On Robust Stability of Limit Cycles for Hybrid Systems With Multiple Jumps

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Abstract: In this paper, we address stability and robustness properties of hybrid limit cycles for a class of hybrid systems with multiple jumps in one period. The main results entail equivalent characterizations of stability of hybrid limit cycles for hybrid systems. The hybrid limit cycles may have multiple jumps in one period and the jumps are allowed to occur on sets. Conditions guaranteeing robustness of hybrid limit cycles are also presented.

Keywords: Hybrid systems; limit cycles; Poincaré map; stability; robustness

1. INTRODUCTION

In recent years, the study of limit cycles in nonlinear hybrid systems has received substantial attention. One reason for this is the existence of hybrid limit cycles in many engineering applications, such as walking robots (see, Grizzle et al. (2001)), genetic regulatory networks (see, Shu and Sanfelice (2014)), among others. The literature shows a variety of techniques for the study of limit cycles for systems with impulsive behaviors; see, e.g., Grizzle et al. (2001), Nersesov et al. (2002), Hiskens and Redd (2007), Morris and Grizzle (2009), and Barreiro et al. (2014). In particular, the existence and stability properties of a periodic orbit of nonlinear systems with impulsive effects via the method of Poincaré sections are established in Grizzle et al. (2001). In Nersesov et al. (2002), the Poincaré's method is generalized to analyze limit cycles for left-continuous hybrid impulsive dynamical systems. In Hiskens and Redd (2007), stability of limit cycles for differential-algebraic equations with impulses was studied via trajectory sensitivity analysis. Motivated by robotic applications, the design of state-feedback controllers that render limit cycles stable for nonlinear systems with hybrid zero dynamics was studied in Morris and Grizzle (2009). More recently, in Barreiro et al. (2014), the existence and stability of limit cycles in reset control systems were investigated via techniques that rely on the linearization of the Poincaré map about its fixed point.

The above works are only suitable for hybrid systems that have limit cycles with only one jump and under nominal noise free conditions. In fact, the results therein do not characterize the robustness properties to perturbations of stable hybrid limit cycles, which is a very challenging problem due to the impulsive behavior in such systems. In this work, we consider hybrid limit cycles that may contain multiple jumps within one period as well as their stability and robustness properties. The main contributions of this paper include the following:

1) We introduce a notion of hybrid limit cycle with one or more jumps in one period for a class of hybrid systems in Goebel et al. (2012). Also, we define the notion of flow periodic solution and asymptotic stability of the hybrid limit cycle for such hybrid systems. [1]
2) We establish sufficient and necessary conditions for guaranteeing stability properties of hybrid limit cycles for a class of hybrid systems. We construct impact functions and Poincaré maps that cope with multiple jumps in one period of a hybrid limit cycle.
3) Via perturbation analysis for hybrid systems, we show that asymptotic stability of a hybrid limit cycle is robust to small perturbations.

The remainder of the paper is organized as follows. Section 2 presents a motivational example. Section 3 provides some preliminaries on hybrid systems. Section 4 introduces the definition of hybrid limit cycle with multiple jumps, stability notions, and the Poincaré map. In addition, sufficient and necessary conditions for stability of hybrid limit cycles are established. Section 5 provides results on general robustness of stability to perturbations. Due to space constraints, the proofs will be published elsewhere.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geqslant 0} := [0, +\infty)$. $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \cdots\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. The equivalent notation $[x^\top \ y^\top]^\top$, $[x \ y]^\top$, and (x, y) are used for the same vector. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of h at x in the direction of f is denoted by $L_f h(x) := \langle \nabla h(x), f(x) \rangle$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$ when $\mathcal{A}$ is closed; $\overline{\mathcal{A}}$ (respectively, $\overline{\text{co}}\,\mathcal{A}$) denotes its closure (respectively, closed convex hull). $\mathbb{B}$ denotes a closed unit ball in Euclidean

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[1] In this work, a hybrid limit cycle is given by a closed set, while the limit cycle defined in Grizzle et al. (2001) is given by an open set due to the right continuity assumption in the definition of solutions.

space (of appropriate dimension). Given $\delta > 0$ and $x \in \mathbb{R}^n$, $x + \delta\mathbb{B}$ denotes a closed ball centered at x with radius δ. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing; it belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if, in addition, is unbounded. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each $t \geqslant 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s,t) = 0$ and, for each $s \geqslant 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \to \infty} \beta(s,t) = 0$.

2. MOTIVATIONAL EXAMPLE

The following example of hybrid systems with limit cycles that have more than one jump within one period is used throughout the paper to illustrate our results.

Example 2.1. (a two-gene network with binary hysteresis) Consider the genetic regulatory network with two genes (a and b) proposed in Shu and Sanfelice (2014) and shown in Fig. 1. The dynamics of such genetic network are given by

$$\begin{cases} \dot{x}_1 = k_1 s^-(x_2, \theta_2) - \gamma_1 x_1 \\ \dot{x}_2 = k_2 s^+(x_1, \theta_1) - \gamma_2 x_2 \end{cases} \tag{1}$$

where $x_1 \geqslant 0$ (or $x_2 \geqslant 0$, respectively) represents the concentration of protein A (or protein B, respectively). The constants θ_1, θ_2 are the thresholds associated with concentrations of protein A and B, respectively. In this model, gene a is expressed at a growing rate k_1 when x_2 is above the threshold θ_2. Similarly, gene b is expressed at a growing rate k_2 when x_1 is above the threshold θ_1. Degradations of both proteins are assumed to be proportional to their own concentrations, a mechanism that is captured by $-\gamma_1 x_1$ and $-\gamma_2 x_2$, respectively. The constants γ_1 and γ_2 represent the degradation rates of the protein A and protein B, respectively. The step functions $s^+(x_i, \theta)$ and $s^-(x_i, \theta)$ are defined as

$$s^+(x_i, \theta) = \begin{cases} 1 & \text{if } x_i \geqslant \theta \\ 0 & \text{if } x_i < \theta \end{cases}, \; s^-(x_i, \theta) = 1 - s^+(x_i, \theta), \tag{2}$$

where $i = 1, 2$, $s^+(x_i, \theta)$ represents the logic for gene expression when the protein concentration exceeds a threshold θ, while $s^-(x_i, \theta)$ represents the logic for gene inhibition. In order to incorporate binary hysteresis between

