A Hybrid Controller for Global Uniform Exponential Stabilization of Linear Systems with Singular Input Constraints

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Abstract— In this paper, we devise a controller that achieves global uniform exponential stabilization of linear systems while avoiding singular input constraints, that is, the proposed controller never crosses a given input value g. We show that it is possible to solve this problem using two linear controllers and an appropriate switching logic, which leads to a hybrid controller.

I. INTRODUCTION

The problem of stabilizing linear systems subject to input constraints is very well documented in [12] and [5]. These works discuss controller design strategies for linear plants with magnitude constraints on the input and analyze their stability properties. Due to the class of input constraints considered in the aforementioned works, it is natural that exponential stability is mostly a local result. In this paper, we are interested in milder input constraints that allow for global exponential stabilization. In particular, given $g \in \mathbb{R}^m \backslash \{0\}$, we address the problem of globally uniformly exponentially stabilizing the origin of a linear system of the form

$$\dot{x} = Ax + Bu(x), \tag{1}$$

subject to the constraint $u(x(t)) \neq g$ for each solution $x(t)$ to (1) and for all $t \geq 0$. Moreover, if the system has a single input, then we are able to tackle the stronger constraint: $|u(x(t))| \neq g$ for each solution $x(t)$ to (1) and for all $t > 0$.

It must be mentioned that if we were to consider global asymptotic stability or semi-global exponential stability instead of global exponential stability, then it would be possible to avoid $|u(x)| = g$ using continuous-time controllers, under certain conditions. The works [15] and [11] highlight some key obstructions to the global stabilization of linear systems with saturating controllers. In [2], a saturated control law that globally asymptotically stabilizes the double integrator system is provided, but the resulting closed-loop system is

not robust to arbitrarily small disturbances, as shown in [14]. If we were to consider semi-global exponential stability, then we would have to restrict the set of initial conditions or decrease the controller gains. In this situation, the presence of disturbances could potentially compromise the assumptions made during the controller design. In both cases, the lower the value of g, the more limited the controller would be.

To avoid the limitations of continuous controllers mentioned above, we devise a hybrid controller for the global uniform exponential stabilization of (1) while a given input value g is avoided. Its usefulness is best seen in single input systems, but it may also be applied to multi-input systems. The hybrid controller that we propose consists of two linear state-feedback controllers indexed by the switching variable q, namely $u(x, q) = K_q x$, and an appropriate switching logic which guarantees that: 1) global exponential stability is not compromised due to switching, 2) the controller avoids any given input value $g \in \mathbb{R}^m \backslash \{0\}$, and 3) the system is robust to disturbances. Also, since we are considering hybrid systems, we make use of the concept of global uniform exponential stability which is provided in [10]. This concept has been explored thoroughly on the description of solutions to switched systems, namely to characterize the stability of switched linear systems under arbitrary switching (see [8]). More recently, the paper [16] has presented a different concept of exponential stability for hybrid systems that is not used in this paper.

The remainder of the paper is organized as follows. In Section II, we introduce the notation that is used throughout the paper. In Section III, we provide a formal definition of the problem at hand. We present a particular construction of a hybrid controller which solves the problem and in Section V we show how the proposed controller can be used for the global uniform exponential stabilization of the double integrator system. Finally, some concluding remarks can be found in Section VI.

II. PRELIMINARIES & NOTATION

A. Differentiable Functions, Euclidian Spaces & Linear Maps

The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} : t \geq 0\}$. $C^n(M)$ denotes the set of functions $f : M \to \mathbb{R}$ that are continuously differentiable up to order n and, more generally, $C^n(M, N)$ denotes the set of functions $f : M \to N$ that are continuously differentiable up to order n. The symbols $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and integers, respectively. Additionally, we define the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with inner product $\langle x, y \rangle := x^\top y$ for each

$x, y \in \mathbb{R}^n$. This inner product induces a norm on $\mathbb{R}^n$, given by $|x| := \sqrt{\langle x, x \rangle}$. $e_i \in \mathbb{R}^n$ is a vector whose entries are all zeros except for the i-th entry, which is 1. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, we define the distance from a point $x \in \mathbb{R}^n$ to $\mathcal{A}$ as follows: $|x|_{\mathcal{A}} = \inf\limits_{y \in \mathcal{A}} |x - y|$.

If a matrix P is positive definite (semidefinite) we use the notation $P \succ 0$ ($P \succeq 0$). Similarly, if a matrix P is negative definite (semidefinite) we use the notation $P \prec 0$ ($P \preceq 0$).

