Results on Stability and Robustness of Hybrid Limit Cycles for A Class of Hybrid Systems

Xuyang Lou, Yuchun Li and Ricardo G. Sanfelice

***Abstract*— This work addresses stability and robustness properties of hybrid limit cycles for a class of hybrid systems, which combine continuous dynamics on a flow set and discrete dynamics on a jump set. Under some mild assumptions, we show that the stability of hybrid limit cycles for a hybrid system is equivalent to the stability of a fixed point of the associated Poincaré map. As a difference to related efforts for systems with impulsive effects, we also explore conditions under which the stability properties of the hybrid limit cycles are robust to small perturbations. The spiking Izhikevich neuron is presented to illustrate the notions and results throughout the paper.**

I. Introduction

Hybrid systems are models having state variables that can evolve continuously (flows) and/or discretely (jumps). In recent years, the study of limit cycles in nonlinear hybrid systems has received substantial attention. One reason is the existence of hybrid limit cycles in many engineering applications, such as walking robots [1], genetic regulatory networks [2], among others. Stability of hybrid limit cycles is often a fundamental requirement for their practical value in applications. The literature shows a variety of techniques for the study of hybrid limit cycles; see, e.g., [1], [3], [4]-[7]. In particular, Grizzle et al. established the existence and stability properties of a periodic orbit of nonlinear systems with impulsive effects via the method of Poincaré sections [1]. Nersesov et al. generalized the Poincaré's method to analyze limit cycles for left-continuous hybrid impulsive dynamical systems [4]. The trajectory sensitivity approach in [5] was employed to develop sufficient conditions for stability of limit cycles in a differential-algebraic impulsive switched model. Motivated by robotics applications, a state-feedback controller to render limit cycles stable within the hybrid zero dynamics is designed in [6]. More recently, in [7], the existence and stability of limit cycles in reset control systems were investigated via techniques that rely on the linearization of the Poincaré map about its fixed point.

To the best of our knowledge, all of the aforementioned results about limit cycles are only suitable for hybrid systems that have jumps on switching surfaces and under nominal noise free conditions. In fact, the results therein do not characterize the robustness properties to perturbations of stable hybrid limit cycles, which is a very challenging problem due to the impulsive behavior in such systems. Motivated by this open question, the contributions of this paper include the following:

- We introduce a notion of hybrid limit cycle with one jump per period for a class of hybrid systems in [8]. Also, we define the notion of flow periodic solution and asymptotic stability of the hybrid limit cycle for such hybrid systems.[1]
- We establish sufficient and necessary conditions for guaranteeing stability properties of hybrid limit cycles for a class of hybrid systems. We construct impact functions and Poincaré maps that cope with one jump per period of a hybrid limit cycle.
- Via perturbation analysis for hybrid systems, we show that asymptotic stability of a hybrid limit cycle is robust to small perturbations.

The organization of the paper is as follows. Section II presents a motivational example. Section III gives some preliminaries on hybrid systems. Section IV presents the definition of hybrid limit cycle, stability notions, and Poincaré map. In addition, sufficient and necessary conditions for stability of hybrid limit cycles are established. Section V provides results on general robustness of stability to perturbations. Due to space constraints, the proofs will be published elsewhere.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geqslant 0} := [0, +\infty)$. $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \cdots\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of h at x in the direction of f is denoted by $L_f h(x) := \langle \nabla h(x), f(x)\rangle$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by dom f, i.e., dom $f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$ when $\mathcal{A}$ is closed; $\overline{\mathcal{A}}$ (respectively, $\overline{\text{co}}\, \mathcal{A}$) denotes its closure (respectively, closed convex hull). $\mathbb{B}$ denotes a closed unit ball in Euclidean space (of appropriate dimension). Given $\delta > 0$ and $x \in \mathbb{R}^n$, $x + \delta\mathbb{B}$ denotes a closed ball centered at x with radius δ. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each $t \geqslant 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s, t) = 0$ and, for each $s \geqslant 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.

[1] In this work, a hybrid limit cycle is given by a closed set, while the limit cycle defined in [1] is given by an open set due to the right continuity assumption in the definition of solutions.

X. Lou is with Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi 214122, China. Email: Louxy@jiangnan.edu.cn Y. Li and R. G. Sanfelice are with the Department of Computer Engineering, University of California, Santa Cruz, CA 95064, USA. Email: yuchunli,ricardo@ucsc.edu. Research by X. Lou has been supported by National Natural Science Foundation of China (61473136). Research by Y. Li and R. G. Sanfelice has been partially supported by NSF Grant no. ECS-1150306 and by AFOSR Grant no. FA9550-12-1-0366.

II. Motivational Example

The following example motivates the study of limit cycles for hybrid systems in this paper.

