On Notions and Sufficient Conditions for Forward Invariance of Sets for Hybrid Dynamical Systems

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***Abstract*— Forward invariance for hybrid dynamical systems modeled by differential and difference inclusions with state-depending conditions enabling flows and jumps is studied. Several notions of forward invariance are considered and sufficient conditions in terms of the objects defining the system are introduced. In particular, we study forward invariance notions that apply to systems with nonlinear dynamics for which not every solution is unique or may exist for arbitrary long hybrid time. Such behavior is very common in hybrid systems. Lyapunov-based conditions are also proposed for the estimation of invariant sets. Applications and examples are given to illustrate the results. In particular, the results are applied to the estimation of weakly forward invariant sets, which is an invariance property of interest when employing invariance principles to study convergence of solutions.**

I. Introduction

A forward invariant set for a dynamical system is a set that has solutions evolving within the set. The property of forward invariance is important for the analysis and control design of dynamical systems since it characterizes regions of the state space from which solutions start and stay for all future time. Forward invariance properties are inherently used in the study of reachability, safety, and asymptotic stability of sets. Several articles and books introducing notions of forward invariance are available in the literature; see, e.g., [1], [2], [3]. Conditions to guarantee forward invariance of a set and methods to construct sets that are forward invariant are introduced in [2], [4], [5]. For systems with an input, forward invariance can be employed as a tool for control design. Referred to as forward invariance-based control, the use of forward invariance for analysis and control design include results for the stabilization of constrained systems [6], cascades of nonlinear systems [7], and for the study of robustness [8].

Forward invariance for hybrid dynamical systems is more intricate than for classical systems. Among the main reasons is the fact that such systems include both continuous and discrete behaviors. In addition, the dynamics of hybrid systems are typically governed by set-valued, nonlinear maps, which lead to nonunique solutions and increase the complexity in predicting the behavior of the system. Recent contributions to the understanding of forward invariance for hybrid systems without inputs include those for impulsive differential inclusions [9] and for hybrid automata [10], [11], [12] (forward invariance is also employed in the analysis of hybrid inclusions [13], [14]). Same as for classical systems, the interest in forward invariance is driven by several applications featuring hybrid systems. The use of forward invariance for analysis and control design for hybrid systems include periodic motion analysis with impacts [15], reachability [16], determining safe sets for switched systems [17], and hybrid control design [18].

In this paper, we study the forward invariance properties of sets for hybrid systems modeled within the hybrid inclusion framework of [13]. Hybrid inclusions are defined by differential and difference inclusions with state-depending conditions enabling flows and jumps. This broad class of hybrid systems may have nonunique solutions and nonlinear dynamics. Motivated by these properties, forward invariance notions based on the existence and completeness of solutions are introduced. For each notion, we propose sufficient conditions that guarantee the said forward invariance property. We employ these sufficient conditions to estimate weakly forward invariant sets for hybrid systems using Lyapunov-like functions. Our notions include those in [9], in particular, the viability notion defined therein, while our results provide sufficient conditions for hybrid systems given in terms of hybrid inclusions.

The remainder of the paper is organized as follows. Forward invariance notions and sufficient conditions are presented in Section II and Section III, respectively. The results for an estimation of weakly forward invariant sets are given in Section IV. Applications and examples are given in Section V. Due to space constraints, the proofs will be published elsewhere.

II. Forward Invariance Notions for Hybrid Systems

In this paper, we follow the framework in [13], in which, a hybrid system $\mathcal{H} = (C, F, D, G)$ is given by

$$\mathcal{H} \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{1}$$

where F is the flow map which governs the continuous evolution of the state on the flow set C, and G is the jump map, which governs the discrete evolution from the jump set D. A solution to the hybrid system $\mathcal{H}$ is parameterized by the ordinary time variable $t \in \mathbb{R}_{\geq 0} := [0, \infty)$ and by the discrete jump variable $j \in \mathbb{N} := \{0, 1, 2, ...\}$, and defined on a hybrid time domain $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$; see [13, Definition 2.3]. The set E is a hybrid time domain if, for each $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ can be written as $\cup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \; ... \leq t_J$. A hybrid arc ϕ is a function

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on a hybrid time domain if, for each $j \in \mathbb{N}$, $t \mapsto \phi(t,j)$ is absolutely continuous on the interval $\{t \ : \ (t,j) \in \operatorname{dom}\phi \ \}$. A solution to $\mathcal{H}$ is a hybrid arc $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ that satisfies the dynamics of $\mathcal{H}$, where $\operatorname{dom}\phi$ is a hybrid time domain E; see [13, Definition 2.6].