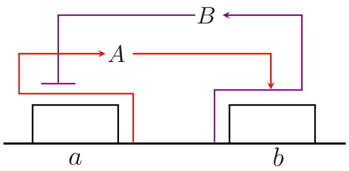

Fig. 1. A genetic regulatory network of two genes (a and b), each encoding for a protein (A and B). Lines ending in arrows represent genetic expression triggers, while lines ending in flatheads refer to genetic inhibition triggers.

the interaction between gene a and gene b, two discrete logic variables, q_1 and q_2, are used to model the genetic network. The dynamics of the logic variables depend on the thresholds, θ_1 and θ_2, respectively. The constants θ_1 and θ_2 that inferred from biological data, are specified to satisfy $0 < \theta_1 < \theta_1^{\max}$, $0 < \theta_2 < \theta_2^{\max}$, where $\theta_1^{\max}$ and $\theta_2^{\max}$ are the maximal values of the concentration of protein A and of the protein B, respectively.

The discrete dynamics of the hybrid system is described as follows. When $q_i = 0$ and $x_i = \theta_i + r_i$, the state q_i is updated to 1, i.e., $q_i^+ = 1$, where r_i, $i = 1, 2$, are given positive constants. When $q_i = 1$ and $x_i = \theta_i - r_i$, the state q_i is updated to 0, i.e., $q_i^+ = 0$, where $i = 1, 2$. Note that at jumps, the continuous states x_1 and x_2 do not change, i.e., $x_1^+ = x_1$ and $x_2^+ = x_2$. We can express the conditions for continuous and discrete behavior in a compact form using the following functions:

$$\eta_1(x_1, q_1) := (2q_1 - 1)(-x_1 + \theta_1 + (1 - 2q_1)r_1),$$
$$\eta_2(x_2, q_2) := (2q_2 - 1)(-x_2 + \theta_2 + (1 - 2q_2)r_2).$$

Then, the condition for continuous evolution is given by

$$\eta_1(x_1, q_1) \leqslant 0 \text{ and } \eta_2(x_2, q_2) \leqslant 0,$$

and the condition for discrete evolution is given by

$$\eta_1(x_1, q_1) = 0 \text{ or } \eta_2(x_2, q_2) = 0.$$

Parameters of the model include positive constants $k_1, k_2, \gamma_1, \gamma_2, \theta_1, \theta_2, r_1, r_2$, which satisfy $\theta_1 + r_1 < \theta_1^{\max}$, $\theta_2 + r_2 < \theta_2^{\max}$, $\theta_1 - r_1 > 0$, $\theta_2 - r_2 > 0$.

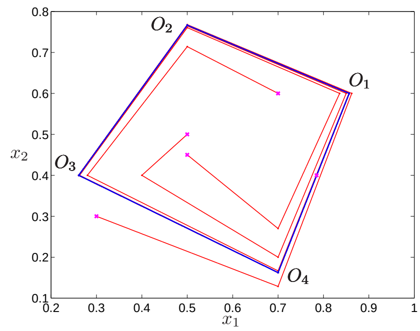

Fig. 2. Phase plot of solutions to the genetic network in (1) and (2) (projection to (x_1, x_2) plane). The point O_1 is given by $(x_1, x_2) = (0.8567, 0.6)$, the point O_2 is $(x_1, x_2) = (0.5, 0.7666)$, the point O_3 is $(x_1, x_2) = (0.2609, 0.4)$, and the point O_4 is $(x_1, x_2) = (0.7, 0.1624)$.

A simulation to the system with parameters $\theta_1 = 0.6$, $\theta_2 = 0.5, \gamma_1 = \gamma_2 = 1$, $k_1 = k_2 = 1$, and $r_1 = r_2 = 0.1$ is depicted in Fig. 2. The trajectory (blue line) in Fig. 2 shows a hybrid limit cycle $\mathcal{O}$ defined by the solution to the hybrid genetic network system with initial condition $(0.785, 0.4, 1, 0)$ that jumps at the points O_i, $i = 1, 2, 3, 4$, and flows in between points. As suggested from the simulation in Fig. 2, the hybrid limit cycle $\mathcal{O}$ is asymptotically stable for the system (more rigorous analysis is performed at a later section). A more detailed discussion of this example can be found in Shu and Sanfelice (2014). △

Motivated by this example, the interest of this work is in developing analysis tools to study the stability and robustness properties of hybrid limit cycles with continuous behavior and multiple jumps within one period for a class of hybrid systems.