Example 2.1: (Izhikevich neuron) Consider the Izhikevich neuron model [10] given by

$$\left\{ \begin{array}{l} \dot{v} = 0.04v^2 + 5v + 140 - w + I_{\text{ext}} \\ \dot{w} = a(bv - w) \end{array} \right. \tag{1}$$

where v is the membrane potential, w is the recovery variable, and I_{ext} represents the synaptic current or injected DC current. When the membrane voltage of a neuron increases and reaches a threshold (30 millivolts – see [10]), the membrane voltage and the recovery variable are instantaneously reset following the resetting rule

$$\text{if } v \geqslant 30, \text{ then} \left\{ \begin{array}{l} v^+ = c \\ w^+ = w + d. \end{array} \right. \tag{2}$$

The value of the input I_{ext} and the model parameters $a, b, c,$ and d are used to determine the neuron type, that is, the model can exhibit a specific firing pattern (of all known types) of cortical neurons when these parameters are appropriately chosen [10]. For instance, when the input is $I_{\text{ext}} = 10$ and the parameters are chosen as $a = 0.02, b = 0.2, c = -55, d = 4$, the neuron model exhibits intrinsic bursting behavior (see the blue lines shown in Fig. 1). This corresponds to a hybrid limit cycle $\mathcal{O}$ defined by the solution to (1)-(2) that jumps from point A to point B, and then flows back to A.

As suggested in Fig. 1, the hybrid limit cycle $\mathcal{O}$ is asymptotically stable. In particular, solutions initialized close to the set $\mathcal{O}$ stay close for all time and converge to the set $\mathcal{O}$ as time gets large. For instance, the trajectory (black line) of a solution starting from $(-54.76, -3.5)$ (the point C in the subfigure), which is close to the point B, remains close to the hybrid limit cycle $\mathcal{O}$ and approaches it eventually. However, solutions initialized relatively far away from the set $\mathcal{O}$ may not stay close for all time. For instance, as shown in Fig. 1, the trajectory (red line) of a solution starting from $(-54.5, -3.5)$ (the point D in the subfigure) that is close to the point B first goes far away from the hybrid limit cycle $\mathcal{O}$ and approaches it eventually.

Interestingly, solutions to the neuron model with state perturbations, in particular, solutions to (1)-(2) with an admissible state perturbation, may not be always close to the nominal solutions. For instance, an additive perturbation $e = (0.24, 0)$ (or $e = (0.5, 0)$, respectively) to the jump map after a jump from the point A would result in a state value equal to the point C (or the point D, respectively) instead of the point B. As shown above, the trajectory (black line) from the point C remains close to the hybrid limit cycle $\mathcal{O}$ and approaches it eventually, while the trajectory (red line) from the point D does not stay close to the one from the point B. This suggests that the hybrid limit cycle $\mathcal{O}$ has a small margin of robustness to perturbations. △

Motivated by this example, our interest is in developing analysis tools that can be applied to such systems so as to determine the stability and robustness properties of the limit cycle with hybrid dynamics.

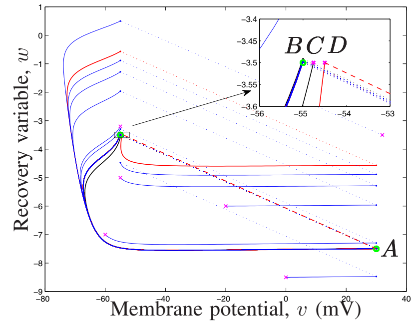


Fig. 1. Phase plot of several solutions to the Izhikevich model in (1)-(2) with different conditions. The point A corresponds to $(v, w) = (30, -7.5)$, the point B corresponds to $(v, w) = (-55, -3.5)$, the point C corresponds to $(v, w) = (-54.76, -3.5)$, and the point D corresponds to $(v, w) = (-54.5, -3.5)$.

III. Preliminaries on Hybrid Systems

Consider a hybrid system $\mathcal{H}$ in [8], which is given by

$$\mathcal{H} : \left\{ \begin{array}{lcll} \dot{x} & = & f(x) & x \in C \\ x^+ & = & g(x) & x \in D \end{array} \right. \tag{3}$$

where $x \in \mathbb{R}^n$ denotes the state of the system. The function $f : C \to \mathbb{R}^n$ (respectively, $g : D \to \mathbb{R}^n$) is a single-valued map describing the continuous evolution (respectively, the discrete jumps) while $C \subset \mathbb{R}^n$ (respectively, $D \subset \mathbb{R}^n$) is the set on which the flow map f is effective (respectively, from which jumps can occur). The data of a hybrid system $\mathcal{H}$ is given by $\mathcal{H} = (C, f, D, g)$. A solution to $\mathcal{H}$ is parameterized by ordinary time t and a counter j for jumps. It is given by a hybrid arc[2] ϕ : dom $\phi \to \mathbb{R}^n$. A solution ϕ to $\mathcal{H}$ is said to be complete if dom ϕ is unbounded. It is Zeno if it is complete and the projection of dom ϕ onto $\mathbb{R}_{\geqslant 0}$ is bounded. It is said to be maximal if it is not a truncated version of another solution. The set of maximal solutions to $\mathcal{H}$ from the set K is denoted as

$$\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi \text{ is a maximal solution to } \mathcal{H} \text{ with } \phi(0,0) \in K\}.$$

We define $t \mapsto \phi^f(t, x_0)$ as a solution of the flow dynamics

$$\dot{x} = f(x) \quad x \in C$$

from $x_0 \in C$. A hybrid system $\mathcal{H}$ is said to be well-posed if it satisfies the hybrid basic conditions [8, Assumption 6.5]. For more details about this hybrid systems framework, we refer the readers to [8].