Following [13, Section 2.2, Section 2.3], we list the types of solutions that will be used in this paper.

Definition 2.1: (Types of Solutions to a Hybrid System) A solution ϕ to the hybrid system $\mathcal{H} = (C, F, D, G)$ is called

1) nontrivial if $\operatorname{dom}\phi$ contains at least two points;
2) complete if $\operatorname{dom}\phi$ is unbounded;
3) maximal if there does not exist another solution ψ to $\mathcal{H}$ such that $\operatorname{dom}\phi$ is a proper subset of $\operatorname{dom}\psi$ and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \operatorname{dom}\phi$.

For convenience, we define the set of maximal solutions to $\mathcal{H}$ from the set K as $\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi$ is a maximal solution to $\mathcal{H}$ with $\phi(0,0) \in K\}$. We also define the range of a solution ϕ to a hybrid system $\mathcal{H}$ as $\operatorname{rge}\phi = \{x \in \mathbb{R}^n : x = \phi(t,j), (t,j) \in \operatorname{dom}\phi\}$. To formulate our results, we will need the following result [13, Proposition 2.10].

Proposition 2.2: (Basic Existence) Consider the hybrid system $\mathcal{H} = (C, F, D, G)$. Let $\xi \in \overline{C} \cup D$. If $\xi \in D$ or

(VC) there exist $\varepsilon > 0$ and an absolutely continuous function $z : [0, \varepsilon] \to \mathbb{R}^n$ such that $z(0) = \xi, \dot{z}(t) \in F(z(t))$ for almost all $t \in [0, \varepsilon]$ and $z(t) \in C$ for all $t \in (0, \varepsilon]$,

then there exists a nontrivial solution ϕ to $\mathcal{H}$ with $\phi(0,0) = \xi$. If (VC) holds for every $\xi \in \overline{C} \setminus D$, then there exists a nontrivial solution to $\mathcal{H}$ from every point of $\overline{C} \cup D$, and every $\phi \in \mathcal{S}_{\mathcal{H}}$ satisfies exactly one of the following:

(a) ϕ is complete;

(b) ϕ is not complete and "ends with flow", with $(T,J) = \sup\operatorname{dom}\phi$, the interval I^J has nonempty interior; and either

(b.1) I^J is closed, in which case $\phi(T,J) \in \overline{C} \setminus (C \cup D)$; or

(b.2) I^J is open to the right, in which case $(T,J) \notin \operatorname{dom}\phi$, and there does not exist an absolutely continuous function $z : \overline{I^J} \to \mathbb{R}^n$ satisfying $\dot{z}(t) \in F(z(t))$ for almost all $t \in I^J, z(t) \in C$ for all $t \in \operatorname{int} I^J$, and such that $z(t) = \phi(t,J)$ for all $t \in I^J$;

(c) ϕ is not complete and "ends with jump": for $(T,J) = \sup\operatorname{dom}\phi$, one has $\phi(T,J) \notin \overline{C} \cup D$.

Furthermore, if $G(D) \subset \overline{C} \cup D$, then (c) above does not occur.

Solutions to $\mathcal{H}$ might not be unique due to F and G being set-valued or due to C and D overlapping. Hence, a set may enjoy weak or strong forward invariance properties for a given hybrid system. Moreover, maximal solutions to $\mathcal{H}$ might not be complete due to the directions of flow determined by F, the new values after the jumps allowed by G, or the geometry of the sets C and D. In this section, we define several forward invariance notions that, in particular, apply in situations where not every maximal solution is complete and unique, which is very common in hybrid systems.

