^+ & \in & Q_c(x, q) & (x, q) \in D_c \end{array} \qquad (4)$$

where $Q_c(x, q) :=$

$$\left\{ \begin{array}{ll} \{\alpha \in Q \mid \alpha > q \ , x \in \Omega_\alpha\} & \text{if } x \in D_q \setminus (\cup_{\alpha<q} \Omega_\alpha) \\ \{\alpha \in Q \mid x \in \Omega_\alpha\} & \text{if } x \in D_q \cap (\cup_{\alpha<q} \Omega_\alpha) \end{array} \right. .$$

It follows from the fact that $\cup_{\alpha<q}\Omega_\alpha$ is closed that Q_c is outer semicontinuous. It also follows that Q_c is nonempty on D_c. Indeed, if $x \in D_q$ then, since the sets Ω_α are closed and $\Omega_q \subset C_q$ and $D_q = \overline{\mathbb{R}^n \setminus C_q}$, we have $x \in \cup_{\alpha \neq q} \Omega_\alpha$. Thus $x \in D_q \setminus (\cup_{\alpha<q}\Omega_\alpha)$ implies $x \in \cup_{\alpha>q}\Omega_\alpha$.

Theorem 6.2: Under Assumption 6.1, the closed-loop system (4) has the set $\{0\} \times Q$ globally asymptotically stable.

VII. Example: Inverted pendulum on a cart

We consider the problem of swinging a pendulum on a cart to the upright position by acting on the cart and simultaneously stabilizing the cart to the neutral position. Swing up of the pendulum has been considered frequently in the literature. See, for example, [2], [20], [4]. The inverted pendulum on a cart system, after a preliminary state feedback of the form[2] $\tilde{u} = \psi(x) + \phi(x)u$, is given by (see e.g. [18])

$$\dot{x}_1 = x_2, \ \dot{x}_2 = \sin(x_1) + \cos(x_1)u, \ \dot{x}_3 = x_4, \ \dot{x}_4 = u \quad (5)$$

where x_1 represents the angle of the pendulum from the up vertical position, x_2 is the angular velocity, x_3 is the cart position and x_4 is the cart velocity. Note that for simplicity, we have normalized the constants.

We consider a hybrid swing-up strategy that chooses the appropriate feedback control law depending on the location of the pendulum. Let W be the energy of the pendulum, $W(x) = \frac{1}{2}x_2^2 + \cos(x_1)$, and let c_1, c_2 be constants that are sufficiently close to but larger than $\min_{x\in\mathbb{R}^4} W(x)$ and satisfy $c_1 > c_2$. Take U_{3a} and U_{3b}, $U_{3a} \subset U_{3b}$, to be closed neighborhoods of the origin in $\mathbb{R}^2$ such that for the system $\dot{x} = f(x, u)$, there exists a state feedback law $\tilde{\kappa}$ that renders the origin (in $\mathbb{R}^4$) locally asymptotically stable with basin of attraction containing $U_{3b} \times \mathbb{R}^2$ and such that solutions starting in $U_{3a} \times \mathbb{R}^2$ do not reach the boundary of $U_{3b} \times \mathbb{R}^2$. Such a construction is given in [18] for example. Then, for each $q \in Q := \{1, 2, 3\}$, we define sets Ω_q and C_q ($\Omega_q \subset C_q$) as follows

$$\begin{array}{rcl} \Omega_1 & = & C_1 = \left\{x \in \mathbb{R}^4 \mid W(x) \leq c_1\right\}, \\ \Omega_3 & = & U_{3a} \times \mathbb{R}^2, \quad C_3 = U_{3b} \times \mathbb{R}^2, \\ \Omega_2 & = & \left\{x \in \overline{\mathbb{R}^2 \setminus U_{3a}} \times \mathbb{R}^2 \mid W(x) \geq c_1\right\}, \\ C_2 & = & \left\{x \in \overline{\mathbb{R}^2 \setminus U_{3a}} \times \mathbb{R}^2 \mid W(x) \geq c_2\right\}. \end{array}$$