IV. Hybrid Limit Cycles and Basic Properties

A. Definitions

In this section, we introduce the notion of hybrid limit cycles and, in the next section, reveal their basic properties. We consider a class of flow periodic solutions defined as follows.

Definition 4.1: (flow periodic solution) A complete solution ϕ^* to $\mathcal{H}$ is *flow periodic with period T^* and one jump in each period* if there exists $T^* \in (0, \infty)$ such that $\phi^*(t + T^*, j + 1) = \phi^*(t, j)$ for all $(t, j) \in$ dom ϕ^*.

[2] A hybrid arc is a function ϕ defined on a hybrid time domain and for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous. A *compact hybrid time domain* is a set $\mathcal{E} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ of the form $\mathcal{E} = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_J$; the set $\mathcal{E}$ is a *hybrid time domain* if for all $(T, J) \in \mathcal{E}$, $\mathcal{E} \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain.

Remark 4.2: The definition of a flow periodic solution ϕ^* with period $T^* > 0$ and one jump per period above implies that if $(t, j) \in \text{dom } \phi^*$, then $(t + T^*, j + 1) \in \text{dom } \phi^*$.

In fact, a flow periodic solution to $\mathcal{H}$ as in Definition 4.1 generates a hybrid limit cycle.

Definition 4.3: (hybrid limit cycle) A flow periodic solution ϕ^* with period T^* and one jump in each period defines a *hybrid limit cycle* $\mathcal{O} = \{x \in \mathbb{R}^n : x = \phi^*(t, j), (t, j) \in \text{dom } \phi^*\}$. [3]

Remark 4.4: The definition of a hybrid limit cycle $\mathcal{O}$ with period $T^* > 0$ and one jump per period implies that $\mathcal{O}$ is nonempty and contains more than two points; in particular, a hybrid arc that generates $\mathcal{O}$ cannot be discrete. A hybrid limit cycle $\mathcal{O}$ is restricted to have one jump per period, but extensions to more complex cases are possible.

The following example illustrates the notion of hybrid limit cycles in Definition 4.3.

Example 4.5: (Izhikevich neuron, revisited) Consider the Izhikevich neuron system in Example 2.1. This neuron system is slightly modified and written as a hybrid system as in (3), for which we denote as $\mathcal{H}_{\text{I}}$, and is given by

$$\mathcal{H}_{\text{I}} := \begin{cases} \dot{x} \ = f(x) := \begin{bmatrix} f_1(x) \\ a(bv - w) \end{bmatrix} & x \in C_{\text{I}} \\ x^+ = g(x) := \begin{bmatrix} c \\ w + d \end{bmatrix} & x \in D_{\text{I}} \end{cases} \tag{4}$$

where $x = (v, w)$, $f_1(x) = 0.04v^2 + 5v + 140 - w + I_{\text{ext}}$, $C_{\text{I}} = \{x \in \mathbb{R}^2 : v \leqslant 30\}$, $D_{\text{I}} = \{x \in \mathbb{R}^2 : v = 30, f_1(x) \geqslant 0\}$, where $f_1(x) \geqslant 0$ models the fact that the spikes occur when the membrane potential v grows to the threshold (30 mV).

The neuron system has a flow periodic solution with T^* and one jump per period. However, due to the form of the flow map, an analytic expression is not obvious. By numerical calculation, an approximate value of it can be obtained, that is, the solution ϕ^* to $\mathcal{H}_{\text{I}}$ from $\phi^*(0, 0) = (-55, -3.5)$ is a flow periodic solution with $T^* = 31.24$ ms and one jump per period. $\triangle$

B. Basic Properties of Hybrid Limit Cycles

In what follows, we focus on a class of hybrid systems that satisfies the following assumption.

Assumption 4.6: For a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$, there exists a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ such that

1) the flow set is given as $C = \{x \in \mathbb{R}^n : h(x) \geqslant 0\}$ and the jump set is given as $D = \{x \in \mathbb{R}^n : h(x) = 0, L_f h(x) \leqslant 0\}$;
2) the flow map f is continuously differentiable on an open neighborhood of $M \cap C$, and the jump map g is continuous on $M \cap D$;
3) $L_f h(x) < 0$ for all $x \in M \cap D$ and $g(M \cap D) \cap (M \cap D) = \emptyset$;
4) $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ has a flow periodic solution ϕ^* with period $T^* > 0$ and one jump per period that defines a hybrid limit cycle $\mathcal{O} \subset M \cap (C \cup D)$.

[3] For some $t_s \in \mathbb{R}_{\geqslant 0}$, it can be written as $\{x \in \mathbb{R}^n : x = \phi^*(t, j), t \in [t_s, t_s + T^*], (t, j) \in \text{dom } \phi^*\}$.