Given a matrix $A \in \mathbb{R}^{n \times m}$, we define

$$\text{Im}\,(A) := \{y \in \mathbb{R}^n : y = Ax \text{ for some } x \in \mathbb{R}^m\}, \quad (2a)$$

$$\text{Ker}\,(A) := \{x \in \mathbb{R}^m : Ax = 0\}. \quad (2b)$$

Two vectors $v, w \in \mathbb{R}^n$ are orthogonal if $\langle v, w \rangle = 0$ and we use the notation $v \perp w$ to represent this property. Similarly, two sets $X, Y \subset \mathbb{R}^n$ are orthogonal if $x \perp y$ for each $x \in X$ and for each $y \in Y$. Given a set $Z \subset \mathbb{R}^n$, its orthogonal complement is the set $Z^{\perp} := \{x \in \mathbb{R}^n : x \perp z \text{ for each } z \in Z\}$. Given a matrix $A \in \mathbb{R}^{n \times m}$ with rank r, its singular value decomposition is given by

$$A = U \begin{bmatrix} B & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix} W^{\top},$$

where $U \in \mathbb{R}^{n \times n}$ and $W \in \mathbb{R}^{m \times m}$ are orthogonal matrices, $B = \text{diag}(\sigma_1(A), \sigma_2(A), \dots, \sigma_r(A))$, $\sigma_i(A)$ denotes the i-th singular value such that $\sigma_j(A) \geq \sigma_i(A)$ for each $j, i \in \{1, 2, \dots, r\}$ satisfying $j \geq i$. The generalized inverse of A is given by

$$A^{-} = W \begin{bmatrix} B^{-1} & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} U^{\top}.$$

B. Set-Valued Maps

Set-valued maps are at the core of hybrid system models. In this section we present the very important definition of outer semicontinuity.

Definition 1 (Outer semicontinuity). *A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semicontinuous (osc) at $x \in \mathbb{R}^m$ if for every sequence of points x_i convergent to x and any convergent sequence of points $y_i \in M(x_i)$, one has $y \in M(x)$, where $\lim\limits_{i \to \infty} y_i = y$. The mapping M is outer semicontinuous if it is outer semicontinuous at each $x \in \mathbb{R}^m$. Given a set $S \subset \mathbb{R}^m$, $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to S if the set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $M(x)$ for $x \in S$ and $\emptyset$ for $x \notin S$ is outer semicontinuous at each $x \in S$.* $\square$

Definition 2 (Local Boundedness). *A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^n$ is bounded. The mapping M is locally bounded if it is locally bounded at each $x \in \mathbb{R}^m$. Given a set $S \subset \mathbb{R}^m$, the mapping M is locally bounded relative to S if the set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $M(x)$ for $x \in S$ and $\emptyset$ for $x \notin S$ is locally bounded at each $x \in S$.* $\square$

C. Hybrid Systems

A hybrid system $\mathcal{H}$ in $\mathbb{R}^n$ is defined as follows:

$$\mathcal{H} : \begin{cases} \xi \in C & \dot{\xi} \in F(\xi) \\ \xi \in D & \xi^{+} \in G(\xi) \end{cases}, \quad (3)$$

where $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ denotes the jump set and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map. A solution x to $\mathcal{H}$ is parametrized by (t, j), where t denotes ordinary time and j denotes the jump time, and its domain dom $x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain: for each $(T, J) \in$ dom x, dom $x \cap ([0, T] \times \{0, 1, \dots J\})$ can be written in the form $\cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$, where $I_j := [t_j, t_{j+1}]$ and the t_j's define the jump times. A solution ξ to a hybrid system, is said to be: maximal if it cannot be extended by flowing nor jumping and complete if its domain is unbounded.

If a hybrid system satisfies the so-called hybrid basic conditions, then its set of solutions has good structural properties, which, in particular, enabled the development of a robust stability theory for hybrid systems [4]. These conditions are as follows:

Definition 3. *The hybrid basic conditions are:*

(A1) C and D are closed subsets of $\mathbb{R}^n$;

(A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C, $C \subset$ dom F, and $F(x)$ is convex for every $x \in C$;

(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, and $D \subset$ dom G.

The reader is referred to [4] for more details on hybrid systems.

III. PROBLEM SETUP

A formal definition of the problem that we tackle in this paper is given next.