When the pendulum is in the region q, the control law is given by $\kappa_c(\cdot, q)$, $q \in Q$. The control law for C_1 is given by κ_1 which drives the system away from the resting condition. We simply choose $\kappa_1 \equiv 1$. In C_2, the control law should inject enough energy into the system so that U_{3a} is reached. For that purpose, we let κ_2 be a feedback law that stabilizes W to the value one, e.g. $\kappa_2(x) = -x_2 \cos(x_1)(W(x) - 1)$. We design the control law for region $q = 3$ so that it satisfies the properties of $\tilde{\kappa}$ above. We construct it by feedback linearizing the system (5), computing the basin of attraction with a quadratic Lyapunov function, and extending the linearized controller so that the cart position and velocity are stabilized to the origin:

$$\begin{array}{rcl} \kappa_3(x) & = & \dfrac{\sin(x_1) + x_1 + x_2}{\cos(x_1)} + \mathrm{sat}_{\lambda_1}\{z_4 + \mathrm{sat}_{\frac{\lambda_1}{2}}\{z_3 + z_4\}\} \\ U_{3a} & = & \left\{ x \in \mathbb{R}^4 \;\middle|\; \frac{1}{2}(x_1^2 + x_2^2) \leq c_3 \right\} \\ U_{3b} & = & \left\{ x \in \mathbb{R}^4 \;\middle|\; \frac{1}{2}(x_1^2 + x_2^2) \leq c_4 \right\}, \end{array}$$

where $z_4 = x_4 + 2x_2 + x_1$, $z_3 = x_3 + x_4 + x_2 + 2x_1$, and $c_4 > c_3 > 0$. Finally, define $\kappa_c(\cdot, q) = \kappa_q(\cdot)$.

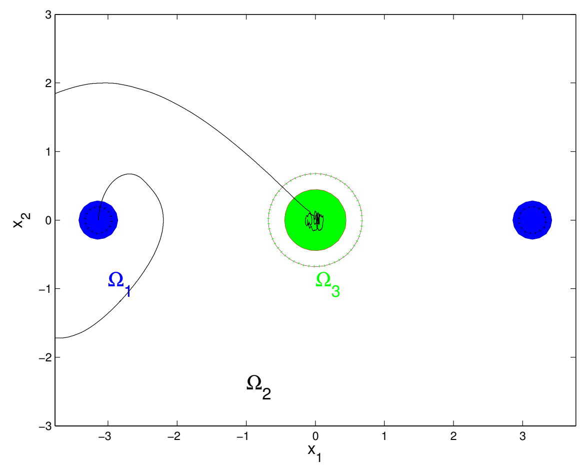

Fig. 3. Solution with normally distributed measurement noise with variance $\sigma = 0.1$ starts in Ω_1, the mode switches when the solution hits the boundary of C_1 and C_2 and then stays in Ω_3. The noise is not able to keep the system away from a neighborhood of the upright condition.

Note that this construction meets the specifications of the controller proposed in Section VI. Moreover, every solution to the closed loop defined by the inverted pendulum and the hybrid controller described above is non-Zeno. Then, temporal regularization is not required for the closed-loop with fast sensor dynamics and control smoothing.

We implement the closed-loop system including fast sensor dynamics and control smoothing as discussed in Section V-B. Since there are three different modes, the control smoothing is modeled as

$$u = u_s(x, \zeta_u) := \sum_{q=1}^{3} \lambda_q(L_u\zeta_u)u_q(x)$$

where the selection functions $\lambda_q : \mathbb{R} \to [0, 1]$, for each $q \in Q$, are continuous and $\lambda_q(q) = 1$.

[2]This state feedback is also subject to measurement noise, but the effect of measurement noise at this location is as in standard ordinary differential equations. We will ignore this subtlety for simplicity.

