Observer-based Control Design for Linear Systems in the Presence of Limited Measurement Streams and Intermittent Input Access*

Francesco Ferrante, Frédéric Gouaisbaut, Ricardo G. Sanfelice and Sophie Tarbouriech

***Abstract*— We consider the problem of stabilizing a linear time-invariant system in the presence of sporadic output measurements and intermittent access to the plant input. The plant is equipped with a zero-order hold device which stores the value of the input between transmissions. We propose an observer-based controller composed consisting of a measurement-triggered observer which experiences jumps in its state whenever a new measure is available, a state-feedback control law computed from the estimated state, and a copy of the zero-order hold device feeding the plant, which jumps whenever the control input is transmitted to the plant. The closed-loop system is modeled as a hybrid system that includes two timers triggering the two different events. The resulting hybrid system is analyzed as the cascade of hybrid systems and its asymptotic stability properties are established through a separation principle. In addition, an efficient design procedure is presented and illustrated in an example.**

I. Introduction

Motivated by their versatility and low cost, the use of embedded devices in control systems has become widely popular in recent years. Unfortunately, digital devices introduce time delays, quantization, sampling, and limited data rate constraints, which can significantly affect the performance and stability of a feedback loop [1], [2]. One of the major drawbacks induced by digital devices pertains to the intermittent availability of resources [3], [4], which makes the (by now classical) periodic sampling paradigm, widely studied in the literature unrealistic; see, e.g., [5], [6], [7].

The relevance of this issue in real-world applications has lead researchers to study the problem of synthesis and analysis of control systems in the presence of aperiodic sampling from different angles. In [8], an approach based on time-varying delay systems is proposed and sufficient conditions for stability based on Krasovskii-Lyapunov functionals were introduced. In [10], [11], an approach based on the Lyapunov theorem in discrete time is proposed to handle asynchronous samplings. More recently, hybrid techniques have been employed to develop protocols that guarantee closed-loop stability under aperiodic sampling and, in particular, characterize the maximum allowable sampling period; see [2], [12], [13] to just list a few. A common assumption in the literature of aperiodic sampling is that the sample and hold operations, of the measured output and of the control input occur synchronously. A notable exception is [9], where the authors, by pursuing a time-delay approach, propose a design for an output feedback controller, guaranteeing an $\mathcal{H}_\infty$ performance, in the presence of aperiodic and asynchronous sampling and holding operations. However, the proposed approach therein is to some extent intrinsically conservative due to the coarseness introduced by modeling the sampling and holding operations as processes with time delay.

This paper pertains to the modeling and design of an observer-based controller to stabilize a linear time-invariant system (LTI) in the presence of sporadic measurements and intermittent access to the plant control input. Building from the hybrid observer in [14], we propose an observer-based controller whose structure takes into account both the limitation afflicting the input channel and the output channel, and does not assume that such events are synchronized. Since the evolution of the considered observer-based controller has variables that exhibit both continuous-time behavior and instantaneous updating (which are not periodic), we provide a hybrid model of such a system that captures all the dynamics induced by the occurrence of the sampling and hold events. By relying on the well-posedness of the hybrid closed-loop system, as defined in [15], we show that a separation principle applies for the design of the proposed scheme. More precisely, inspired by [16], we treat the closed-loop system as the cascade of hybrid systems and then following similar arguments as those for upper triangular nonlinear systems, we establish a global asymptotic stability property of the closed-loop system by exploiting the properties of the individual system defining the cascade. Note that the design of the proposed observer-based controller is not carried out using an emulation approach, namely, it is not designed by building from an asymptotically stable loop and analyzing the effect of sampling and hold as bounded perturbations, cf. [17].

The paper is organized as follows. Section II presents the system under consideration, the stabilization problem we intend to solve, and the hybrid modeling of the proposed controller as well as of the closed-loop system. Section III is dedicated to the main results, which provide a solution to the stabilization problem and design procedures for the proposed observer-based controller. In Section IV, the effectiveness of the approach is illustrated in a numerical example. Due to space limitations, proofs of the results will be published elsewhere.

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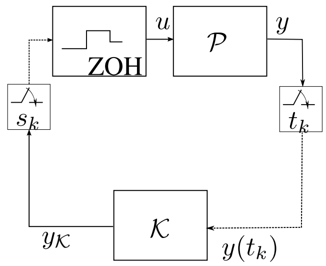


Fig. 1: Continuous-time plant $\mathcal{P}$ controlled by the controller $\mathcal{K}$, which has intermittent access to the input channel and sporadic available measurements of the output y.