We start by defining weak forward pre-invariance of a set.

Definition 2.3: (Weak Forward pre-Invariance) The set $K \subset \mathbb{R}^n$ is said to be weakly forward pre-invariant for $\mathcal{H}$ if for every $x \in K$ there exists at least one solution $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ such that $\operatorname{rge}\phi \subset K$.

The weak forward pre-invariance notion requires that at least one solution exists from every point in K. Such a solution can be trivial ($\operatorname{dom}\phi$ with only one point) or nontrivial, but at least one maximal solution from each point in the set has to stay in the set for all future hybrid time. Note that the prefix "pre" captures the fact that the solution staying in K may not be complete.

Next, we define a weak forward invariant notion, which is equivalent to the notion in [13, Definition 6.19] and in [14, Definition 3.1].

Definition 2.4: (Weak Forward Invariance) The set $K \subset \mathbb{R}^n$ is said to be weakly forward invariant for $\mathcal{H}$ if for every $x \in K$ there exists at least one complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ with $\operatorname{rge}\phi \subset K$.

Next, we define forward pre-invariance of a set as the property that every maximal solution starting from K stays in K. This notion was introduced in [13, Definition 6.25] in the context of invariance principles.

Definition 2.5: (Forward pre-Invariance) The set $K \subset \mathbb{R}^n$ is said to be forward pre-invariant for $\mathcal{H}$ if for every $x \in K$ there exists at least one solution, and for every solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$, $\operatorname{rge}\phi \subset K$.

Finally, we define the strongest version of forward invariance properties, which requires not only that every maximal solution starting from K stays in K, but also requires completeness of all maximal solutions.

Definition 2.6: (Forward Invariance) The set $K \subset \mathbb{R}^n$ is said to be forward invariant for $\mathcal{H}$ if for every $x \in K$ there exists at least one solution, and every solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ is complete and satisfies $\operatorname{rge}\phi \subset K$.

The relationship among the four notions is summarized in the diagram in Figure 1.

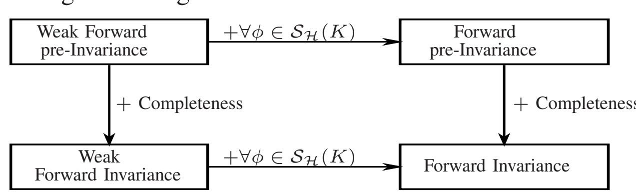

Fig. 1. Relationships of the notions of forward invariance for a set K.

Remark 2.7: In [9], viable and invariant sets concepts are introduced for hybrid systems that are modeled in term of impulsive differential inclusions. The viability property in [9] is equivalent to the weak forward invariance in Definition 2.4, while the invariance property in [9] is equivalent to the definition of forward pre-invariance in Definition 2.5 and the definition of positively invariant in [2]. Note that in addition, we introduce the strongest forward invariance notion in Definition 2.6.

III. Sufficient Conditions for the Forward Invariance Properties

In general, it is very difficult to directly check forward invariance of a set from the definitions, as that would require checking solutions explicitly. In this section, when possible, solution independent conditions to check if a set satisfies each notion are given. To this end, we use the concept of tangent cone to a set K (see, e.g.,[13, Definition 5.12]), which is also known as the Bouligand tangent cone or contingent cone.

Definition 3.1: (Tangent Cone) The tangent cone to a set $K \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted as $T_K(x)$, is the set of all vectors $\omega \in \mathbb{R}^n$ for which there exist sequences $x_i \in K$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \searrow 0$ and $\omega = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$.
In comparison to the Clarke tangent cone [19, Remark 4.7], the tangent cone defined in Definition 3.1 includes all vectors that point inward to the set K or that are tangent to the boundary of K.

For a given set K and a hybrid system $\mathcal{H} = (C, F, D, G)$ for the study of forward invariance, the sufficient conditions we present in this paper require the following mild assumptions on K, C, D, and F.

Assumption 3.2: The sets K, C, and D are such that $K \subset \overline{C} \cup D$ and that $K \cap C$ is closed. The map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded relative to $K \cap C$, and $F(x)$ is convex for every $x \in K \cap C$. Furthermore, $C \subset \operatorname{dom} F$ and $D \subset \operatorname{dom} G$.