Problem 1. *Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $g \in \mathbb{R}^m \backslash \{0\}$, design a hybrid controller of the form*

$$\underbrace{\dot{q} = 0}_{(x,q) \in C} \quad \underbrace{q^{+} \in G_Q(x, q)}_{(x,q) \in D} \quad (4)$$

with controller state $q \in \mathbb{R}^r$, for some $r \in \mathbb{N}$, jump set $D \subset \mathbb{R}^n \times \mathbb{R}^r$, flow set $C \subset \mathbb{R}^n \times \mathbb{R}^r$, jump map $G_Q : \mathbb{R}^n \times \mathbb{R}^r \rightrightarrows \mathbb{R}^r$, and output $w : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^m$ such that:

- *the maximal solutions to the closed-loop hybrid system resulting from the interconnection of (1) and (4), given by*

$$\begin{array}{ll} \dot{x} &= Ax + Bw(x, q) \\ \dot{q} &= 0 \end{array} \Bigg\} \quad (x, q) \in C \quad (5a)$$

$$\begin{array}{ll} x^{+} &= x \\ q^{+} &\in G_Q(x, q) \end{array} \Bigg\} \quad (x, q) \in D \quad (5b)$$

are complete and satisfy

$$|(x, q)(t, j)|_{\mathcal{A}} \leq k \exp(-\gamma t) |(x, q)(0, 0)|_{\mathcal{A}} \quad (6)$$

for each initial condition $(x,q)(0,0)$ and for some $k, \gamma > 0$, with

$$\mathcal{A} := \{(x,q) \in \mathbb{R}^n \times \mathbb{R}^r : x = 0, \ q \in \mathcal{Q}\}, \quad (7)$$

for some closed set $\mathcal{Q} \subset \mathbb{R}^r$;
- $(x, G_Q(x,q)) \cap D = \emptyset$ *for each* $(x,q) \in D$;
- *the control law w satisfies $w(x,q) \neq g$ for every* $(x,q) \in C$. $\square$

Notice that the hybrid system (5) is, in particular, a switched system and, in this case, we say that if the condition (6) is satisfied then the set (7) is globally uniformly exponentially stable for (5), in the sense that the norm of the plant state has an exponential upper bound, uniformly over any piecewise constant switching sequence (see [10, Equation 5]).

In the sequel, we study the particular controller structure for (5) defined by:

$$C := \{(x,q) \in \mathbb{R}^n \times Q : \max_{h \in Q} V(x,q) - V(x,h) \leq \delta\}, \tag{8a}$$

$$D := \{(x,q) \in \mathbb{R}^n \times Q : \max_{h \in Q} V(x,q) - V(x,h) \geq \delta\}, \tag{8b}$$

for some compact subset Q of $\mathbb{Z}$ and for some $V \in C^1(\mathbb{R}^n \times Q)$, and

$$G_Q(x,q) := \arg\max_{h \in Q} V(x,q) - V(x,h), \tag{9}$$

such that the closed-loop system is given by

$$\begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} = F(x,q) := \begin{pmatrix} Ax + Bw(x,q) \\ 0 \end{pmatrix} \quad \forall (x,q) \in C,$$

$$\begin{pmatrix} x^+ \\ q^+ \end{pmatrix} \in G(x,q) := \begin{pmatrix} x \\ G_Q(x,q) \end{pmatrix} \quad \forall (x,q) \in D. \tag{10}$$

With respect to the original linear system (1), the hybrid system (10), has an additional controller state, in the form of the logic variable q, which indexes a bank of continuous state-feedback controllers $w(x,q)$. This particular controller structure requires a decrease of the function V during jumps by, at least, an amount $\delta > 0$.

Before proceeding into the controller design we assert some important properties of the hybrid system (10) and its solutions. Firstly, we show that the hybrid system (10) satisfies the hybrid basic conditions given in Definition 3. These conditions are of significant importance for hybrid systems, because they ensure that the hybrid system is nominally well-posed and, in particular, it is robust to small measurement noise (see [4] for more details on the properties of nominally well-posed hybrid systems).

Lemma 4. *Given a compact set $Q \subset \mathbb{Z}$, if $w \in C^0(\mathbb{R}^n \times Q, \mathbb{R}^m)$ and $V \in C^1(\mathbb{R}^n \times Q)$, then:*

1) the map

$$\mu(x,q) := \max_{h \in Q} V(x,q) - V(x,h)$$

is continuous;
2) the hybrid system (10) satisfies the hybrid basic conditions (A1)–(A3).