Notation: The set $\mathbb{N}_0$ is the set of the positive integers including zero and $\mathbb{R}_{\geq 0}$ represents the set of the nonnegative real scalars. The identity matrix is denoted by $\mathbf{I}$, whereas the null matrix is denoted by $\mathbf{0}$. For a matrix $A \in \mathbb{R}^{n\times m}$, A' denotes the transpose of A, $\|A\|$ denotes the induced 2-norm and $\mathrm{He}(A) = A + A'$. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. In partitioned symmetric matrices, the symbol $\star$ stands for symmetric blocks. The matrix $\mathrm{diag}\{A_1, \dots, A_n\}$ is the block-diagonal matrix having $A_1, \dots, A_n$ as diagonal blocks. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. Given two vectors x, y, we denote $(x, y) = [x' \ y']'$. Given a set X, $\mathrm{co}\{X\}$ represents the convex hull of X, $\delta\mathbb{B}$ is the closed ball with radius δ of appropriate dimension in the Euclidean norm. Given a hybrid system $\mathcal{H} = (C, F, D, G)$ with state in $\mathbb{R}^\ell$ and a set $\mathcal{J} \subset \mathbb{R}^\ell$, $\mathcal{H}|_{\mathcal{J}} = (C \cap \mathcal{J}, F, D \cap \mathcal{J}, G)$ is the restriction of $\mathcal{H}$ to $\mathcal{J}$. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A}$, the distance of x from $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y\in\mathcal{A}} \|x - y\|$. For any function $z\colon \mathbb{R} \to \mathbb{R}^n$, we denote $z(t^+) := \lim_{s\to t^+} z(s)$.

II. Problem Statement

A. System Description

Consider the following continuous-time linear system:

$$\mathcal{P}\colon \begin{cases} \dot{z} = Az + Bu \\ y = Mz \end{cases} \tag{1}$$

where $z \in \mathbb{R}^n$, $y \in \mathbb{R}^q$ and $u \in \mathbb{R}^p$ are, respectively, the state, the measured output, and the input of the system, while A, B and M are constant matrices of appropriate dimensions. Now, let us suppose that both the input channel and the output channel of system (1) are accessible in an intermittent fashion. Specifically, let us assume that the output of system (1) is gathered only at time instances t_k, $k \in \mathbb{N}_0$, not known *a priori* and that the input channel grants its access only at time instances s_k, $k \in \mathbb{N}_0$, not known *a priori*. Specifically, suppose that $\{t_k\}_0^{+\infty}$ and $\{s_k\}_0^{+\infty}$ are two strictly increasing unbounded real sequences of times and assume that there exist four positive real scalars $T_1^{\mathcal{O}} \leq T_2^{\mathcal{O}}, T_1^{\mathcal{U}} \leq T_2^{\mathcal{U}}$, such that

$$T_1^{\mathcal{O}} \leq t_{k+1} - t_k \leq T_2^{\mathcal{O}} \qquad \forall k \in \mathbb{N}_0 \tag{2}$$

$$T_1^{\mathcal{U}} \leq s_{k+1} - s_k \leq T_2^{\mathcal{U}} \qquad \forall k \in \mathbb{N}_0 \tag{3}$$

The problem studied in this paper consists of designing an observer-based controller that asymptotically stabilizes the resulting closed-loop system for any given sequences satisfying (2) and (3) providing respectively measurements of the plant output and input access. To solve this problem, since the information of the output y is available in an impulsive fashion and the input channel is available sporadically, motivated by [14], we design an observer-based controller with jumps in its state $(\hat{z}, \hat{u})$, given by

$$\mathcal{K}\colon \begin{cases} \left.\begin{array}{rcl} \dot{\hat{z}} &=& A\hat{z} + B\hat{u} \\ \dot{\hat{u}} &=& 0 \end{array}\right\} t \notin \{t_k\}_0^{+\infty} \cup \{s_k\}_0^{+\infty} \\ \left.\begin{array}{rcl} \hat{z}(t^+) &=& \hat{z}(t) \\ \hat{u}(t^+) &=& K\hat{z}(t) \end{array}\right\} t \in \{s_k\}_0^{+\infty} \\ \left.\begin{array}{rcl} \hat{z}(t^+) &=& \hat{z}(t) + LM(z(t) - \hat{z}(t)) \\ \hat{u}(t^+) &=& \hat{u}(t) \end{array}\right\} t \in \{t_k\}_0^{+\infty} \\ y_{\mathcal{K}} = K\hat{z} \end{cases} \tag{4}$$