The following proposition introduces conditions implying that a set is weakly forward pre-invariant.

Proposition 3.3: (Sufficient Conditions for Weak Forward pre-Invariance) Let K and $\mathcal{H} = (C, F, D, G)$ satisfy Assumption 3.2. Then, the set K is weakly forward pre-invariant for $\mathcal{H}$ if:

2.1) $\forall x \in K \cap D, G(x) \cap K \neq \emptyset$; and

2.2) For every $\xi \in K \setminus D$, there exists a neighborhood U of ξ such that for every $x \in U \cap K \cap C, F(x) \cap T_{K\cap C}(x) \neq \emptyset$.

Furthermore, for every point in K, there exists a nontrivial solution ϕ satisfying $\operatorname{rge} \phi \subset K$.

Next, building from Proposition 3.3, we provide sufficient conditions for weak forward invariance as given in Definition 2.4, which requires completeness of a maximal solution from each point in the set K.

Proposition 3.4: (Sufficient Conditions for Weak Forward Invariance) Let K and $\mathcal{H} = (C, F, D, G)$ satisfy Assumption 3.2. Then, the set K is weakly forward invariant for $\mathcal{H}$ if:

3.1) Conditions 2.1) and 2.2) of Proposition 3.3 hold; and

3.2) For every $\phi \in \mathcal{S}_\mathcal{H}(x)$, case (b.2) in Proposition 2.2 does not hold.

Remark 3.5: Although, in principle, condition *3.2)* in Proposition 3.4 is a solution-dependent property, it can be guaranteed by verifying that solutions from $C \setminus D$ can flow into C (which is expressed in terms of the tangent cone condition in Proposition 3.3) and that $K \cap C$ is compact or F is bounded on $K \cap C$ (see [20, Chapter 4, Theorem 3]). Under such conditions, solutions cannot escape to infinity in finite time, therefore, case *(b.2)* in Proposition 2.2 would not hold. The following result pertains to this case.

Proposition 3.6: (Sufficient Conditions for Weak Forward Invariance Revised) Let K and $\mathcal{H} = (C, F, D, G)$ satisfy Assumption 3.2. Then, the set K is weakly forward invariant for $\mathcal{H}$ if:

3'.1) Conditions 2.1) and 2.2) of Proposition 3.3 hold; and

3'.2) Either $K \cap C$ is compact or F is bounded on $K \cap C$.

Remark 3.7: Compared to Proposition 3.4, Proposition 3.6 further constraints the data of $\mathcal{H}$ so as to provide a solution-independent condition. In [9, Theorem 1, 2], the map F is assumed to be Marchuad to guarantee the completeness of solutions from the set K, which is a stronger assumption than condition 3'.2) in Proposition 3.6.[1]

In the remainder of this section, we obtain conditions for the two stronger forward invariance properties introduced in Definition 2.5 and Definition 2.6. First, we give sufficient conditions for a set K to be forward pre-invariant.

Proposition 3.8: (Sufficient Conditions for Forward pre-Invariance) Let K and $\mathcal{H} = (C, F, D, G)$ satisfy Assumption 3.2, and suppose F is locally Lipschitz on C.[2] Then, the set K is forward pre-invariant for $\mathcal{H}$ if:

4.1) $G(K \cap D) \subset K$; and

4.2) For every $\xi \in K \cap C$, there exists a neighborhood U of ξ such that for every $x \in U \cap K \cap C, F(x) \subset T_{K\cap C}(x)$.

Furthermore, every maximal solution $\phi \in \mathcal{S}_\mathcal{H}(K)$ is nontrivial and satisfies $\operatorname{rge} \phi \subset K$.

Remark 3.9: Note that the condition in the well known Nagumo Theorem (see a rewriten version in [2, Theorem 3.1]), describes a special case of condition 4.2) in Proposition 3.3. In particular, the Nagumo Theorem provides a necessary and sufficient condition for a set to be positively invariant (defined similarly to our forward pre-invariant notion) for the continuous-time system $\dot{x} = f(x)$. The condition in [2, Equation (12)] for autonomous discrete-time system is similar to condition 4.1).