Proof. Since w is continuous and dom $F = \mathbb{R}^n \times Q$, then F is continuous and, consequently, it satisfies (A2). Since $x^+ = x$ and $q^+ \in Q$ with Q compact, then G is locally bounded relative to D and $D \subset$ dom G. If μ is continuous then the hybrid basic conditions are satisfied for (10), because, in that case, G is outer semicontinuous and C, D are the pre-images of closed sets under a continuous map, which are closed.

To show the continuity of $\mu(x,q) = V(x,q) - \min_{h \in Q} V(x,h)$ it suffices to show the continuity of $\min_{h \in Q} V(x,h)$. Suppose that there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ which converges to $x \in \mathbb{R}^n$. Then, for each $i \in \mathbb{N}$, we define

$$h_i \in \arg\min_{h \in Q} V(x_i, h). \tag{11}$$

It follows from the compactness of Q that there exists a convergent subsequence $\{h_{k(i)}\}_{i \in \mathbb{N}}$ of $\{h_i\}_{i \in \mathbb{N}}$, which converges to some $h \in Q$. From the continuity of V, it follows that $V(x_{k(i)}, h_{k(i)})$ converges to $V(x,h)$. Suppose that there exists $h^\star$ such that $V(x, h^\star) < V(x,h)$, then, by the continuity of V, for i large enough we should have $V(x_i, h^\star) < V(x_i, h_{k(i)})$. However, this is a contradiction since $h_{k(i)} \in \arg\min_{h \in Q} V(x_i, h)$. It follows that $\lim_{i \to \infty} \min_{h \in Q} V(x_i, h) = \min_{h \in Q} V(x, h)$, thus proving continuity. On the other hand, $\arg\min_{h \in Q} V(x,h)$ is outer semicontinuous because for every sequence $\{x_i\}_{i \in \mathbb{N}}$ convergent to $x \in \mathbb{R}^n$ and for each subsequence $h_{k(i)} \in \arg\min_{h \in Q} V(x_i, h)$ convergent to $h \in Q$, it follows that $h \in \arg\min_{h \in Q} V(x,h)$. $\square$

In the next lemma, we show that maximal solutions to (10) are complete. [1]

Lemma 5. *Given a compact set $Q \subset \mathbb{Z}$, if $w \in C^0(\mathbb{R}^n \times Q, \mathbb{R}^m)$, then every maximal solution to the hybrid system (10) is complete.*

Proof. Since w is continuous, it follows from standard existence of solutions (see e.g. [9, Theorem 3.1]) that the viability condition (VC) in [4, Proposition 2.10] is satisfied for every $(x,q) \in C \backslash D$. Since $G(D) \subset C$ then solutions do not satisfy condition (c) in [4, Proposition 2.10]. Suppose that dom (x,q) is bounded for each solution $(x,q)(t,j)$ to the hybrid system, then there exists $J = \sup_j$dom $(x,q) < \infty$, where $\sup_j E = \sup\{j \in \mathbb{N}_0 : \exists t \in \mathbb{R} \geq 0$ such that $(t,j) \in E\}$, for a hybrid time domain E.

In particular, this implies that the solution is allowed to flow for every $t \geq t_J$. However, solutions to continuous-time linear systems do not blow up in finite time, hence $t \to \infty$ which contradicts the assumption that dom (x,q) is bounded. Therefore we conclude that condition (b) in [4, Proposition 2.10] does not occur and, consequently, solutions to (10) are complete. $\square$

The previous results establish important properties of the hybrid system (10), but they do not provide much insight on the design of V and w, other than that the former is continuously differentiable and the latter is continuous.

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[1] If solutions to (10) were not complete, then it would not be possible to achieve global exponential stability of (7). At best, we would be able to prove global *pre*-exponentially stability. We refer the reader to [4] for more information on the concepts of *pre*-stability.

Theorem 6. *Let Q denote a compact subset of $\mathbb{Z}$ such that $0 \in Q$ and consider the set*

$$\mathcal{A} := \{(x, q) \in \mathbb{R}^n \times Q : x = q = 0\} \quad (12)$$

If there exist $\underline{\alpha}, \overline{\alpha}, \gamma > 0$, $V \in C^1(\mathbb{R}^n \times Q)$ and $w \in C^0(\mathbb{R}^n \times Q, \mathbb{R}^m)$ such that

$$\underline{\alpha}|(x,q)|_{\mathcal{A}}^2 \leq V(x,q) \leq \overline{\alpha}|(x,q)|_{\mathcal{A}}^2 \ \forall (x,q) \in C \cup D, \quad (13a)$$

$$\langle \nabla V(x,q), F(x,q) \rangle \leq -\gamma V(x,q) \ \forall (x,q) \in C, \quad (13b)$$

$$V(x',q') \leq V(x,q) \ \forall (x,q) \in D, (x',q') \in G(x,q) \quad (13c)$$

then the condition (6) holds for each solution $(x, q)(t, j)$ to (10).