Finally, we implement the closed loop in Simulink with

$$A_u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix},\ B_u = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix},\ L_u = [1 \quad 0 \quad 0]$$

$$\varepsilon = 0.01,\ A_s = -I,\ B_s = I,\ L_s = I,$$

$$c_1 = -0.96,\ c_2 = -0.98,\ c_3 = 0.1,\ c_4 = 0.23,\ \lambda_1 = 0.5.$$

Figure 3 shows a closed-loop solution in the (x_1, x_2) plane starting at $x^0 = [-\pi\ 0\ 0\ 0]^T$, $q^0 = 1$, $\zeta_u^0 = [0\ 0\ 0]^T$, and with normally distributed noise on each measurement with $\sigma = 0.1$. In the same figure, we also plotted the sets Ω_q in solid and the sets C_q with dashes lines.

To highlight the robustness property to measurement noise, we increased the magnitude of the noise by setting $\sigma = 1$. The results are shown in Figure 4 and 5. When the noise is able to kick the solution, for example, outside the set C_3, the controller reaction is to switch the mode from $q = 3$ to $q = 2$. Then, it drives the solution back to Ω_3 by switching the mode back to $q = 3$. The time between switches in Figure 5 shows that the controller reacts relatively fast.

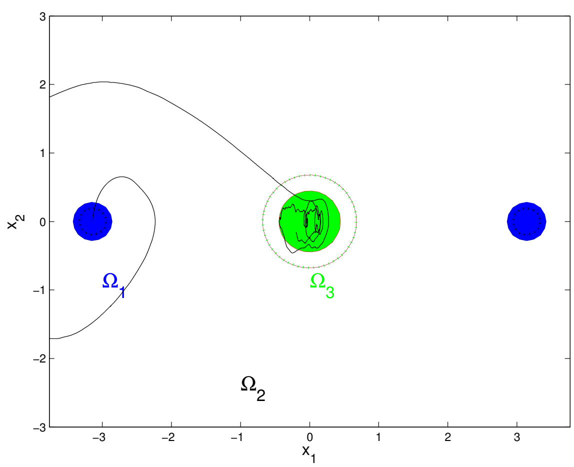

Fig. 4. Solution starting with the same initial conditions as before but with ten times larger measurement noise. The noise is able to drive the solution outside C_3 and therefore, perturb the pendulum from the straight-up position, but the hybrid controller reacts and steers it back in.

VIII. Conclusion

Hybrid systems can have very poor properties with respect to small perturbations that enter the flow and/or jump equations. In particular, measurement noise can have a dramatic negative effect on the very existence of solutions. These problems can be alleviated to a large degree by introducing measurement filters and exploiting the robustness to perturbations of hybrid control systems that satisfy certain basic conditions. Here, we have pointed out that hybrid control systems can withstand filtered measurements, a class of singular perturbation, and the continuous-time implementation of the control signal. These behaviors have been illustrated on the problem of swinging up a pendulum on a cart. Simulations files are available at http://www.ccec.ece.ucsb.edu/∼rsanfelice/ .

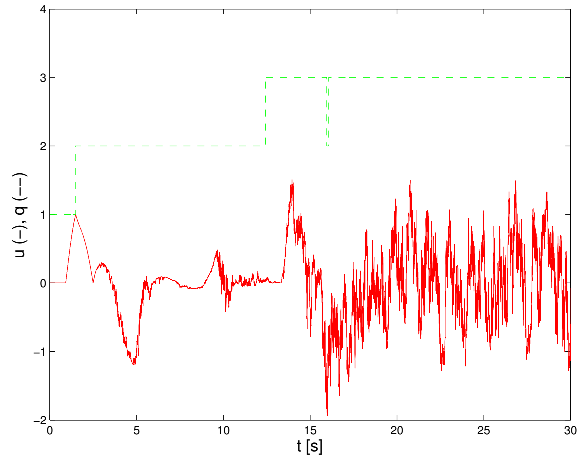

Fig. 5. Control law and discrete mode for large noise. The mode changes rapidly between $q = 3$ and $q = 2$ when the noise perturbs the mode.

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