where L and K are two matrices of appropriate dimensions to be designed. The variable $\hat{z}$ represents the estimated state of the plant generated by the observer by means of the measured plant output y, while $\hat{u}$ stores the last value of the control input transmitted to the plant. Indeed, whenever a new sample of the control value is sent to the plant, the controller accordingly updates its internal variable $\hat{u}$ so as to memorize the signal applied to the plant input u. Furthermore, the plant is equipped with a zero-order hold device which stores the value of the last received input between two transmissions, see Fig. 1. Thus, the input applied to the plant is piecewise constant, and specifically, for every integer $k \geq 1$, $u(t) = K\hat{z}(s_k)$ for $t \in [s_k, s_{k+1})$, while $u(t) = u(0)$ for $t \in [0, t_1)$, where $u(0)$ denotes the initial condition of the zero-order hold device. Moreover, notice that if $t \in \{t_k\}_0^\infty \cap \{s_k\}_0^\infty$ then both $\hat{z}$ and $\hat{u}$ need to be updated.

B. Hybrid Modeling

The fact that the closed-loop system experiences jumps when a new measurement is available or when the input channel grants access to the controller suggests that the dynamics of the closed-loop system can be described via a hybrid system. We provide a hybrid model that captures not only the behavior due to a single pair of sequences $\{t_k\}_0^\infty$, $\{s_k\}_0^\infty$, but each possible evolution generated by any sequence satisfying (2) and (3). This is a unique approach that, while leads to nonunique solutions, allows to established a strong result for a family of sequences t_k and s_k. The proposed modeling approach requires to model the time-driven mechanism governing the availability of measurements or of access to the plant input. To this end, following [14], and in a similar fashion as in [12], we add two auxiliary timer variables τ_1 and τ_2 to keep track of the duration of flows and to trigger jumps according to the mechanism in (4). In particular, this modeling procedure leads to a model that can be efficiently represented by the framework for hybrid systems proposed in [15].

To accomplish that, we make τ_1 and τ_2 decrease as ordinary time t increases and, whenever $\tau_1 = 0$ or $\tau_2 = 0$, reset it to any point in $[0, T_2^{\mathcal{U}}]$ or $[0, T_2^{\mathcal{O}}]$ respectively, so as to enforce (2) and (3), respectively. Then, after a jump occurs, the two timers are reset according to the following jump rule:

$$\begin{bmatrix}\tau_1^+\\ \tau_2^+\end{bmatrix} \in \begin{cases} \begin{bmatrix}[T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]\\ \tau_2\end{bmatrix} & \text{if } \tau_1 = 0, \tau_2 \neq 0\\ \begin{bmatrix}\tau_1\\ [T_1^{\mathcal{O}}, T_2^{\mathcal{O}}]\end{bmatrix} & \text{if } \tau_1 \neq 0, \tau_2 = 0\\ \left\{\begin{bmatrix}[T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]\\ \tau_2\end{bmatrix}, \begin{bmatrix}\tau_1\\ [T_1^{\mathcal{O}}, T_2^{\mathcal{O}}]\end{bmatrix}\right\} & \text{if } \tau_1 = \tau_2 = 0 \end{cases}$$

where τ_1^+ and τ_2^+ represent the value, respectively of τ_1 and τ_2, after a jump occurs. To capture this mechanism, we define a hybrid system $\mathcal{H}$ within the framework in [15]. Let $\tilde{x} = (z, u, \tau_1, \hat{z}, \hat{u}, \tau_2)$ be the state vector. Define the flow map as

$$F(x) := \begin{bmatrix} Az + Bu\\ 0\\ -1\\ A\hat{z} + B\hat{u}\\ 0\\ -1 \end{bmatrix}$$

for each $x \in C = \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2^{\mathcal{U}}] \times \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2^{\mathcal{O}}]$, where C is the flow set. For each $x \in D$, define the jump mas as

$$G(x) := \begin{cases} G_1(x) & \text{if } x \in D_1 \setminus D_2\\ G_2(x) & \text{if } x \in D_2 \setminus D_1\\ \{G_1(x), G_2(x)\} & \text{if } x \in D_1 \cap D_2 \end{cases}$$

where for each $x \in D = D_1 \cup D_2$,

$$G_1(x) = \begin{bmatrix} z\\ K\hat{z}\\ [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]\\ \hat{z}\\ K\hat{z}\\ \tau_2 \end{bmatrix}, G_2(x) = \begin{bmatrix} z\\ u\\ \tau_1\\ \hat{z} + LM(z - \hat{z})\\ \hat{u}\\ [T_1^{\mathcal{O}}, T_2^{\mathcal{O}}] \end{bmatrix} \tag{5}$$

$$\begin{aligned} D_1 &= \mathbb{R}^n \times \mathbb{R}^p \times \{0\} \times \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2^{\mathcal{O}}]\\ D_2 &= \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2^{\mathcal{U}}] \times \mathbb{R}^n \times \mathbb{R}^p \times \{0\}. \end{aligned} \tag{6}$$