Proof. It follows from (13b) and from the comparison lemma [3, Lemma C.1] that for each solution $(t, j) \mapsto (x, q)(t, j)$ to (10)

$$V((x,q)(t,j)) \leq \exp(-\gamma t))V((x,q)(0,0)), \quad (14)$$

for every $(t, j) \in \text{dom}\,(x, q)$. From Lemma 4 and Lemma 5, it follows that (10) satisfies the hybrid basic conditions and each solution $(x, q)(t, j)$ is complete and, from (14), it is bounded. Since $G(D) \cap D = \emptyset$, then [13, Lemma 2.7] implies that the time between jumps is uniformly lower bounded. Consequently, if for each solution $(t, j) \mapsto (x, q)(t, j)$ we have that $t + j \to \infty$, then $t \to \infty$.

From (13a), it follows that for each $(t, j) \in \text{dom}\,(x, q)$

$$|(x,q)(t,j)|_{\mathcal{A}} \leq \sqrt{\frac{\overline{\alpha}}{\underline{\alpha}}} \exp\left(-\frac{\gamma t}{2}\right)|(x,q)(0,0)|_{\mathcal{A}}. \quad (15)$$

We conclude that (6) holds for each solution to (10). $\square$

In the next section, we address the details of the controller design.

IV. CONSTRUCTING $w(x, q)$ AND $V(x, q)$

To solve Problem 1, we propose the continuously differentiable function

$$V(x,q) := \alpha q^2 + x^\top P_q x, \quad (16)$$

where $\alpha \in \mathbb{R}$, $q \in Q$, and $P_q \in \mathbb{R}^{n \times n}$ is a symmetric matrix for each $q \in Q$, and the control law

$$w(x,q) = K_q x, \quad (17)$$

where $K_q \in \mathbb{R}^{m \times n}$ for each $q \in Q$. Next, we show that a solution to Problem 1 is achieved for $Q := \{0, 1\}$, that is, using solely two controllers.

It is straightforward to check that $w(x, q) \neq g$ for each $(x, q) \in C$ is equivalent to $w^{-1}(g) \subset \text{int}\,D$ (where w^{-1} denotes the preimage of w), which motivates the following result.

Lemma 7. *Let V and w be given by (16) and (17), respectively, with $Q := \{0, 1\}$ and let[2]*

$$V_\star(K_q, g) := g^\top (K_q^-)^\top (P_0 - P_1) K_q^- g. \quad (18)$$

[2]Recall that A^- denotes the generalized inverse of a matrix A.

Given $g \in \mathbb{R}^m \backslash \{0\}$ for each $q \in Q$, there exist $\alpha \in \mathbb{R}$ and $\delta > 0$ such that

$$V(x,q) - V(x,h) > \delta, \quad (19)$$

for every $(x, q) \in w^{-1}(g)$ and for some $h \in Q$ if

$$\text{Im}\,(P_0 - P_1) = \text{Im}\,(K_q^\top) \ \forall q \in Q \quad (20a)$$

$$V_\star(K_1, g) < V_\star(K_0, g). \quad (20b)$$

Proof. Since w is a continuous function, the pre-image of a closed set is closed, thus $w^{-1}(g)$ is closed. Then, there exists $\delta > 0$ such that (19) holds if and only if for every $(x, q) \in w^{-1}(g)$

$$V(x,q) - V(x, 1 - q) > 0. \quad (21)$$

Using the definitions (16) and (17), we see that (21) is equivalent to

$$\Delta_0(g) := \min\{x^\top (P_0 - P_1)x : K_0 x = g\} > \alpha, \quad (22a)$$

$$\Delta_1(g) := \max\{x^\top (P_0 - P_1)x : K_1 x = g\} < \alpha. \quad (22b)$$

Let us apply the bijective transformation of variables

$$x := K_q^- g + v \quad (23)$$

with $v_q \in \mathbb{R}^n$ to the optimization problem $\Delta_q(g)$ for each $q \in Q$, where K_q^- denotes the generalized inverse of K_q. If $g \in \text{Im}\,(K_q)$ for each $q \in Q$, then it follows from [1, Proposition 6.1.7] that, for each $q \in Q$, any x satisfying (23) for some $v_q \in \text{Ker}\,(K_q)$ also satisfies $K_q x = g$. Hence, substituting (23) into (22), it follows that $\Delta_q(g) = V_\star(K_q, g) + \Delta_q(0)$ for each $q \in Q$. Therefore, we conclude that