With the conditions in Proposition 3.8, and if all maximal solutions that start from K are complete, then, the set K is forward invariant for $\mathcal{H}$. The following proposition gives a set of sufficient conditions for a such property.

Proposition 3.10: (Sufficient Conditions for Forward Invariance) Let K and $\mathcal{H} = (C, F, D, G)$ satisfy Assumption 3.2, and suppose F is locally Lipschitz on C. Then, the set K is forward invariant for $\mathcal{H}$ if:

5.1) Conditions 4.1) and 4.2) of Proposition 3.8 hold; and

5.2) For every $\phi \in \mathcal{S}_\mathcal{H}(K)$, case (b.2) in Proposition 2.2 does not hold.

IV. Forward pre-Invariance of Sublevel Sets of Lyapunov-like Functions

In [2, Section 3.3], Lyapunov functions are employed to determine invariant sets for a given system and for

[1] A map F is said to be Marchaud if its graph and its domain are nonempty and closed; $F(x)$ is convex, compact, and nonempty for each $x \in \operatorname{dom} F$; and F has linear growth; see [1, Definition 10.3.2].

[2] *Definition: (Locally Lipschitz) A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally Lipschitz on a set $K \subset \mathbb{R}^n$ if for any $x \in K$, there exist a neighborhood U of x and a constant $\lambda \geq 0$ (the Lipschitz constant) such that for every $\xi \in U \cap \operatorname{dom} F$, $F(x) \subset F(\xi) + \lambda|x - \xi|\mathbb{B}$.*

the design of invariant-based feedback controllers. In this section, also inspired by the Lyapunov stability result for hybrid systems in [13, Theorem 3.18], we characterize the forward invariance properties of sets that are sublevel sets of Lyapunov-like functions.

Proposition 4.1: (Forward pre-Invariance of Sublevel Sets) Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ *in* (1)*. Let* $c \geq 0$ *and* $W : \mathbb{R}^n \to \mathbb{R}$ *be continuously differentiable on an open set containing*[3] $\overline{C} \cap L_W(c)$ *and such that it satisfies*

$$\langle \nabla W(x), \eta \rangle \leq 0 \qquad \forall x \in C \cap L_W(c), \eta \in F(x), \qquad (2)$$

$$W(\eta) - W(x) \leq 0 \quad \forall x \in D \cap L_W(c), \eta \in G(x), \qquad (3)$$

$$G(x) \subset K \qquad \forall x \in D \cap L_W(c), \qquad (4)$$

where $K = L_W(c) \cap (C \cup D)$*. In addition, let the set* K *and* $\widetilde{\mathcal{H}} = (K \cap C, F, D \cap K, G)$ *satisfy Assumption 3.2. Then, the set* K *is forward pre-invariant for* $\widetilde{\mathcal{H}}$*.*

Proposition 4.1 establishes a forward pre-invariance property of sublevel sets of Lyapunov-like functions for a modified version of a hybrid system $\mathcal{H}$, namely $\widetilde{\mathcal{H}}$. In particular, $\widetilde{\mathcal{H}}$ has the same flow and jump map as the original system $\mathcal{H}$, but its flow set and jump set are intersected by the sublevel set $L_W(c)$. We provide sufficient conditions for the set K to be forward invariant for $\widetilde{\mathcal{H}}$ by applying Proposition 3.10.

Proposition 4.2: (Forward Invariance of a Sublevel Set for $\widetilde{\mathcal{H}}$*) Consider a hybrid system* $\mathcal{H} = (C, F, D, G)$*,* $c \geq 0$*, and* W *(as well as* K*) be such that the conditions in Proposition 4.1 hold. Then, the set* K *is forward invariant for* $\widetilde{\mathcal{H}} = (C \cap K, F, D \cap K, G)$ *if at least one of the following condition holds:*

- *For every* $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}(K)$*, case (b.2) in Proposition 2.2 does not hold;*
- *Either* $K \cap C$ *is compact or* F *is bounded on* $K \cap C$*.*

In addition to the results from Proposition 4.1, Proposition 4.2 states that if every solution $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}(K)$ is complete, the set K is forward invariant for the modified hybrid system $\widetilde{\mathcal{H}}$. With these results, we provide a result that can be used to estimate weakly forward invariant sets of the original hybrid system $\mathcal{H}$.