The set D is the jump set. These objects define a hybrid system $\mathcal{H} = (C, F, D, G)$ that represents the dynamics of the closed-loop system. Notice that this hybrid system is well-posed; see [15] for more details and consequences of well-posedness. Concerning the existence of solutions to system $\mathcal{H}$, by relying on the concept of solution proposed in [15, Definition 2.6], it is straightforward to check that for every initial condition $\phi(0, 0) \in C \cup D$, every maximal solution to ϕ, (that is a solution whose domain is not the truncation of the domain of any other solution) is complete, *i.e.*, $\sup \operatorname{dom} \phi = \infty$. Moreover, the following properties hold:

- For every $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$ and $\phi(t, j) \in D_2 \setminus D_1$, one has $(t, j + 2) \notin \operatorname{dom} \phi$,
- For every $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$ and $\phi(t, j) \in D_1 \setminus D_2$, one has $(t, j + 2) \notin \operatorname{dom} \phi$,
- For every $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$ and $\phi(t, j) \in D_1 \cap D_2$, we have either $\phi(t, j + 1) \in D_1 \setminus D_2$ or $\phi(t, j + 1) \in D_2 \setminus D_1$,

that is, at most two jumps can occur consecutively without flowing. Moreover, for every maximal solution ϕ to $\mathcal{H}$, due to (2) and (3), every $(t, j) \in \operatorname{dom} \phi$ such that $(t, s) \in \operatorname{dom} \phi$, for some $s \in \{j + 1, j + 2\}$, implies that $\big\{[t, t + \min\{T_1^{\mathcal{O}}, T_1^{\mathcal{U}}\}] \times \{s\}\big\} \subset \operatorname{dom} \phi$. Essentially, the domain of the solutions to $\mathcal{H}$ manifests an average dwell-time property, with dwell time $\tau_D = \min\{T_1^{\mathcal{O}}, T_1^{\mathcal{U}}\}$ and offset $N_0 = 2$; see, *e.g.*, [15, Example 2.15]. Such a property imposes a strictly positive uniform lower bound on the length of every flow interval, preventing from the existence of Zeno solutions.

Now, for the purpose of stabilization, consider the following change of coordinates:

$$(z, u, \tau_1, \varepsilon, \tilde{u}, \tau_2) = (z, u, \tau_1, z - \hat{z}, u - \hat{u}, \tau_2) = x_e$$

which leads to the following model of the closed-loop system $\mathcal{H}$:

$$\begin{cases} \dot{x}_e = F_e(\tilde{x}_e) & x_e \in C_e\\ x_e^+ \in G_e(x_e) & x_e \in D_e \end{cases} \tag{7a}$$

where $C_e = C, D_e = D_{1e} \cup D_{2e}, D_{1e} = D_1, D_{2e} = D_2$ and

$$F_e(x_e) = \begin{bmatrix} Az + Bu\\ 0\\ -1\\ A\varepsilon + B\tilde{u}\\ 0\\ -1 \end{bmatrix}, G_{1e}(x_e) = \begin{bmatrix} z\\ K(z - \varepsilon)\\ [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]\\ \varepsilon\\ 0\\ \tau_2 \end{bmatrix}$$

$$G_{2e}(x_e) = \begin{bmatrix} z\\ u\\ \tau_1\\ (I - LM)\varepsilon\\ \tilde{u}\\ [T_1^{\mathcal{O}}, T_2^{\mathcal{O}}] \end{bmatrix}$$

$$G_e(x_e) = \begin{cases} G_{1e}(x_e) & \text{if } x_e \in D_{1e} \setminus D_{2e}\\ G_{2e}(x_e) & \text{if } x_e \in D_{2e} \setminus D_{1e}\\ \{G_{1e}(x_e), G_{2e}(x_e)\} & \text{if } x_e \in D_{1e} \cap D_{2e}. \end{cases}$$