$$\Delta_0(g) = \begin{cases} +\infty & \text{if } g \notin \text{Im}\,(K_0) \\ V_\star(K_0, g) + \Delta_0(0) & \text{if } g \in \text{Im}\,(K_0) \end{cases} \quad (24a)$$

$$\Delta_1(g) = \begin{cases} -\infty & \text{if } g \notin \text{Im}\,(K_1) \\ V_\star(K_1, g) + \Delta_1(0) & \text{if } g \in \text{Im}\,(K_1) \end{cases} \quad (24b)$$

If $g \notin \text{Im}\,(K_q)$ for each $q \in Q$, then $w^{-1}(g) = \emptyset$ and, consequently, (19) holds. If $g \in \text{Im}\,(K_q)$ for each $q \in Q$, it follows from (20a) and [1, Theorem 2.4.3] that $\text{Ker}\,(P_0 - P_1) = \text{Ker}\,(K_q)$ for each $q \in Q$, hence $\Delta_0(0) = 0$ and $\Delta_1(0) = 0$. Thus, if $g \in \text{Im}\,(K_1)$ and $g \in \text{Im}\,(K_0)$, the inequalities (22) hold for some $\alpha \in \mathbb{R}$ if (20b) holds. In particular, we may select any α satisfying

$$\alpha \in (V_\star(K_1, g), V_\star(K_0, g)). \quad (25)$$

$\square$

From the set of conditions (20), one notices that the eigenvectors of P_q are partially determined by the controller gains K_q for each $q \in Q$, and vice versa.

Theorem 8. *Consider the system (10) with V and w given by (16) and (17), respectively. Given $g \in \mathbb{R}^m \backslash \{0\}$, each solution to (10) satisfies (6) for (12) and $w(x, q) \neq g$ for all $(x, q) \in C$ if conditions (20), (25) and*

$$(A + BK_q)^\top P_q + P_q(A + BK_q) \prec 0 \ \forall q \in Q, \quad (26a)$$

$$P_q \succ 0 \ \forall q \in Q, \quad (26b)$$

$$\alpha > \delta > 0, \quad (26c)$$

are satisfied.

Proof. By construction of (10), the constraint $w(x, q) \neq g$ for all $(x, q) \in C$ and $V(x^+, q^+) < V(x, q)$ for each $(x, q) \in D$ and $(x^+, q^+) \in G(x, q)$ are satisfied if the conditions (20) are satisfied, as proved in Lemma 7. Therefore, (13c) is satisfied.

The function (16) satisfies

$$\alpha q^2 + \lambda_{\min}(P_q)|x|^2 \leq V(x, q) \leq \alpha q^2 + \lambda_{\max}(P_q)|x|^2, \tag{27}$$

where $\lambda_{\min}(P_q)$ and $\lambda_{\max}(P_q)$ denote the minimum and the maximum eigenvalues of P_q, respectively. Therefore (13a) in Theorem 6 is satisfied with $\overline{\alpha} = \max(\alpha, \lambda_{\max}(P_0), \lambda_{\max}(P_1))$ and $\underline{\alpha} = \min(\alpha, \lambda_{\min}(P_0), \lambda_{\min}(P_1))$. Then, conditions (26b) and (26c) imply that $\underline{\alpha}, \overline{\alpha} > 0$, which, in turn, implies (13a). Condition (13b) follows from (26a) and (26c), as shown next.

The derivative of (16) subject to the flow map F in (3), using (17), is given by

$$\langle \nabla V(x, q), F(x, q) \rangle = x^\top \left((A + BK_q)^\top P_q + P_q(A + BK_q) \right) x. \tag{28}$$