3. PRELIMINARIES ON HYBRID SYSTEMS

Consider a hybrid system $\mathcal{H}$ in Goebel et al. (2012), which is given by

$$\mathcal{H} : \begin{cases} \dot{x} &= f(x) \quad x \in C \\ x^+ &= g(x) \quad x \in D \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$ denotes the state of the system. The function $f : C \to \mathbb{R}^n$ (respectively, $g : D \to \mathbb{R}^n$) is a single-valued map describing the continuous evolution (respectively, the discrete jumps) while $C \subset \mathbb{R}^n$ (respectively, $D \subset \mathbb{R}^n$) is the set on which the flow map f is effective (respectively, from which jumps can occur). The data of a hybrid system is given by $\mathcal{H} = (C, f, D, g)$. A solution to $\mathcal{H}$ is parameterized by ordinary time t and a counter j for jumps. It is given by a hybrid arc[2] $\phi : \mathrm{dom}\,\phi \to \mathbb{R}^n$. A solution ϕ to $\mathcal{H}$ is said to be Zeno if it is complete and the projection of $\mathrm{dom}\,\phi$ onto $\mathbb{R}_{\geqslant 0}$ is bounded. It is said to be maximal if it is not a truncated version of another solution. It is complete if $\mathrm{dom}\,\phi$ is unbounded. The set of maximal solutions to $\mathcal{H}$ from the set K is denoted as

$$\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi \text{ is a maximal solution to } \mathcal{H},\ \phi(0,0) \in K\}.$$

We define $t \mapsto \phi^f(t, x_0)$ as a solution of the flow dynamics

$$\dot{x} = f(x) \quad x \in C$$

from $x_0 \in C$. A hybrid system $\mathcal{H}$ is said to be well-posed if it satisfies the hybrid basic conditions (Goebel et al., 2012, Assumption 6.5). For more details about this hybrid systems framework, we refer the reader to Goebel et al. (2012).

4. HYBRID LIMIT CYCLES AND BASIC PROPERTIES

4.1 Definitions

In this section, we introduce the notion of hybrid limit cycles and reveal their basic properties. We consider a class of flow periodic solutions defined as follows.

Definition 4.1. (flow periodic solution) A complete solution ϕ^* to $\mathcal{H}$ is *flow periodic with period* $T^* \in (0, \infty)$ *and* $N^* \in \mathbb{N} \setminus \{0\}$ *jumps in each period* if $\phi^*(t + T^*, j + N^*) = \phi^*(t, j)$ for all $(t, j) \in \mathrm{dom}\,\phi^*$.

A flow periodic solution to $\mathcal{H}$ as in Definition 4.1 generates a hybrid limit cycle.

Definition 4.2. (hybrid limit cycle) A flow periodic solution ϕ^* with period $T^* \in (0, \infty)$ and $N^* \in \mathbb{N} \setminus \{0\}$ jumps in each period defines a *hybrid limit cycle* $\mathcal{O} = \{x \in \mathbb{R}^n : x = \phi^*(t, j), (t, j) \in \mathrm{dom}\,\phi^*\}$.[3]

Remark 4.3. The definition of a hybrid limit cycle $\mathcal{O}$ with period $T^* \in (0, \infty)$ and $N^* \in \mathbb{N} \setminus \{0\}$ jumps per period implies that $\mathcal{O}$ is nonempty and contains more than two points. In particular, a hybrid arc that generates the hybrid limit cycle $\mathcal{O}$ cannot be discrete. A hybrid limit cycle $\mathcal{O}$ may have more than one jump per period. Moreover, to define a proper hybrid limit cycle with N^* jumps, the parameter N^* should be chosen as the smallest integer such that the condition in Definition 4.1 is satisfied. Furthermore, if $N^* = 0$, the corresponding hybrid limit cycle would be continuous, and the results for the study of limit cycles for continuous-time systems would also apply; see Guckenheimer and Holmes (1983).

The following example illustrates the notion of hybrid limit cycles in Definition 4.2.

Example 4.4. (a two-gene network with binary hysteresis, revisited) Consider the hybrid genetic network system in Example 2.1. On the region $M_g := \big\{z := (x_1, x_2, q_1, q_2) \in \mathbb{R}^2_{\geqslant 0} \times \{0,1\}^2 : [0, \theta_1 + r_1] \times [0, \theta_2] \times \{0\} \times \{0\} \cup [\theta_1, \theta_1^{\max}] \times [0, \theta_2 + r_2] \times \{1\} \times \{0\} \cup [\theta_1 - r_1, \theta_1^{\max}] \times [\theta_2, \theta_2^{\max}] \times \{1\} \times \{1\} \cup [0, \theta_1] \times [\theta_2 - r_2, \theta_2^{\max}] \times \{0\} \times \{1\}\big\}$ (later, the set M_g will be part of our analysis), it can be described as a hybrid system $\mathcal{H}_{\mathrm{N}}$ as follows:

$$\mathcal{H}_{\mathrm{N}} : \begin{cases} \dot{z} = f_{\mathrm{N}}(z) := \begin{bmatrix} k_1(1 - q_2) - \gamma_1 x_1 \\ k_2 q_1 - \gamma_2 x_2 \\ 0 \\ 0 \end{bmatrix} & z \in C_{\mathrm{N}} \cap M_g \\ z^+ = g_{\mathrm{N}}(z) & z \in D_{\mathrm{N}} \cap M_g \end{cases} \tag{4}$$

where $C_{\mathrm{N}} := \{z \in \mathbb{R}^2_{\geqslant 0} \times \{0,1\}^2 : \eta_1(x_1, q_1) \leqslant 0, \eta_2(x_2, q_2) \leqslant 0\}$, $D_{\mathrm{N}} := \{z \in C_{\mathrm{N}} : \eta_1(x_1, q_1) = 0 \text{ or } \eta_2(x_2, q_2) = 0\}$. The jump map g is given by

$$g_{\mathrm{N}}(z) := \begin{cases} g_1(z) & \text{if } \eta_1(x_1, q_1) = 0, \eta_2(x_2, q_2) < 0 \\ g_2(z) & \text{if } \eta_1(x_1, q_1) < 0, \eta_2(x_2, q_2) = 0 \end{cases} \tag{5}$$

where

$$g_1(z) := \begin{bmatrix} x_1 \\ x_2 \\ 1 - q_1 \\ q_2 \end{bmatrix}, g_2(z) := \begin{bmatrix} x_1 \\ x_2 \\ q_1 \\ 1 - q_2 \end{bmatrix}. \tag{6}$$