Item 1) in Assumption 4.6 implies that flows occur when h is nonnegative and jumps only occur at points in the zero level set of h. Note that since h is continuous and f is continuously differentiable, the flow set and the jump set are closed. The continuity property of f in item 2) of Assumption 4.6 is further required for the existence of solutions to $\dot{x} = f(x)$ according to [8, Proposition 2.10]. Moreover, item 2) also guarantees that solutions to $\dot{x} = f(x)$ continuously depend on initial conditions. Items 3) and 4) in Assumption 4.6 allow us to restrict the analysis of a hybrid system $\mathcal{H}$ to a region of the state space $M \subset \mathbb{R}^n$. As we will show later, the set M is appropriately chosen for each specific system such that it guarantees completeness of solutions to $\mathcal{H}_M$ and the existence of periodic solutions.

It can be shown that a hybrid limit cycle generated by periodic solutions as in Definition 4.3 is closed and bounded, as established in the following result.

Lemma 4.7: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ and a closed set M satisfying Assumption 4.6. Then, any hybrid limit cycle $\mathcal{O}$ for $\mathcal{H}$ is compact.

Remark 4.8: By items 1) and 2) of Assumption 4.6, the data of $\mathcal{H}_M$ satisfies the hybrid basic conditions [8, Assumption 6.5]. Then, using item 3) of Assumption 4.6, by [9, Lemma 2.7], for any precompact solution ϕ to $\mathcal{H}_M$, there exists $r > 0$ such that $t_{j+1} - t_j \geqslant r$ for all $j \geqslant 1$, $(t_j, j), (t_{j+1}, j) \in \text{dom } \phi$ (i.e., the elapsed time between two consecutive jumps is uniformly bounded below by a positive constant). These conditions guarantee that two successive jumps without flow in between do not happen.

Remark 4.9: Since a hybrid limit cycle $\mathcal{O}$ to $\mathcal{H}_M$ is compact, for any solution ϕ to $\mathcal{H}_M$, the distance $|\phi(t, j)|_{\mathcal{O}}$ is well-defined for all $(t, j) \in \text{dom } \phi$.

The following example illustrates Assumption 4.6.

Example 4.10: (Izhikevich neuron, revisited) Consider the Izhikevich neuron system introduced in Example 4.5. By design, the sets C_{I} and D_{I} are closed; f is continuously differentiable; and g is continuous. Define a function $h : \mathbb{R}^2 \to \mathbb{R}$ as $h(x) = 30 - v$. Then, C_{I} and D_{I} can be written as $C_{\text{I}} = \{x \in \mathbb{R}^2 : h(x) \geqslant 0\}$ and $D_{\text{I}} = \{x \in \mathbb{R}^2 : h(x) = 0, L_f h(x) \leqslant 0\}$. Consider the closed set $M := \{x \in \mathbb{R}^2 : w \leqslant 325 + I_{\text{ext}}\}$. Then, for each $x \in D_{\text{I}} \cap M$ we have $L_f h(x) = -f_1(x) = -(0.04v^2 + 5v + 140 - w + I_{\text{ext}}) \leqslant -(326 - (325 + I_{\text{ext}}) + I_{\text{ext}}) = -1 < 0$.

For $\mathcal{H}_{\text{I}_M} = (C_{\text{I}} \cap M, f, D_{\text{I}} \cap M, g)$, it can be verified that $g(D_{\text{I}} \cap M) \cap (D_{\text{I}} \cap M) = \emptyset$. Note that $g(D_{\text{I}} \cap M) \subset M \cap (D_{\text{I}} \cup C_{\text{I}})$. Furthermore, for any point $\bar{x} = (\bar{v}, \bar{w}) \in M \cap (D_{\text{I}} \cup C_{\text{I}})$, if $\bar{x}$ belongs to the boundary of the set $M \cap (D_{\text{I}} \cup C_{\text{I}})$ and $\bar{w} = 325 + I_{ext}$, then, $b\bar{v} < \bar{w}$ (recall $b = 0.2$), we have $a(b\bar{v} - \bar{w}) < 0$, and the w component of the vector field is negative, therefore, $T_{M \cap (D_{\text{I}} \cup C_{\text{I}}) \setminus D_{\text{I}}}(\bar{x}) \cap f(\bar{x}) = \{f(\bar{x})\} \neq \emptyset$.[4] If $\bar{x}$ belongs to the interior of the set $M \cap (D_{\text{I}} \cup C_{\text{I}}) \setminus D_{\text{I}}$, $T_{M \cap (D_{\text{I}} \cup C_{\text{I}}) \setminus D_{\text{I}}}(\bar{x}) \cap f(\bar{x}) = \{f(\bar{x})\} \neq \emptyset$. When $\bar{x} \in M \cap D_{\text{I}} \cap C_{\text{I}}$, $f_1(\bar{x}) > 0$, solutions cannot be extended via flow. By [8, Proposition 6.10], every maximal

[4] $T_{M \cap (D_{\text{I}} \cup C_{\text{I}}) \setminus D_{\text{I}}}(\bar{x})$ denotes the tangent cone to the set $M \cap (D_{\text{I}} \cup C_{\text{I}}) \setminus D_{\text{I}}$ at $\bar{x}$, see [8, Definition 5.12].

solution to $\mathcal{H}_{\mathrm{I}_M}$ is complete. Therefore, the neuron system $\mathcal{H}_{\mathrm{I}}$ on $\mathbb{R}^2$ satisfies Assumption 4.6 and has a flow periodic solution ϕ^* with period T^* and one jump per period, which defines a unique hybrid limit cycle $\mathcal{O} \subset M \cap (C_{\mathrm{I}} \cup D_{\mathrm{I}})$. △

The following result establishes a transversality[5] property of a hybrid limit cycle for $\mathcal{H}$.