If (26a) holds, there exists $\beta > 0$ such that

$$(A + BK_q)^\top P_q + P_q(A + BK_q) \preceq -\beta I_n, \tag{29}$$

therefore it follows from (28) that

$$\langle \nabla V(x, q), F(x, q) \rangle \leq -\frac{\beta}{\lambda_M} \left(\alpha q^2 + \lambda_M |x|^2 \right) + \frac{\beta}{\lambda_M} \alpha q^2, \tag{30}$$

where $\lambda_M := \max_{q \in Q}(\lambda_{max}(P_q))$. Let $\underline{V} := \min_{(x,1) \in C} V(x, 1)$ and notice that $\underline{V}$ exists and is a finite number greater or equal to $\alpha > 0$ because C is closed and $V(x, 1)$ is a convex function greater or equal than α. We have that $\lambda_M |x|^2 \geq x^\top P_q x$ for each $q \in Q$, $\alpha q^2 \leq \alpha V(x, q)/\underline{V}$ and, consequently,

$$\langle \nabla V(x, q), F(x, q) \rangle \leq -\frac{\beta}{\lambda_M} \left(1 - \frac{\alpha}{\underline{V}} \right) V(x, q). \tag{31}$$

Since $V(x, 1) = \alpha$ if and only if $x = 0$ and since $\delta < \alpha$ by (26c), we have that $V(0, 1) - V(0, 0) = \alpha > \delta$, thus $(0, 1) \in D$. Consequently, we have that $\underline{V} > \alpha$ and (13b) holds with

$$\gamma = \frac{\beta}{\lambda_M} \left(1 - \frac{\alpha}{\underline{V}} \right). \tag{32}$$

The desired result follows from Theorem 6. □

In the following corollary, we show that for the single input system we obtain a stronger result than for general multi-input systems.

Corollary 9. *Consider the hybrid system* (10) *with $B \in \mathbb{R}^n$ and the definitions of V and w given in* (16) *and* (17), *respectively, satisfying conditions* (20) *and* (26), *then $|w(x, q)| \neq g$ for each $(x, q) \in C$ and* (6) *holds for the set* (12).

Proof. In Theorem 8, we prove that $w(x, q) \neq g$ for each $(x, q) \in C$. In order to verify that $|w(x, q)| \neq g$ for each

$(x, q) \in C$ we check that every $(x, q) \in w^{-1}(-g)$ belongs to D. This follows directly from the fact that $V_\star(K, g) = V_\star(K, -g)$ for any $K \in \mathbb{R}^{1 \times n}$, thus condition (20b) holds also for $-g$. □

In the next section, we study the application of these results to the double integrator system.

V. NUMERICAL EXAMPLE – DOUBLE INTEGRATOR

The double integrator system is modeled by (1), with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{33}$$

The application of the controller designed in Section IV is very interesting not only because this system appears very often in dynamic models of mechanical systems, but also because it allows us to showcase some interesting properties of the closed loop hybrid system.

The conditions for global exponential stability and avoidance of the input value g given in (20) and (26) are not convex, in general. However, if we are given a set of controllers K_q such that $A + BK_q$ is asymptotically stable for each $q \in Q$, then we may run an convex optimization program so as to find appropriate matrices P_q for each $q \in Q$, as stated in the next lemma.

Lemma 10. *For the system matrices* (33), *conditions* (20) *and* (26) *hold if there exist $\lambda_0 > \lambda_1 > 1$ and $X = X^\top \succ 0$ such that*

$$K_1 = \lambda_1 K_0, \tag{34a}$$

$$P_0 = \lambda_0 K_0^\top K_0 + X, \tag{34b}$$

$$P_1 = K_1^\top K_1 + X, \tag{34c}$$

$$(A + BK_q)^\top P_q + P_q(A + BK_q) \prec 0 \; \forall q \in Q, \tag{34d}$$

Proof. Condition (34a) implies that $\text{Im}\left(K_0^\top\right) = \text{Im}\left(K_1^\top\right)$ as required by (20a). Then, if (34b) and (34c) hold, we have that

$$P_0 - P_1 = (\lambda_0 - \lambda_1)K_0^\top K_0,$$

thus (20a) holds. Notice that $\text{Im}(K_q) = \mathbb{R}$ for each $q \in Q$, otherwise $K_q = 0$ for each $q \in Q$ and, in this case, condition (34d) would not be verified. Notice that the generalized inverse of K_q is

$$K_q^- = \frac{K_q^\top}{|K_q|^2},$$

for each $q \in Q$, because [1, (6.1.7)–(6.1.10)] hold. Hence, from (18) we have that

$$V_\star(K_q, g) = \frac{g^2}{|K_q|^4}(\lambda_0 - \lambda_1)K_0 K_q^\top, \tag{35}$$

for each $q \in Q$. From the assumption $\lambda_0 > \lambda_1 > 1$ and some straightforward computations, it follows that (20b) holds.