It follows from Shu and Sanfelice (2014) that when the conditions

$$\theta_1 + r_1 < k_1/\gamma_1 < \theta_1^{\max},\ \theta_2 + r_2 < k_2/\gamma_2 < \theta_2^{\max} \tag{7}$$

hold, there exists a hybrid limit cycle $\mathcal{O}$ for the hybrid system $\mathcal{H}_{\mathrm{N}}$. △

4.2 Basic Properties of Hybrid Limit Cycles

In this section, we focus on a class of hybrid systems that satisfies the following assumption.

Assumption 4.5. For a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$, there exist N^* continuously differentiable functions $h_i : \mathbb{R}^n \to \mathbb{R}$ such that

1) the flow set C and the jump set D are given by $C = \bigcap_{i=1}^{N^*} C_i$, and $D = \bigcup_{i=1}^{N^*} D_i$, where $C_i = \{x \in \mathbb{R}^n : h_i(x) \geqslant 0\}$ and $D_i = \{x \in \mathbb{R}^n : h_i(x) = 0, L_f h_i(x) \leqslant 0\}$ for each $i \in \{1, 2, \cdots, N^*\}$;
2) the flow map f is continuously differentiable on an open neighborhood of $M \cap C$, and the jump map g is continuous on $M \cap D$;
3) for each $i, k \in \{1, 2, \cdots, N^*\}$, $L_f h_i(x) < 0$ for all $x \in M \cap D_i$ and $g(M \cap D_i) \cap (M \cap D) = \emptyset$, and $(M \cap D_i) \cap (M \cap D_k) = \emptyset$ for $i \neq k$;
4) $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ has a flow periodic solution ϕ^* with period $T^* > 0$ and $N^* \in \mathbb{N} \setminus \{0\}$ jumps per period that defines a hybrid limit cycle $\mathcal{O} \subset M \cap (C \cup D)$.

Item 1) in Assumption 4.5 implies that flows occur when every h_i is nonpositive and jumps only occur at points in zero level sets of h_i. Note that since every h_i is continuous and f is continuously differentiable, the flow set and the jump set are closed. The continuity property of f in item 2) of Assumption 4.5 is further required for the existence of solutions to $\dot{x} = f(x)$ according to (Goebel et al., 2012, Proposition 2.10). Items 3) and 4) in Assumption 4.5 allow us to restrict the analysis of a hybrid system $\mathcal{H}$ to a region of the state space $M \subset \mathbb{R}^n$. As we will show later, the set M is appropriately chosen for each specific system such that it guarantees completeness of solutions to $\mathcal{H}_M$ and the existence of periodic solutions.

It can be shown that a hybrid limit cycle generated by such periodic solutions is closed and bounded, as established in the following result.

[2] A hybrid arc is a function ϕ defined on a hybrid time domain, and for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous. A *compact hybrid time domain* is a set $\mathcal{E}$ of the form $\mathcal{E} = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_J$; the set $\mathcal{E}$ is a *hybrid time domain* if, for all $(T, J) \in \mathcal{E}$, $\mathcal{E} \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain.

[3] For some $t_s \in \mathbb{R}_{\geqslant 0}$, it can be written as $\{x \in \mathbb{R}^n : x = \phi^*(t, j), t \in [t_s, t_s + T^*], (t, j) \in \mathrm{dom}\,\phi^*\}$.

Lemma 4.6. Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ satisfying Assumption 4.5. Then, any hybrid limit cycle $\mathcal{O}$ for $\mathcal{H}$ is compact.

Remark 4.7. By items 1) and 2) of Assumption 4.5, the data of $\mathcal{H}_M$ satisfies the hybrid basic conditions (Goebel et al., 2012, Assumption 6.5). Then, using item 3) of Assumption 4.5, by (Sanfelice et al., 2007, Lemma 2.7), for any precompact solution ϕ to $\mathcal{H}_M$, there exists $r > 0$ such that $t_{j+1} - t_j \geqslant r$ for all $j \geqslant 1$, (t_j, j), $(t_{j+1}, j) \in \operatorname{dom}\phi$ (i.e., the elapsed time between two consecutive jumps is uniformly bounded below by a positive constant). These conditions guarantee that two successive jumps without flow in between do not happen.

Remark 4.8. Since a hybrid limit cycle $\mathcal{O}$ to $\mathcal{H}_M$ is compact, for any solution ϕ to $\mathcal{H}_M$, the distance $|\phi(t,j)|_{\mathcal{O}}$ is well-defined for all $(t,j) \in \operatorname{dom}\phi$.

The following result establishes a transversality[4] property of a hybrid limit cycle for $\mathcal{H}$.

Lemma 4.9. Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.5. Any hybrid limit cycle $\mathcal{O} \subset M \cap (C \cup D)$ for $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is transversal to $M \cap D$ at every jump.

The following example illustrates the properties of a hybrid system $\mathcal{H}$ under the satisfaction of Assumption 4.5.