Remark 1: A notable property enforced by timer τ_1 is that for every maximal solution to (7), there exists $(T, J) \in \operatorname{dom} \phi$ satisfying $T + J \leq T_2^{\mathcal{U}}$, such that $\phi(T, J) \in D_{2e}$, which implies that $\tilde{u}(T, J+1) = 0$. Then, since the set $\mathbb{R}^n \times \mathbb{R}^p \times [0, T_2^{\mathcal{U}}] \times \mathbb{R}^n \times \{0\} \times [0, T_2^{\mathcal{O}}]$ is strongly forward invariant for (7) (see [15, Definition 6.25.] for a formal definition of strong forward invariance for hybrid systems), it follows that for every initial condition $\phi(0,0) \in C_e \cup D_e$, $\tilde{u}$ converges to zero in finite hybrid time. Moreover, notice that to make the hybrid system (7) an accurate description of the real time-triggered phenomenon, which governs the update process, τ_1 and τ_2 have to belong to the intervals $[0, T_2^{\mathcal{U}}]$ and $[0, T_2^{\mathcal{O}}]$ respectively, which is a property that is guaranteed by the definition of C_e and D_e.

In this paper, we consider the following notions for a general hybrid system $\mathcal{H}$ with state in $\mathbb{R}^\ell$.

Definition 1: ([15, Definition 7.1.]) Let $\mathcal{A} \subset \mathbb{R}^\ell$ be compact. The set $\mathcal{A}$ is

- stable for $\mathcal{H}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that every solution to $\mathcal{H}$ with $|\phi(0,0)|_\mathcal{A} \leq \delta$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \epsilon$ for all $(t,j) \in \operatorname{dom}\phi$;
- locally attractive for $\mathcal{H}$ if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_\mathcal{A} \leq \mu$ is complete and satisfies $\lim_{t+j\to+\infty} |\phi(t,j)|_\mathcal{A} = 0$;
- locally asymptotically stable (LAS) for $\mathcal{H}$, if it is both stable and locally attractive for $\mathcal{H}$;
- globally asymptotically stable (GAS) for $\mathcal{H}$, if it is both stable and locally attractive for $\mathcal{H}$ for every $\mu > 0$.

Then, by introducing the set

$$\mathcal{A} = \{0\} \times \{0\} \times [0, T_2^{\mathcal{U}}] \times \{0\} \times \{0\} \times [0, T_2^{\mathcal{O}}] \qquad (8)$$

for which, for every $x_e \in C_e \cup D_e \cup G_e(D_e)$, $|x_e|_\mathcal{A} = \|(z,\, u,\, \varepsilon,\, \tilde{u})\|$, the problem we solve is as follows:

Problem 1: Given the matrices A, B, and M of appropriate dimensions and four positive scalars $T_1^{\mathcal{U}} \leq T_2^{\mathcal{U}}$, $T_1^{\mathcal{O}} \leq T_2^{\mathcal{O}}$, design matrices $L \in \mathbb{R}^{n\times q}$ and $K \in \mathbb{R}^{p\times n}$ such that the set $\mathcal{A}$ in (8) is globally asymptotically stable for the hybrid system (7).

To cope with this problem, we treat (7) as the cascade of two hybrid systems (modulo the coupling effect, yet vanishing in finite time, as shown in Remark 1, induced by $\tilde{u}$ on the ε dynamics). Namely, this cascade is composed by the ε dynamics along with its timer τ_2, which enters into the (z, u, τ_1) dynamics.

III. Main Results

A. A Solution via a Separation Principle

In this section, we provide a solution to Problem 1 that relies on the properties inherited from the components of the closed-loop system, namely, the observer and the controller subsystems.

Assumption 1 (Observer subsystem): The hybrid system

$$\left\{\begin{array}{l} \left.\begin{array}{rcl} \dot{\varepsilon} &=& A\varepsilon \\ \dot{\tau}_2 &=& -1 \end{array}\right\} \quad (\varepsilon,\tau_2) \in C_o \\ \left.\begin{array}{rcl} \varepsilon^+ &=& (I - LM)\varepsilon \\ \tau_2^+ &\in& [T_1^{\mathcal{O}}, T_2^{\mathcal{O}}] \end{array}\right\} \quad (\varepsilon,\tau_2) \in D_o \end{array}\right. \qquad (9a)$$

where

$$C_o = \mathbb{R}^n \times [0, T_2^{\mathcal{O}}],\ D_o = \mathbb{R}^n \times \{0\} \qquad (9b)$$

has the set $\mathcal{A}_o = \{(\varepsilon,\tau_2) \in \mathbb{R}^{n+1} \colon \varepsilon = 0, \tau_2 \in [0, T_2^{\mathcal{O}}]\}$ GAS. △