Theorem 4.3: (Weak forward invariance of a set for $\mathcal{H}$*) Consider the hybrid system* $\mathcal{H}$ *in* (1)*. For each* $i \in \{1, 2, ..., N\}$*, let* c_i *and* K_i *satisfy the conditions in Proposition 4.2 for some function* W_i*. Then, the set*

$$K = \bigcup_{i \in \{1,2,\dots,N\}} K_i$$

is weakly forward invariant for $\mathcal{H}$ *if 2.2) in Proposition 3.3 holds.*

Remark 4.4: Note that $\mathcal{S}_{\mathcal{H}}(K)$ may include more solutions than $\bigcup_{i \in \{1,2,\dots,N\}} \mathcal{S}_{\widetilde{\mathcal{H}}_i}(K_i)$, due to $C_i = K_i \cap C$ and $D_i = K_i \cap D$ for each i, where $\widetilde{\mathcal{H}}_i = (K_i \cap C, F, K_i \cap D, G)$. These extra solutions may be allowed to flow or jump outside of K, therefore, we cannot guarantee forward invariance of the set K for $\mathcal{H}$. On the other hand, if for every $x \in K$, solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ is unique, and we can conclude that K is forward invariant for $\mathcal{H}$.

V. Applications and Examples

In this section, examples and applications of our results are presented. The following three examples illustrate the proposed notions and sufficient conditions for hybrid systems $\mathcal{H}$ in Section II and Section III.

Example 5.1 (Weak Forward pre-Invariant Set): Consider the hybrid system $\mathcal{H} = (C, f, D, G)$ in $\mathbb{R}^2$ with system data given by[4]

$$f(x) := [1 + x_1^2 \;\; 0]^\top \quad \forall x \in C := \{x \in \mathbb{R}^2 : x_1 \in [0, \infty), x_2 \in [-1, 1]\};$$

$$G(x) := [x_1 + \mathbb{B} \;\; x_2]^\top \quad \forall x \in D := \{x \in \mathbb{R}^2 : x_1 \in [0, \infty), x_2 = 0\}.$$

We can observe that solutions to $\mathcal{H}$ from $K = C$ are not complete, since their x_1 component escapes to infinity in finite time, though they stay inside K. Thus, we argue that K is weakly forward pre-invariant for $\mathcal{H}$. We verify this by showing that system $\mathcal{H}$ and the set K satisfy Assumption 3.2, and by applying Proposition 3.3. Since $G(K \cap D) \cap K \neq \emptyset$, we have that condition *2.1)* holds. Condition *2.2)* holds because for every $x \in K \cap C$, $f(x) \in T_{K \cap C}(x)$ since it is pointing horizontally, and $T_{K \cap C}(x) =$

$$\begin{cases} \mathbb{R} \times \mathbb{R}_{\leq 0} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 \in (0, \infty), x_2 = 1\} \\ \mathbb{R} \times \mathbb{R} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 \in (0, \infty), x_2 \in (-1, 1)\} \\ \mathbb{R} \times \mathbb{R}_{\geq 0} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 \in (0, \infty), x_2 = -1\} \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0} & \text{if } x = (0, 1) \\ \mathbb{R}_{\geq 0} \times \mathbb{R} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \in (-1, 1)\} \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \text{if } x = (0, -1). \end{cases}$$

In addition, if we consider a new jump map G', namely $x^+ \in G'(x) =: [2x_1 \;\; x_2]^\top$, and maintain the other system data C, f, D, then the set $K = C$ is forward pre-invariant for $\mathcal{H}$, since the solutions that are able to jump out of K now are only allowed to jump towards $+\infty$ on the x_1-axis. This can be verified by checking Proposition 3.8: the flow condition is met as discussed above, and the jump map $G'(D \cap K) \subset K$.