Example 4.10. (a two-gene network with binary hysteresis, revisited) Consider the hybrid genetic network system $\mathcal{H}_{\mathrm{N}}$ in Example 4.4. The sets C_{N} and D_{N} can be rewritten as $C_{\mathrm{N}} := \{z \in \mathbb{R}^2_{\geqslant 0} \times \{0,1\}^2 : h_i(z) \geqslant 0, i = 1,2,3,4\}$, $D_{\mathrm{N}} := \bigcup_{i=1}^4 D_{\mathrm{N}_i}$, where

$$D_{\mathrm{N}_1} := \{z \in C_{\mathrm{N}} : h_1(z) = 0, (1-2q_1)f_1(z) \geqslant 0\},$$
$$D_{\mathrm{N}_2} := \{z \in C_{\mathrm{N}} : h_2(z) = 0, (1-2q_1)f_1(z) \geqslant 0\},$$
$$D_{\mathrm{N}_3} := \{z \in C_{\mathrm{N}} : h_3(z) = 0, (1-2q_2)f_2(z) \geqslant 0\},$$
$$D_{\mathrm{N}_4} := \{z \in C_{\mathrm{N}} : h_4(z) = 0, (1-2q_2)f_2(z) \geqslant 0\},$$
$$f_1(z) := k_1(1-q_2) - \gamma_1 x_1,\ f_2(z) := k_2 q_1 - \gamma_2 x_2,$$

and the four functions $h_i : C_{\mathrm{N}} \cup D_{\mathrm{N}} \to \mathbb{R}$, $i \in \{1,2,3,4\}$, are defined as

$$h_1(z) = h_2(z) := (1-2q_1)(-x_1 + \theta_1 + (1-2q_1)r_1),$$
$$h_3(z) = h_4(z) := (1-2q_2)(-x_2 + \theta_2 + (1-2q_2)r_2).$$

Consider the closed set M_g introduced in Example 4.4. The system $\mathcal{H}_{\mathrm{N}}$ can be rewritten as $\mathcal{H}_{\mathrm{N}_M} = (M_g \cap C_{\mathrm{N}}, f_{\mathrm{N}}, M_g \cap D_{\mathrm{N}}, g_{\mathrm{N}})$. Then, using the conditions in (7), we obtain that for all $z \in M_g \cap D_{\mathrm{N}_i}$ and $i = 1,2$, $L_f h_i(z) < 0$. By definition, the sets C_{N} and D_{N} are closed, f_{N} is continuous on $M_g \cap C_{\mathrm{N}}$, f_{N} is differentiable on a neighborhood of $M_g \cap C_{\mathrm{N}}$, and g_{N} is continuous on $M_g \cap D_{\mathrm{N}}$. Moreover, it can be verified that $g_{\mathrm{N}}(M_g \cap D_{\mathrm{N}_i}) \cap (M_g \cap D_{\mathrm{N}}) = \emptyset$ and $(M_g \cap D_{\mathrm{N}_i}) \cap (M_g \cap D_{\mathrm{N}_k}) = \emptyset$, for all $i, k \in \{1,2,3,4\}$, $i \neq k$. Therefore, Assumption 4.5 holds. △

4.3 Stability of Hybrid Limit Cycles

In this section, we present stability properties of hybrid limit cycles for $\mathcal{H}$. Following the stability notion in (Goebel et al., 2012, Definition 3.6), we employ the following notion for stability of hybrid limit cycles for $\mathcal{H}$.

Definition 4.11. Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. A hybrid limit cycle $\mathcal{O}$ is said to be

- *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{O}} \leqslant \delta$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;
- *globally attractive* for $\mathcal{H}$ if every maximal solution ϕ to $\mathcal{H}$ from $\overline{C} \cup D$ is complete and satisfies $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *globally asymptotically stable* for $\mathcal{H}$ if it is both stable and globally attractive;
- *locally attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ starting from $|\phi(0,0)|_{\mathcal{O}} \leqslant \mu$ is complete and satisfies $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *locally asymptotically stable* for $\mathcal{H}$ if it is both stable and locally attractive.

We will also employ the following stability notion.

Definition 4.12. ($\mathcal{KL}$ asymptotic stability) Let $\mathcal{H}$ be a hybrid system on $\mathbb{R}^n$, $\mathcal{A} \subset \mathbb{R}^n$ be a compact set, and $\mathcal{B}_{\mathcal{A}}$ be the basin of attraction of the set $\mathcal{A}$[5]. The set $\mathcal{A}$ is $\mathcal{KL}$ asymptotically stable on $\mathcal{B}_{\mathcal{A}}$ for $\mathcal{H}$ if for every proper indicator ω of $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$, there exists a function $\beta \in \mathcal{KL}$ such that for every solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{B}_{\mathcal{A}})$

$$\omega(\phi(t,j)) \leqslant \beta(\omega(\phi(0,0)), t+j) \quad \forall (t,j) \in \operatorname{dom}\phi. \tag{8}$$

Before presenting the main results, let us introduce the *time-to-impact function* and the Poincaré map for hybrid systems. Following the construction in Grizzle et al. (2001), for a hybrid system $\mathcal{H}$, and for each $i \in \{1, 2, \cdots, N^*\}$, the *time-to-impact function with respect to* D_i is defined by $T_{D_i} : C \cup D \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where[6]

$$T_{D_i}(x) := \inf\{t \geqslant 0 : \phi(t,j) \in D_i,\ \phi \in \mathcal{S}_{\mathcal{H}}(x)\} \tag{9}$$

for each $x \in C \cup D$. Under item 1) of Assumption 4.5, without using the function s in footnote 4, suppose that $T_{D_i}(x)$ ($i \in \{1, 2, \cdots, N^*\}$) follows the order $0 < T_{D_1}(x) < T_{D_2}(x) < \cdots < T_{D_{N^*}}(x)$.

Inspired by (Grizzle et al., 2001, Lemma 3), we show that for each $i \in \{1, 2, \cdots, N^*\}$, the function $x \mapsto T_{D_i}(x)$ is continuous on a subset of $M \cap (C \cup D)$.