Assumption 2 (Controller subsystem): The hybrid system

$$\left\{\begin{array}{l} \left.\begin{array}{rcl} \dot{z} &=& Az + Bu \\ \dot{u} &=& 0 \\ \dot{\tau}_1 &=& -1 \end{array}\right\} \quad (z,u,\tau_1) \in C_\mathcal{K} \\ \left.\begin{array}{rcl} z^+ &=& z \\ u^+ &=& Kz \\ \tau_1^+ &\in& [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}] \end{array}\right\} \quad (z,u,\tau_1) \in D_\mathcal{K} \end{array}\right. \qquad (10a)$$

where

$$C_\mathcal{K} = \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2^{\mathcal{U}}],\ D_\mathcal{K} = \mathbb{R}^n \times \mathbb{R}^p \times \{0\} \qquad (10b)$$

has the set $\mathcal{A}_\mathcal{K} = \{0\} \times \{0\} \times [0, T_2^{\mathcal{U}}]$ GAS. △

A sufficient condition guaranteeing that Assumption 1 holds is given in [14, Corollary 1], while a sufficient condition for Assumption 2 and Assumption 3 to hold is given in Proposition 1; see Section III-C for constructive design procedures for L and K, and also [15, Example 3.21]

The following result establishes asymptotic stability of the set $\mathcal{A}$ for the closed-loop system (7) under the two aforementioned assumptions.

Theorem 1: Let Assumption 1 and Assumption 2 hold. Then, the set $\mathcal{A}$ defined in (8) is LAS for system (7). Furthermore, its basin of attraction contains every initial condition such that the resulting solutions are bounded. □

The above result establishes LAS for the set $\mathcal{A}$ and suggests that if every solution to the closed-loop system is bounded, then the asymptotic stability holds globally (since, in that case, the basin of attraction would include $C_e \cup D_e$). Given a solution to the closed-loop system, boundedness of τ_1, τ_2 and $\tilde{u}$ is guaranteed by construction of the controller, while Assumption 1 guarantees boundedness of ε. On the other hand, boundedness of the z and u components requires further conditions to hold, in particular, due to ε entering the dynamics of z and u as an additive disturbance through the jumps of u; see (7). The following assumption imposes a boundedness property on z and u under vanishing disturbances. Such an assumption is also needed in cascades of continuous-time systems; see, *e.g.*, [19, Theorem 10.3.1 and Corollary 10.3.3].

Assumption 3 (Boundedness for vanishing inputs): The solutions to

$$\left\{\begin{array}{l} \left.\begin{array}{rcl} \dot{z} &=& Az + Bu \\ \dot{u} &=& 0 \\ \dot{\tau}_1 &=& -1 \end{array}\right\} \quad (z,u,\tau_1) \in C_\mathcal{K} \\ \left.\begin{array}{rcl} z^+ &=& z \\ u^+ &=& Kz - Ku_0 \\ \tau_1^+ &\in& [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}] \end{array}\right\} \quad (z,u,\tau_1) \in D_\mathcal{K} \end{array}\right. \qquad (11)$$

are bounded for every u_0 s.t. $\lim_{t+j\to\infty} u_0(t,j) = 0$. △

A checkable sufficient condition for Assumption 3 is in the next section; see Proposition 1. Next, we provide a solution to Problem 1.

Theorem 2: Given four positive scalars $T_1^{\mathcal{O}} \leq T_2^{\mathcal{O}}$ and $T_1^{\mathcal{U}} \leq T_2^{\mathcal{U}}$, if there exist a matrix $L \in \mathbb{R}^{q\times n}$ such that Assumption 1 holds and a matrix $K \in \mathbb{R}^{p\times n}$ such that Assumption 2 and Assumption 3 hold, then the set $\mathcal{A}$ defined in (8) is GAS for the hybrid system (7). □

B. Sufficient Conditions

Now, we provide sufficient conditions guaranteeing that that stated assumptions hold.

The observer gain L can be already designed to satisfy Assumption 1 via [14, Corollary 1]. To design the controller K ensuring that Assumption 2 and Assumption 3 are verified, the following constructing method is provided. It uses ideas from [15, Example 3.21].