Notice that the proposed tangent cone condition is able to handle the nonlinear dynamics in f. △

To illustrate Proposition 3.4, we present the following example on $\mathbb{R}^2$.

Example 5.2 (Weak Forward Invariant Set): Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ in $\mathbb{R}^2$ given by

$$F(x) := \begin{cases} [1 \;\; 1]^\top & \text{if } x_2 > 1 - x_1 \\ \overline{\text{con}}\, \{[1 \;\; 1]^\top, [-1 \;\; -1]^\top\} & \text{if } x_2 = 1 - x_1 \\ [-1 \;\; -1]^\top & \text{if } x_2 < 1 - x_1 \end{cases}$$

$$\forall x \in C := [0, 1] \times [0, 1];$$

$$G(x) := \begin{cases} [\frac{1}{2} + \frac{1}{4}\mathbb{B} \;\; \frac{1}{2}]^\top & \text{if } x \in D_1 \\ \{[\frac{1}{2} + \frac{1}{4}\mathbb{B} \;\; \frac{1}{2}]^\top, [\frac{1}{2} \;\; \frac{1}{2} + \frac{1}{4}\mathbb{B}]^\top\} & \text{if } x \in D_2 \\ [\frac{1}{2} \;\; \frac{1}{2} + \frac{1}{4}\mathbb{B}]^\top & \text{if } x \in D_3, \end{cases}$$

[3]The c-sublevel set of the function $W : \mathbb{R}^n \to \mathbb{R}$ is denoted by $L_W(c) = \{x : W(x) \leq c\}$.

[4]$\mathbb{B}$ denotes the closed unit ball in a Euclidean space centered at the origin.

where $D = D_1 \cup D_2 \cup D_3$, and
$D_1 := \{x \in \mathbb{R}^2 : x_1 \in (0,1), x_2 \in \{0,1\}\}$,
$D_2 := \{x \in \mathbb{R}^2 : x_1 \in \{0,1\}, x_2 \in \{0,1\}\}$,
$D_3 := \{x \in \mathbb{R}^2 : x_1 \in \{0,1\}, x_2 \in (0,1)\}$.

We consider the set $K = [\frac{1}{2},1] \times [\frac{1}{2},1]$. According to the first piece of the definition of F, every solution that starts from the set $\left((\frac{1}{2},1) \times (\frac{1}{2},1)\right) \bigcup \left(\{\frac{1}{2}\} \times (\frac{1}{2},1)\right) \bigcup \left((\frac{1}{2},1) \times \{\frac{1}{2}\}\right)$, initially flows within K with vector field $[1\ \ 1]^\top$. According to the definition of G, points in set $\left(\{1\} \times [\frac{1}{2},1]\right) \bigcup \left([\frac{1}{2},1] \times \{1\}\right)$, are mapped via G to either outside of K (to a point in $\{x \in \mathbb{R}^2 : x_1 \in [\frac{1}{4},\frac{1}{2}), x_2 = \frac{1}{2}\} \bigcup \{x \in \mathbb{R}^2 : x_2 \in [\frac{1}{4},\frac{1}{2}), x_1 = \frac{1}{2}\}$) or mapped inside K (to a point in $\{x \in \mathbb{R}^2 : x_1 \in [\frac{1}{2},\frac{3}{4}], x_2 = \frac{1}{2}\} \bigcup \{x \in \mathbb{R}^2 : x_2 \in [\frac{1}{2},\frac{3}{4}], x_1 = \frac{1}{2}\}$). Finally, a solution that starts from $(\frac{1}{2},\frac{1}{2})$, can flow either inside or outside of K due to the second piece in the definition of F. In summary, using a solution-based approach, from every point in K, there exists at least one complete solution that stays in K.