Lemma 4.11: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.6. Any hybrid limit cycle $\mathcal{O} \subset M \cap (C \cup D)$ for $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is transversal to $M \cap D$.

Remark 4.12: In [1], the authors extend the Poincaré method to analyze the stability properties of a periodic orbit to nonlinear systems with impulsive effects. In particular, the solutions to the systems considered therein are right-continuous over (not necessarily closed) intervals of flow. In particular, the model considered in [1] requires that $C \cap D = \emptyset$ while Example 4.5 does not satisfy this condition. The model employed in [4] suffers similar drawbacks.

C. Stability of Hybrid Limit Cycles

In this section, we present stability properties of hybrid limit cycles for $\mathcal{H}$. Following the stability notion introduced in [8, Definition 3.6], we employ the following notion for stability of hybrid limit cycles.

Definition 4.13: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a compact hybrid limit cycle $\mathcal{O}$. Then, the hybrid limit cycle $\mathcal{O}$ is said to be

- *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{O}} \leqslant \delta$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \varepsilon$ for all $(t,j) \in \mathrm{dom}\ \phi$;
- *globally attractive* for $\mathcal{H}$ if every maximal solution ϕ to $\mathcal{H}$ from $\bar{C} \cup D$ is complete and satisfies $\lim\limits_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *globally asymptotically stable* for $\mathcal{H}$ if it is both stable and globally attractive.
- *locally attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ starting from $|\phi(0,0)|_{\mathcal{O}} \leqslant \mu$ is complete and satisfies $\lim\limits_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *locally asymptotically stable* for $\mathcal{H}$ if it is both stable and locally attractive.

We will also employ the following stability notion.

Definition 4.14: ($\mathcal{KL}$ asymptotic stability) Let $\mathcal{H}$ be a hybrid system on $\mathbb{R}^n$, $\mathcal{A} \subset \mathbb{R}^n$ be a compact set, and $\mathcal{B}_{\mathcal{A}}$ be the basin of attraction of the set $\mathcal{A}$.[6] The set $\mathcal{A}$ is $\mathcal{KL}$ asymptotically stable on $\mathcal{B}_{\mathcal{A}}$ for $\mathcal{H}$ if for every proper indicator ω of $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$, there exists a function $\beta \in \mathcal{KL}$ such that for every solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{B}_{\mathcal{A}})$

$$\omega(\phi(t,j)) \leqslant \beta(\omega(\phi(0,0)), t+j) \quad \forall (t,j) \in \mathrm{dom}\ \phi. \quad (5)$$

Before presenting the main results, let us introduce the *time-to-impact function* and the Poincaré map for hybrid systems. Following the definition in [1], for a hybrid system $\mathcal{H} = (C, f, D, g)$, the *time-to-impact function with respect to* D is defined by $T_I : C \cup D \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where[7]

$$T_I(x) := \inf\{t \geqslant 0 : \phi(t,j) \in D,\ \phi \in \mathcal{S}_{\mathcal{H}}(x)\} \quad (6)$$

for each $x \in C \cup D$.

Inspired by [1, Lemma 3], we show that the function $x \mapsto T_I(x)$ is continuous on a subset of $M \cap (C \cup D)$.

Lemma 4.15: Suppose a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfy Assumption 4.6 and every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, T_I is continuous at points in $\mathcal{X} := \{x \in M \cap (C \cup D) : 0 < T_I(x) < \infty\}$.

The Poincaré map $P : M \cap D \to M \cap D$ given by

$$P(x) := \left\{ \begin{array}{l} \phi(T_I(g(x)), j) : \phi \in \mathcal{S}_{\mathcal{H}}(g(x)), \\ (T_I(g(x)), j) \in \mathrm{dom}\ \phi \end{array} \right\} \quad \forall x \in M \cap D$$

is well-defined and continuous on $\mathcal{X}$ due to the continuity of T_I on $\mathcal{X}$. The importance of the hybrid Poincaré map in (7) is that it allows one to determine the stability of hybrid limit cycles. Let P^k denote k compositions of the Poincaré map P with itself.

Definition 4.16: A fixed point x^* of a Poincaré map $P : M \cap D \to M \cap D$ is said to be

- *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in M \cap D$, $|x - x^*| \leqslant \delta$ implies $|P^k(x) - x^*| \leqslant \epsilon$ for all integers $k > 0$;
- *globally attractive* if for all $x \in M \cap D$, $\lim\limits_{k\to\infty} P^k(x) = x^*$;
- *globally asymptotically stable* if it is both stable and globally attractive;
- *locally attractive* if there exists $\mu > 0$ such that for all $x \in M \cap D$, $|x - x^*| \leqslant \mu$ implies $\lim\limits_{k\to\infty} P^k(x) = x^*$;
- *locally asymptotically stable* if it is both stable and locally attractive.