Proposition 1: If there exist a symmetric positive definite matrix $\mathbb{P} \in \mathbb{R}^{(n+p)\times(n+p)}$, and a matrix $K \in \mathbb{R}^{p\times n}$ such that

$$\mathbb{G}' e^{\mathbb{F}' v} \mathbb{P} e^{\mathbb{F} v} \mathbb{G} - \mathbb{P} < \mathbf{0} \qquad \forall v \in [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}], \tag{12}$$

where

$$\mathbb{F} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \mathbb{G} = \begin{bmatrix} \mathbf{I} & 0 \\ K & 0 \end{bmatrix} \tag{13}$$

then Assumption 2 and Assumption 3 are verified. □

C. Design Procedure

Direct computation of the gain K via Proposition 1 is not straightforward. In particular, from a numerical standpoint, (12) has two issues: it is not-convex in $\mathbb{P}$ and K, and it needs to be verified for infinitely many values of v. The relevance of the second issue is evident at a first sight, while the lack of convexity is a severe constraint, since nonconvex problems often lead to NP-hard problems; see, for example, [20]. Thus, to make the problem numerically tractable, some manipulations are needed. To this end, the following results allow to derive a convex design procedure for the proposed controller.

Proposition 2: If there exist a matrix $K \in \mathbb{R}^{p\times n}$, and a symmetric positive definite matrix $P_1 \in \mathbb{R}^{n\times n}$, such that for each $v \in [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]$,

$$\left(e^{Av} + \int_0^v e^{As} ds BK\right)' P_1 \left(e^{Av} + \int_0^v e^{As} ds BK\right) - P_1 < \mathbf{0} \tag{14}$$

then, there exists a symmetric positive matrix $\mathbb{P} \in \mathbb{R}^{n+p\times n+p}$, such that the pair $(K, \mathbb{P})$ satisfies (12). □

Now, we proceed to provide a convex condition in the decision variables K and P_1 that is equivalent to (14).

Proposition 3: The matrices P_1 and K satisfy condition (14), if and only if there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, a matrix $S \in \mathbb{R}^{n\times n}$, and a matrix $Y \in \mathbb{R}^{p\times n}$ such that, $KS = Y$, $S' P_1 S = W$, and for each $v \in [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]$

$$\begin{bmatrix} W + S + S' & -e^{Av} S - \int_0^v e^{As} ds BY \\ \star & -W \end{bmatrix} < \mathbf{0}. \tag{15}$$

□

Proposition 3 provides an equivalent condition to (14), which is convex in the decision variables W and S. Nevertheless, it still has to be verified for infinitely many values of v. This situation often occurs in the literature of sample data systems and impulsive systems, see for example [21] and the reference therein. Obviously, to effectively design the controller gain K, one needs to avoid finding a solution to (15) for infinitely many values of v. A general procedure to overcome this issue consists of embedding the terms e^{Av} and $\int_0^v e^{As} ds$, with $v \in [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]$, in a convex set, (other approaches could be used to cope with this issue; see, *e.g.*, [22]). Namely, one needs to find matrices $R_1, R_2, \ldots, R_\mu, Q_1, Q_2, \ldots, Q_\chi \in \mathbb{R}^{n\times n}$, such that $e^{Av} \in \text{co}\{R_1, R_2, \ldots, R_\nu\}$ and $\int_0^v e^{As} ds \in \text{co}\{Q_1, Q_2, \ldots, Q_\chi\}$ for each $v \in [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]$. Then, by exploiting the convexity of condition (15), one can obtain a finite set of inequalities. Specifically, we want to pursue a similar approach as in [14], where a sufficient convex condition to design the observer gain L is given. To this end, let us consider the following result.

Proposition 4: Let

$$X_1 = \begin{bmatrix} R_1 & Q_1 \\ U_1 & L_1 \end{bmatrix}, X_2 = \begin{bmatrix} R_2 & Q_2 \\ U_2 & L_2 \end{bmatrix}, \ldots, X_\nu = \begin{bmatrix} R_\nu & Q_\nu \\ U_\nu & L_\nu \end{bmatrix} \tag{16}$$

be matrices such that for each $v \in [T_1^{\mathcal{U}}, T_2^{\mathcal{U}}]$,

$$\exp\left(\begin{bmatrix} A & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix} v\right) \in \text{co}\{X_1, X_2, \ldots, X_\nu\}. \tag{17}$$