Moreover, since $V_\star(K_q, g) > 0$, it follows from (25) that (26c) holds for some $\alpha > 0$ and some $\delta \in (0, \alpha)$. Condition (26b) holds because $\lambda_0 > \lambda_1 > 1$ and $X \succ 0$. Finally, condition (26a) is the same as (34d). □

Given K_q satisfying (34a) for some $\lambda_1 > 0$ and such that $A + BK_q$ is Hurwitz for each $q \in Q$, the set of

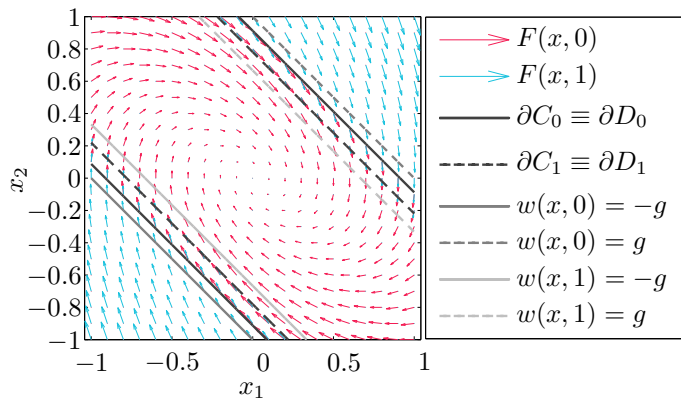Fig. 1. Hybrid phase plane for the double integrator system in the numerical example.

Fig. 2. Simulation results for 10 random initial conditions.

conditions (34) becomes a set of convex constraints, thus amenable to automated feasibility testing by means of a convex optimization problem. For this numerical study we chose,

$$K_0 = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -1.5 & -1.5 \end{bmatrix}. \tag{36}$$

and found the following feasible solution using CVX, a package for specifying and solving convex programs [7], [6]:

$$P_0 = \begin{bmatrix} 15.5910 & 9.9782 \\ 9.9782 & 15.9121 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 15.3410 & 9.7282 \\ 9.7282 & 15.6621 \end{bmatrix}. \tag{37}$$

Also, we chose $\delta \approx 0.03$ and

$$\alpha = \frac{1}{2} \left(V_\star(K_0, g) + V_\star(K_1, g) \right) \approx 0.1806,$$

with $g = 1$. The hybrid phase plane resulting from this design is represented in Fig. 1, which highlights several important properties of the closed-loop hybrid system:

- the sets $w^{-1}(\pm g)$ are straight lines normal to the vectors K_q, as well as the boundaries of the jump sets and flow sets projected onto the x-plane;
- near the origin, the nominal controller $w(x, 0)$ is selected;
- since $\delta > 0$, there exists a region of the state space where both controllers can be selected, depending on the value of q, preventing controller chattering due to noise.

The results of 10 simulations with random initial conditions is given in Fig. 2. In this figure, it is possible to verify that the input jumps over the unwanted values $\pm g$.

VI. CONCLUSION

In this paper, we proposed a hybrid controller that is able to avoid a given input value while globally uniformly asymptotically stabilizing a given linear system. We have shown that if the system has a single input then the controller is able to perform the desired task while avoiding a stronger constraint.
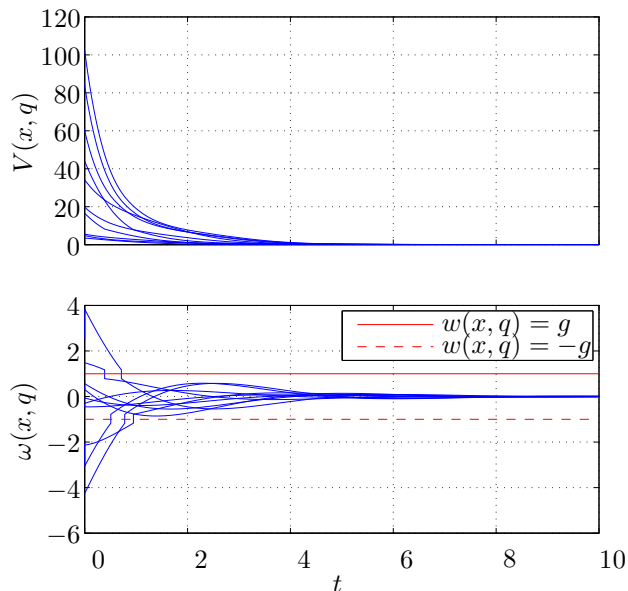
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