For $x \in M \cap (C \cup D)$, define a distance function $d : M \cap (C \cup D) \to \mathbb{R}_{\geqslant 0}$ as

$$d(x) := \sup_{t \in [0, T_I(x)],\ (t,j) \in \mathrm{dom}\ \phi,\ \phi \in \mathcal{S}_{\mathcal{H}}(x)} |\phi(t,j)|_{\mathcal{O}},$$

when $0 \leqslant T_I(x) < \infty$ and

$$d(x) := \sup_{(t,j) \in \mathrm{dom}\ \phi,\ \phi \in \mathcal{S}_{\mathcal{H}}(x)} |\phi(t,j)|_{\mathcal{O}},$$

if $T_I(x) = \infty$. Note that d vanishes on $\mathcal{O}$. Moreover, for $0 \leqslant t \leqslant T_I(x)$, $\phi(t,0) = \phi^f(t,x)$, and hence

$$d(x) = \sup_{t \in [0, T_I(x)], \phi \in \mathcal{S}_{\mathcal{H}}(x)} |\phi(t,0)|_{\mathcal{O}} = \sup_{t \in [0, T_I(x)]} |\phi^f(t,x)|_{\mathcal{O}}.$$

Then, following the ideas in [1, Lemma 4], the following property of the function d can be established.

Lemma 4.17: Suppose a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfy Assumptions 4.6, and every maximal solution to $\mathcal{H}_M$ is complete. Then, the function $d : M \cap (C \cup D) \to \mathbb{R}_{\geqslant 0}$ is well-defined and continuous on $\mathcal{O}$.

It can be shown that the local asymptotic stability of $\mathcal{O}$ leads to a $\mathcal{KL}$ bound as in (5) on its basin of attraction.

Theorem 4.18: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and

[5] A hybrid limit cycle $\mathcal{O}$ is transversal to D if its closure intersects D at exactly one point $\bar{x} := \mathcal{O} \cap D$ with the property $L_f h(\bar{x}) \neq 0$.

[6] $\mathcal{B}_{\mathcal{A}}$ is the set of points $\xi \in \mathbb{R}^n$ such that every complete solution ϕ to $\mathcal{H}_M$ with $\phi(0,0) = \xi$ is bounded and $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$.

[7] In particular, when there does not exist $t \geqslant 0$ such that $\phi^f(t,x) \in D$, we have $\{t \geqslant 0 : \phi^f(t,x) \in D\} = \emptyset$, which gives $T_I(x) = \infty$.

a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.6. If $\mathcal{O}$ is a locally asymptotically stable compact set for $\mathcal{H}_M$, then $\mathcal{O}$ is $\mathcal{KL}$ asymptotically stable on the basin of attraction $\mathcal{B}_{\mathcal{O}}$ of the set $\mathcal{O}$.

Next, a relationship between stability of fixed points of Poincaré maps and the stability of the corresponding hybrid limit cycles is established.

Theorem 4.19: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.6. Suppose every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, the following statements hold:

1) *$x^* \in M \cap D$ is a stable fixed point of the Poincaré map P in (7) if and only if the hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ with period T^* and one jump in each period to $\mathcal{H}_M$ from $\phi(0,0) = g(x^*)$ is stable for $\mathcal{H}_M$,*
2) *$x^* \in M \cap D$ is a globally asymptotically stable fixed point of the Poincaré map P if and only if the unique hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ with period T^* and one jump in each period to $\mathcal{H}_M$ from $\phi(0,0) = g(x^*)$ is globally asymptotically stable for $\mathcal{H}_M$.*

Proof (sketch): The sufficiency part follows from the stability of $\mathcal{O}$ and the completeness of ϕ, and we omit it. To prove the necessity of item 1), suppose that $x^* \in M \cap D$ is a stable point of P. Then, by completeness of maximal solutions, it is implied that there exists $\bar{\delta} > 0$ such that for each $\tilde{x} \in (x^* + \bar{\delta}\mathbb{B}) \cap (M \cap D)$, there exists a complete solution ϕ to $\mathcal{H}_M$ with $\phi(0,0) = g(\tilde{x})$. Furthermore, the distance between ϕ and the hybrid limit cycle $\mathcal{O}$ is bounded by

$$\sup_{(t,j)\in \text{dom } \phi} |\phi(t,j)|_{\mathcal{O}} \leqslant \sup_{x \in (x^*+\bar{\delta}\mathbb{B})\cap(M\cap D)} d \circ g(x).$$

By Lemma 4.17, d is continuous at x^*. Since $\mathcal{O}$ is transversal to $M \cap D$, $\mathcal{O} \cap (M \cap D)$ is a singleton and $g(x^*) \in \mathcal{O}$, $d \circ g$ is continuous at x^*. Moreover, since $d \circ g(x^*) = 0$, it follows that given $\forall \epsilon > 0$, we can pick $\bar{\epsilon}$ and $\bar{\delta}$ such that $0 < \bar{\epsilon} < \epsilon$ and

$$\sup_{x \in (x^*+\bar{\delta}\mathbb{B})\cap(M\cap D)} d \circ g(x) < \epsilon.$$