Lemma 4.13. Suppose a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$ satisfy Assumption 4.5 and every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, for each $i \in \{1, 2, \cdots, N^*\}$, T_{D_i} is continuous at points in $\mathcal{X}_i := \{x \in M \cap (C \cup D) : 0 < T_{D_i}(x) < \infty\}$.

For each $i \in \{1, 2, \cdots, N^*\}$, the hybrid Poincaré map $P_i : M \cap D_i \to M \cap D_i$ is given by

$$P_i(x) := \big\{\phi(T_{D_i}(g(x)), j) : \phi \in \mathcal{S}_{\mathcal{H}}(g(x)), \\ (T_{D_i}(g(x)), j) \in \operatorname{dom}\phi \big\} \quad \forall x \in M \cap D_i \tag{10}$$

is well-defined and continuous on $\mathcal{X}_i$ due to the continuity of T_{D_i} and graphical convergence under hybrid basic conditions. Note that $P_i(x)$ is the value of the solution from x after N^* jumps, which is different from the standard way to define it; cf. Grizzle et al. (2001). The importance of the hybrid Poincaré map in (10) is that it allows one to determine the stability of hybrid limit cycles. Let P_i^k denote k compositions of the Poincaré map P_i with itself.

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[4] A hybrid limit cycle $\mathcal{O}$ with N^* jumps in each period is transversal to D at every jump (where $N^* \in \mathbb{N} \setminus \{0\}$ and D is the union of N^* jump sets, i.e., $D = \bigcup_{i=1}^{N^*} D_i$), if it intersects each jump set D_i at exactly one point $\bar{x}_i := \mathcal{O} \cap D_{s(i)}$ with the property $L_f h_i(\bar{x}_i) \neq 0$, where $i \in \{1, 2, \cdots, N^*\}$, where the function $s : \{1, 2, \cdots, N^*\} \to \{1, 2, \cdots, N^*\}$ defines an order of jumps in the hybrid limit cycle.

[5] $\mathcal{B}_{\mathcal{A}}$ is the set of points $\xi \in \mathbb{R}^n$ such that every complete solution ϕ to $\mathcal{H}_M$ with $\phi(0,0) = \xi$ is bounded and $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$.

[6] When there does not exist $t \geqslant 0$ such that $\phi^f(t,x) \in D$, we have $\{t \geqslant 0 : \phi^f(t,x) \in D_i\} = \emptyset$ for each $i \in \{1, 2, \cdots, N^*\}$, which gives $T_{D_i}(x) = \infty$.

Definition 4.14. A fixed point x^* of a Poincaré map $P : M \cap D \to M \cap D$ is said to be

- *stable* if for each $x \in M \cap D$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x^*| \leqslant \delta$ implies $|P^k(x) - x^*| \leqslant \epsilon$ for all integers $k > 0$;
- *globally attractive* if for all $x \in M \cap D$, $\lim_{k\to\infty} P^k(x) = x^*$;
- *globally asymptotically stable* if it is both stable and globally attractive;
- *locally attractive* if there exists $\mu > 0$ such that for all $x \in M \cap D$, $|x - x^*| \leqslant \mu$ implies $\lim_{k\to\infty} P^k(x) = x^*$;
- *locally asymptotically stable* if it is both stable and locally attractive.

For each $i \in \{1, 2, \cdots, N^*\}$, and for $x \in M \cap (C \cup D_i)$, define a distance function $d_i : M \cap (C \cup D_i) \to \mathbb{R}_{\geqslant 0}$ as

$$d_i(x) := \sup_{t\in[0,T_{D_i}(x)],\ (t,j)\in \text{dom}\ \phi,\ \phi\in\mathcal{S}_{\mathcal{H}}(x)} |\phi(t, j)|_{\mathcal{O}}, \quad (11)$$

when $0 \leqslant T_{D_i}(x) < \infty$ and

$$d_i(x) = \sup_{(t,j)\in \text{dom}\ \phi,\ \phi\in\mathcal{S}_{\mathcal{H}}(x)} |\phi(t, j)|_{\mathcal{O}}.$$

if $T_{D_i}(x) = \infty$. Note that d_i vanishes on $\mathcal{O}$. Then, following the ideas in (Grizzle et al., 2001, Lemma 4), the following property for functions d_i's can be established.

Lemma 4.15. Suppose a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$ satisfy Assumptions 4.5 and every maximal solution to $\mathcal{H}_M$ is complete. Then, for each $i \in \{1, 2, \cdots, N^*\}$, the function $d_i : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ given by (11) is well-defined and continuous on $\mathcal{O}$.

It can be shown that the local asymptotic stability of $\mathcal{O}$ leads to a $\mathcal{KL}$ bound as in (8) on its basin of attraction.

Theorem 4.16. Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.5. If $\mathcal{O}$ is a locally asymptotically stable compact set for $\mathcal{H}_M$, then $\mathcal{O}$ is $\mathcal{KL}$ asymptotically stable on $\mathcal{B}_{\mathcal{O}}$, i.e., the basin of attraction of the set $\mathcal{O}$.

Next, the relationship between stability of fixed points of Poincaré maps and the stability of the corresponding hybrid limit cycles is established.