If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, a matrix $S \in \mathbb{R}^{n\times n}$ and a matrix $Y \in \mathbb{R}^{p\times n}$ such that for every $i = 1, \ldots, \nu$

$$\begin{bmatrix} W + S + S' & S' - R_i S - Q_i Y \\ \star & -W \end{bmatrix} < \mathbf{0}, \tag{18}$$

then $K = YS^{-1}$ ensures Assumption 2 and Assumption 3. □

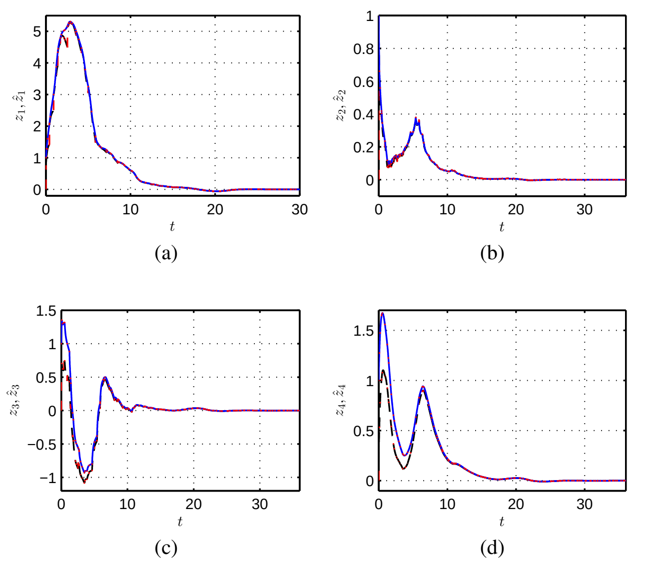
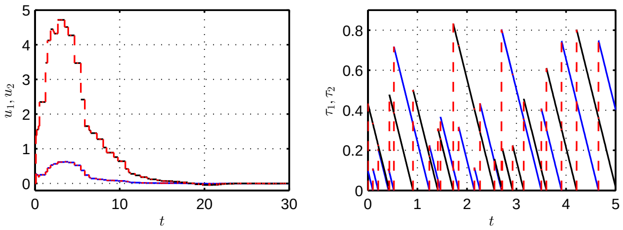


Fig. 2: Evolution of $(z, \hat{z})$ projected onto ordinary time, z (blue) and $\hat{z}$ (dashed black).



(a) u_1 (blue) and u_2 (black). (b) τ_1 (blue) and τ_2 (black).

Fig. 3: Evolution of (u, τ_1, τ_2) projected onto ordinary time.

IV. Numerical Example

Consider the linearized model for the unstable batch reactor in [24], which is described by the following data:

$$A = \begin{bmatrix} 1.38 & -0.208 & 6.71 & -5.68 \\ -0.581 & -4.29 & 0 & 0.675 \\ 1.07 & 4.27 & -6.65 & 5.89 \\ 0.048 & 4.27 & 1.34 & -2.1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 5.68 & 0 \\ 1.14 & -3.15 \\ 1.14 & 0 \end{bmatrix} \quad (19)$$
$$M = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and assume $T_1^{\mathcal{O}} = T_1^{\mathcal{U}} = 0.1$ and $T_2^{\mathcal{O}} = T_2^{\mathcal{U}} = 0.9$. The procedure proposed in [14, Corollary 1] yields

$$L = \begin{bmatrix} 0.8625 & -0.0994 \\ -0.0000 & 1.0000 \\ 0.1340 & 0.2916 \\ -0.0036 & 0.1922 \end{bmatrix}$$

while, via Proposition 4 applied building from the polytopic embedding technique proposed in [14], gives $K = \begin{bmatrix} 0.1074 & 0.3680 & -0.0414 & -0.0702 \\ 0.8996 & 0.5143 & 0.0594 & 0.1486 \end{bmatrix}$.

Fig. 2 depicts the projection onto ordinary time t of the states z and $\hat{z}$, while Fig. 3 reports the evolution of the control variable u, and of the two timers τ_1 and τ_2 projected onto ordinary time. Simulations show the effectiveness of the proposed approach, by stressing that the stabilization is achieved despite the lack of synchronism between the output sampling and input updating, as Fig. 3 suggests.

V. Conclusion

This paper proposed a methodology to model and design, through a convex setup, an observer-based controller in the presence of sporadically available measurement and temporal constrained input access. One of the main contributions provided by this paper consists in showing that a separation principle can be applied to design the observer and the controller. With the proposed observer-based controller, a numerical design procedure based on convex optimization was proposed. Future directions of research include allowing for more complex topologies as well as extensions to nonlinear plants.

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