Therefore, an open neighborhood of $\mathcal{O}$ given by $\mathcal{V} := d^{-1}([0,\epsilon))$ is such that any solution ϕ to $\mathcal{H}_M$ from $\phi(0,0) \in \mathcal{V}$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \epsilon$ for all $(t,j) \in \text{dom } \phi$. Thus, the necessity of item 1) follows immediately. The stability part of item 2) follows similarly from the proof of item 1). ■

Note that, at times, it might be difficult to guarantee the conditions in item 2) of Theorem 4.19, while local asymptotic stability of the fixed point of the Poincaré map P can be readily verified. Such cases are handled by the following corollary.

Corollary 4.20: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.6. Suppose every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, $x^* \in M \cap D$ is a locally asymptotically stable fixed point of the Poincaré map P if and only if the unique hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution ϕ with period T^* and one jump in each period to $\mathcal{H}_M$ from $\phi(0,0) = g(x^*)$ is locally asymptotically stable for $\mathcal{H}_M$.

The following example illustrates the sufficient conditions in Theorem 4.19 by checking the eigenvalues of the Jacobian matrix of the Poincaré map at the fixed point. In this case, we require the Poincaré map P to be differentiable in the interior of its domain.

Example 4.21: (Izhikevich neuron, revisited) Consider the Izhikevich neuron system analyzed in Example 4.10. Suppose the Poincaré map for $\mathcal{H}_{\mathrm{I}_M}$ is given by P with a fixed point x^*. The sufficient condition in Corollary 4.20 can be verified as follows. If x^* is locally asymptotically stable for $\mathcal{H}_{\mathrm{I}_M}$, then the hybrid limit cycle $\mathcal{O}$ of $\mathcal{H}_{\mathrm{I}_M}$ is locally asymptotically stable. To do this, it suffices to check the eigenvalues of the Jacobian matrix of the Poincaré map at the fixed point. However, due to the quadratic form in the flow map of $\mathcal{H}_{\mathrm{I}}$, the calculation of the Jacobian matrix of the Poincaré map is as difficult as to find the solution of the flow map of $\mathcal{H}_{\mathrm{I}}$. Therefore, we apply the shooting method [5] to compute the Jacobian matrix based on an approximate Poincaré map numerically.

For the Izhikevich model (4), consider the case of intrinsic bursting behavior with parameters $a = 0.02, b = 0.2, c = -55, d = 4, I_{\text{ext}} = 10$. Using a numerical method, a fixed point $x^*(0,0) = (30, -7.5)$ and the period time between the jumps is $T^* = 31.218$. The Jacobian matrix of the hybrid Poincaré map at the fixed point is

$$\mathbb{J}_P(x^*) = \begin{bmatrix} 0 & 0 \\ -0.0246 & -0.0246 \end{bmatrix}.$$

The eigenvalues of $\mathbb{J}_P$ are $\lambda_1 = 0$ and $\lambda_2 = -0.0246$, with one eigenvalue at zero and the other one locates inside the unit circle. Therefore, the hybrid limit cycle $\mathcal{O}$ of the Izhikevich model is locally asymptotically stable. The properties of the hybrid limit cycle $\mathcal{O}$ are illustrated numerically in Fig. 1. Note that the hybrid limit cycle $\mathcal{O}$ starting from $(-55, -3.5)$ is locally asymptotically stable, but not globally asymptotically stable. △

V. Robustness of Hybrid Limit Cycles

In this section, we explore the robustness properties of a hybrid limit cycle for $\mathcal{H}$ to generic state perturbations. In the presence of perturbations, there is no guarantee that solutions to the hybrid system would exist, even if such solutions to the nominal hybrid system $\mathcal{H}$ are known to exist for every point in $C \cup D$. However, this situation can be remedied, at least for small noise by properly defining C and D; see [8].

Consider the flow dynamics of the hybrid system $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ with perturbations

$$\dot{x} = f(x + d_1) + d_2,$$

where d_1 corresponds to state noise and d_2 captures unmodeled dynamics. Similarly, we consider the perturbed discrete dynamics

$$x^+ = g(x + d_1) + d_2.$$

Then, denoting by $\tilde{d}_i$ the signals d_i extended to the state space of x, the hybrid system $\mathcal{H}_M$ results in a perturbed hybrid system, which is denoted by $\tilde{\mathcal{H}}_M$, with dynamics

$$\begin{cases} \dot{x} = f(x + \tilde{d}_1) + \tilde{d}_2 & x + \tilde{d}_1 \in M \cap C \\ x^+ = g(x + \tilde{d}_1) + \tilde{d}_2 & x + \tilde{d}_1 \in M \cap D. \end{cases}$$