Theorem 4.17. Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$ satisfying Assumption 4.5. Suppose every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, the following statements hold:

1) *for each $i \in \{1, 2, \cdots, N^*\}$, $x_i^* \in M \cap D_i$ is a stable fixed point of the Poincaré map P_i in (10) if and only if the hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ with period T^* and N^* jumps in each period to $\mathcal{H}_M$ from $\phi(0, 0) = g(x_i^*)$ for each $i \in \{1, 2, \cdots, N^*\}$ is stable for $\mathcal{H}_M$;*
2) *for each $i \in \{1, 2, \cdots, N^*\}$, $x_i^* \in M \cap D_i$ is a globally asymptotically stable fixed point of the Poincaré map P_i if and only if the unique hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ with period T^* and N^* jumps in each period to $\mathcal{H}_M$ from $\phi(0, 0) = g(x_i^*)$ for each $i \in \{1, 2, \cdots, N^*\}$ is globally asymptotically stable for $\mathcal{H}_M$.*

Note that sometimes it might be difficult to guarantee the conditions in statement 2) of Theorem 4.17, while local asymptotic stability of the fixed point of the Poincaré map P_i for each $i \in \{1, 2, \cdots, N\}$ can be readily verified. Such cases are handled by the following corollary.

Corollary 4.18. Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$ satisfying Assumption 4.5. Suppose every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, for each $i \in \{1, 2, \cdots, N^*\}$, $x^* \in M \cap D_i$ is a locally asymptotically stable fixed point of the Poincaré map P_i if and only if the unique hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ with period T^* and N^* jumps in each period to $\mathcal{H}_M$ from $\phi(0, 0) = g(x^*)$ is locally asymptotically stable for $\mathcal{H}_M$.

The following example illustrates the sufficient conditions in Theorem 4.17 by checking the eigenvalues of the Jacobian matrix of each Poincaré map at its fixed point. In this case, we require each Poincaré map to be differentiable in the interior of its domain.

Example 4.19. (a two-gene network with binary hysteresis, revisited) Consider the hybrid genetic network system $\mathcal{H}_{\text{N}_M}$ introduced in Example 4.10. By (Shu and Sanfelice, 2014, Proposition 3.1), every maximal solution to $\mathcal{H}_{\text{N}_M}$ is complete. Therefore, the hybrid genetic network system $\mathcal{H}_{\text{N}}$ on M_g satisfies Assumption 4.5 and has a flow periodic solution ϕ^* with period T^* and four jumps per period, which defines a unique hybrid limit cycle $\mathcal{O} \subset M_g \cap (C_{\text{N}} \cup D_{\text{N}})$.

Suppose the Poincaré maps for $\mathcal{H}_{\text{N}_M}$ are given by P_i with its fixed point z_i^* for all $i \in \{1, 2, 3, 4\}$. The sufficient condition in Corollary 4.18 can be verified as follows. If for all $i \in \{1, 2, 3, 4\}$ z_i^* is locally asymptotically stable for P_i, then the hybrid limit cycle $\mathcal{O}$ of $\mathcal{H}_{\text{N}_M}$ is locally asymptotically stable. To do this, it suffices to check the eigenvalues of the Jacobian matrix of the Poincaré map P_i at the fixed point z_i^* for all $i \in \{1, 2, 3, 4\}$. Due to the linear form of the flow and jump maps, it is possible to obtain the analytic form of the Jacobian matrices of the Poincaré maps. Here, we apply the shooting method in Hiskens and Redd (2007) to compute the Jacobian matrices based on approximate Poincaré maps numerically.

For the hybrid genetic network system in (4), consider the case with parameters $\theta_1 = 0.6$, $\theta_2 = 0.5, \gamma_1 = \gamma_2 = 1$, $k_1 = k_2 = 1$, and $r_1 = r_2 = 0.1$. Using a numerical method, the four fixed points are obtained as $z_1^*(0, 0, 0, 0) = (0.7, 0.1624, 0, 0) \in D_{\text{N}_1}$, $z_2^*(0, 0, 0, 0) = (0.8567, 0.6, 1, 0) \in D_{\text{N}_2}$, $z_3^*(0, 0, 0, 0) = (0.5, 0.7666, 1, 1) \in D_{\text{N}_3}$ and $z_4^*(0, 0, 0, 0) = (0.2609, 0.4, 0, 1) \in D_{\text{N}_4}$, and the period time of the hybrid limit cycle is $T^* = 2.83$. For all $i \in \{1, 2, 3, 4\}$, the Jacobian matrices of the hybrid Poincaré maps P_i at the fixed points z_i^* are

$$\mathbb{J}_{P_1}(z_1^*) = 10^{-3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -3.9 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathbb{J}_{P_2}(z_2^*) = 10^{-3}\begin{bmatrix} 3.5 & 3.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{J}_{P_3}(z_3^*) = 10^{-3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.9 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathbb{J}_{P_4}(z_4^*) = 10^{-3}\begin{bmatrix} 3.5 & 2.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively. All the eigenvalues of $\mathbb{J}_{P_i}(z_i^*)$, $i \in \{1, 2, 3, 4\}$, given by $\lambda_1 = 0.0035$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$, are located inside the unit circle. Therefore, the hybrid limit cycle $\mathcal{O}$ of the hybrid genetic network system is locally asymptotically stable. The properties of the hybrid limit cycle $\mathcal{O}$ are illustrated numerically in Fig. 2 (blue line). △

5. ROBUSTNESS OF HYBRID LIMIT CYCLES

In this section, we study the robustness properties of a hybrid limit cycle of $\mathcal{H}$ to generic state perturbations.