Suppose there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that the two measurable functions can be defined as $\tilde{d}_1, \tilde{d}_2 : \mathbb{R}_{\geqslant 0} \times \mathbb{N} \to \rho(x)\mathbb{B}$. A perturbation of the hybrid system $\tilde{\mathcal{H}}_M$ is the ρ-perturbation of $\mathcal{H}_M$, denoted $\mathcal{H}_M^\rho$, which is given by

$$\begin{cases} \dot{x} & \in & F_\rho(x) & \quad x \in C_\rho \\ x^+ & \in & G_\rho(x) & \quad x \in D_\rho \end{cases}$$

where

$$\begin{aligned} C_\rho &:= \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap (M \cap C) \neq \emptyset\}, \\ F_\rho(x) &:= \overline{\mathrm{co}} f((x + \rho(x)\mathbb{B}) \cap (M \cap C)) + \rho(x)\mathbb{B}, \\ D_\rho &:= \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap (M \cap D) \neq \emptyset\}, \\ G_\rho(x) &:= \{v \in \mathbb{R}^n : v \in \eta + \rho(\eta)\mathbb{B}, \\ &\qquad \eta \in g((x + \rho(x)\mathbb{B}) \cap (M \cap D))\}. \end{aligned}$$

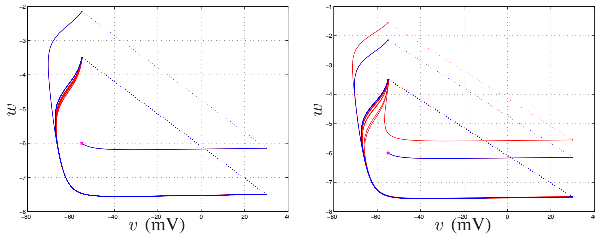
The following result establishes that the stability of $\mathcal{O}$ for $\mathcal{H}_M$ is robust to a class of perturbations defined above.

Theorem 5.1: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.6. If $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}_M$ with basin of attraction $\mathcal{B}_\mathcal{O}$, then $\mathcal{O}$ is semiglobally practically robustly $\mathcal{KL}$ asymptotically stable for $\mathcal{H}_M^\rho$ on $\mathcal{B}_\mathcal{O}$, i.e., given a proper indicator ω of $\mathcal{O}$ on $\mathcal{B}_\mathcal{O}$ there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for every $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}_\mathcal{O}$, there exists $\bar{\rho} > 0$ such that for every continuous function $\rho : \mathbb{R}^n \to \bar{\rho}\mathbb{B}$ that is positive on $K \backslash \mathcal{O}$, every solution ϕ to $\mathcal{H}_M^\rho$ with $\phi(0,0) \in K$ satisfies

$$\omega(\phi(t,j)) \leqslant \tilde{\beta}(\omega(\phi(0,0)), t+j) + \varepsilon \quad \forall (t,j) \in \textit{dom } \phi.$$

Remark 5.2: Robustness results of stability of compact sets for general hybrid systems are available in [8]. Since $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}_M$, Theorem 5.1 can be regarded as a direct consequence of [8, Lemma 7.20]. However, Theorem 5.1 is novel in the context of the literature of Poincaré maps. In particular, if one was to write the systems in [1] and [4] within the framework of [8], then one would not be able to apply the results on robustness for hybrid systems in [8] since the hybrid basic conditions would not be satisfied and the hybrid limit cycle may not be given by a compact set.

Example 5.3: (Izhikevich neuron, revisited) Consider the Izhikevich neuron system in Example 4.5. Theorem 4.19 will be illustrated for the hybrid system $\mathcal{H}_\mathrm{I}$ by plotting the solutions from the initial condition $(-55, -6)$, when an admissible state perturbation $e = (e_1, e_2)$ affects the jump map. The noise is injected as unmodeled dynamics on the jump map as $e = (e_1, e_2) = (\kappa \sin(t), 0)$ where κ is chosen differently in order to verify the robustness. Two simulations are performed with different values of κ. Fig. 2 shows the phase plots for both the perturbed solution (red line) and normal solution (blue line). It is found that the hybrid limit cycle $\mathcal{O}$ is robust to the state perturbation e when $\kappa = 0.24$ as shown in Fig. 2(a), while $\mathcal{O}$ is not robust to the state perturbation e when $\kappa = 0.42$ as shown in Fig. 2(b). A general method to determine the actual margin of robustness guaranteed by Theorem 5.1 requires further investigation. By simulation, it is possible to quantify the relationship between



(a) $\kappa = 0.24$ (b) $\kappa = 0.42$

Fig. 2. Trajectories with initial condition $(-55, -6)$ in Example 5.3

the maximal perturbation parameter κ and the size of the ball where the steady state values converge to. $\triangle$

VI. Conclusion

In this paper, we defined the notions of flow periodic solution and hybrid limit cycle. To investigate the stability properties of the hybrid limit cycles, we also constructed an impact functions inspired by those introduced by Grizzle et al. [1]. Based on these constructions, sufficient and necessary conditions for the stability of hybrid limit cycles were presented. Moreover, comparing to previous results in the literature, we established conditions for robustness of hybrid limit cycles with respect to small perturbations, which is a very challenging problem in systems with impulsive effects. An extension effort that characterizes the situation where a hybrid limit cycle may contain multiple jumps within a period can be found in [11].

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