Consider the flow dynamics of the hybrid system $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ with perturbations

$$\dot{x} = f(x + d_1) + d_2, \tag{12}$$

where d_1 denotes the noise injected on the state x and d_2 captures unmodeled dynamics. Similarly, the perturbed jump map is given by

$$x^+ = g(x + d_1) + d_2. \tag{13}$$

Then, denoting by $\tilde{d}_i$ the signals d_i extended to the state space of x, the hybrid system $\mathcal{H}_M$ results in a perturbed hybrid system, which is denoted by $\widetilde{\mathcal{H}}_M$, with dynamics

$$\begin{cases} \dot{x} \ = f(x + \tilde{d}_1) + \tilde{d}_2 & x + \tilde{d}_1 \in M \cap C \\ x^+ = g(x + \tilde{d}_1) + \tilde{d}_2 & x + \tilde{d}_1 \in M \cap D. \end{cases} \tag{14}$$

Suppose there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that for two measurable functions $\tilde{d}_1, \tilde{d}_2 : \mathbb{R}_{\geqslant 0} \times \mathbb{N} \to \rho(x)\mathbb{B}$, the hybrid system $\widetilde{\mathcal{H}}_M$ can be written as the following ρ-perturbation of $\mathcal{H}_M$, denoted $\mathcal{H}_M^\rho$,

$$\begin{cases} \dot{x} \ \in F_\rho(x) & x \in C_\rho \\ x^+ \in G_\rho(x) & x \in D_\rho \end{cases} \tag{15}$$

where

$$C_\rho := \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap (M \cap C) \neq \emptyset\}, \tag{16}$$

$$F_\rho(x) := \overline{\text{co}} f((x + \rho(x)\mathbb{B}) \cap (M \cap C)) + \rho(x)\mathbb{B}, \tag{17}$$

$$D_\rho := \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap (M \cap D) \neq \emptyset\}, \tag{18}$$

$$G_\rho(x) := \{v \in \mathbb{R}^n : v \in \eta + \rho(\eta)\mathbb{B}, \quad \eta \in g((x + \rho(x)\mathbb{B}) \cap (M \cap D))\}. \tag{19}$$

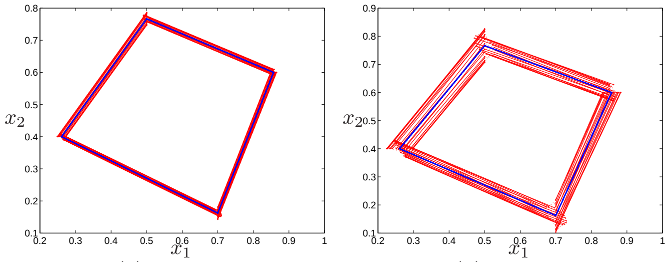
The following result establishes that the stability of $\mathcal{O}$ for $\mathcal{H}_M$ is robust to the class of perturbations defined above.

Theorem 5.1. Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.5. If $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}_M$ with basin of attraction $\mathcal{B}_\mathcal{O}$, then $\mathcal{O}$ is practically robustly $\mathcal{KL}$ asymptotically stable for $\mathcal{H}_M^\rho$ on $\mathcal{B}_\mathcal{O}$, i.e., given a proper indicator ω of $\mathcal{O}$ on $\mathcal{B}_\mathcal{O}$ there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for every $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}_\mathcal{O}$, there exists $\bar{\rho} > 0$ such that for every continuous function $\rho : \mathbb{R}^n \to \bar{\rho}\mathbb{B}$ that is positive on $K \backslash \mathcal{O}$, every solution ϕ to $\mathcal{H}_M^\rho$ with $\phi(0,0) \in K$ satisfies

$$\omega(\phi(t,j)) \leqslant \tilde{\beta}(\omega(\phi(0,0)), t + j) + \varepsilon \quad \forall (t,j) \in \text{dom}\, \phi.$$

Remark 5.2. Robustness results of stability of compact sets for general hybrid systems are available in Goebel et al. (2012). Since $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}_M$, Theorem 5.1 can be regarded as a direct consequence of (Goebel et al., 2012, Lemma 7.20). However, Theorem 5.1 is novel in the context of the literature of Poincaré maps. In particular, if one was to write the systems in Grizzle et al. (2001) and Nersesov et al. (2002) within the framework of Goebel et al. (2012), then one would not be able to apply the results on robustness for hybrid systems in Goebel et al. (2012) since the hybrid basic conditions would not be satisfied.

Example 5.3. (a two-gene network with binary hysteresis, revisited) Consider the hybrid system $\mathcal{H}_{\mathrm{N}_M}$ in Example 4.10. Two simulations are performed. The admissible state perturbation considered is $\tilde{d}_1 = (\kappa \sin(t), 0, 0, 0)$. The unmodeled dynamics considered is $\tilde{d}_2 = (0, \kappa \cos(t), 0, 0)$. As shown in Fig. 3, the solutions to the perturbed system are denoted in red line while solutions to the unperturbed system are denoted in blue line (with initial condition $(0.785, 0.4, 1, 0)$). It is shown that the solutions to the unperturbed system (in blue) and the solutions to the



(a) $\kappa = 0.01$ (b) $\kappa = 0.03$

Fig. 3. The trajectories with initial condition (0.785, 0.4, 1, 0) in Example 5.3

perturbed system (in red) stay close. These validate the result in Theorem 5.1. However, a general method to determine the precise margin of robustness guaranteed by Theorem 5.1 requires further investigation. By simulation, it is possible to find a relationship between the maximal perturbation parameter κ and the region where the solutions to the perturbed system converge to. △

6. CONCLUSION

In this paper, we defined the notions of hybrid limit cycles with multiple jumps in each period for a class of hybrid systems. Asymptotic stability of such hybrid limit cycles for a class of hybrid systems was characterized as a set stabilization problem. In particular, inspired by those in Grizzle et al. (2001), new impact functions and Poincaré maps were defined. Via these constructions, sufficient and necessary conditions for the stability of hybrid limit cycles with multiple jumps in each period were established. These results can be applied to systems that can be formulated as hybrid inclusions.

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