Now, using Proposition 3.4, we verify that the set K is weakly forward invariant for $\mathcal{H}$. First, $\mathcal{H}$ and K satisfy Assumption 3.2. Then, according to above analysis, condition *2.1)* in Proposition 3.3 holds, since for every $x \in K \cap D$, which is $x \in \left(\{1\} \times [\frac{1}{2},1]\right) \bigcup \left([\frac{1}{2},1] \times \{1\}\right)$, $G(x) \cap K \neq \emptyset$. Moreover, we verify that condition *2.2)* in Proposition 3.3 holds: for every point $x \in K \setminus D$, we have

$$T_{K\cap C}(x) = \begin{cases} \mathbb{R} \times \mathbb{R} & \text{if } x \in (\frac{1}{2},1) \times (\frac{1}{2},1) \\ \mathbb{R}_{\geq 0} \times \mathbb{R} & \text{if } x \in \{\frac{1}{2}\} \times (\frac{1}{2},1) \\ \mathbb{R} \times \mathbb{R}_{\geq 0} & \text{if } x \in (\frac{1}{2},1) \times \{\frac{1}{2}\} \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \text{if } x = (\frac{1}{2},\frac{1}{2}). \end{cases}$$

As a result, for every x in a neighborhood U of every $\xi \in K \setminus D$, $F(x) \cap T_{K\cap C}(x) \neq \emptyset$. Then, since F is linear everywhere on C, condition *3.2)* in Proposition 3.4 holds. Therefore, according to Proposition 3.4, K is weakly forward invariant for $\mathcal{H}$. △

We present an example inspired from [13, Example 8.3] to which we apply Proposition 3.4.

Example 5.3 (Forward Invariant Set): Consider the hybrid system $\mathcal{H} = (C, f, D, g)$ in $\mathbb{R}^2$ given by

$$f(x) := \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \qquad \forall x \in C := \{x \in \mathbb{R}^2 : |x| \leq 1, x_2 \geq 0\};$$

$$g(x) := \begin{bmatrix} -0.9x_1 \\ x_2 \end{bmatrix} \quad \forall x \in D := \{x \in \mathbb{R}^2 : x_1 \geq -1, x_2 = 0\}.$$

The set $K = \partial C$ is weakly forward invariant for $\mathcal{H}$ by Proposition 3.4. More precisely, for every $x \in K \cap D$, $g(x) \subset K$; and for every $x \in K \setminus D = \{x \in \mathbb{R}^2 : |x| = 1, x_2 > 0\}$, since $\frac{d}{dt}(x_1^2 + x_2^2) = 2x_1x_2 - 2x_1x_2 = 0$, $f(x) \subset T_{K\cap C}(x)$. In addition, $K \cap C = \partial C$ is compact. Thus, for every $x \in K$, there exists one maximal solution that is complete and stay within K. △

The invariance principle introduced in [13, Theorem 8.2] requires the computation of (the largest) weakly invariant set (inside some particular set) to characterize the set to which solutions that are bounded and complete converge. Proposition 3.4 can be helpful in such computations, in particular, to determine weakly forward invariant sets. The following example illustrates such an application of Proposition 3.4.

Example 5.4 (Determining Largest Invariant Sets): Consider the hybrid system $\mathcal{H} = (C, f, D, g)$ in $\mathbb{R}^2$ given by

$$f(x) := \begin{bmatrix} -x_2 & x_1 \end{bmatrix}^\top \qquad \forall x \in C := \mathbb{R} \times [0, +\infty),$$
$$g(x) := \begin{bmatrix} -x_2 & x_1 \end{bmatrix}^\top \qquad \forall x \in D := \mathbb{R} \times (-\infty, 0].$$

To determine where solutions to $\mathcal{H}$ converge to, using [13, Theorem 8.2], we take the Lyapunov-like function $W(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, and define the functions $u_C(x)$ and $u_D(x)$ as

$$u_C(x) := \begin{cases} \langle \nabla W(x), f(x) \rangle = 0 & \text{if } x \in C \\ -\infty & \text{otherwise} \end{cases}$$

$$u_D(x) := \begin{cases} W(g(x)) - W(x) = 0 & \text{if } x \in D \\ -\infty & \text{otherwise} \end{cases}$$

Then, following [13, Theorem 8.2], we compute the zero level set of u_C and u_D defined above. It follows that $u_C^{-1}(0) = \mathbb{R} \times [0, +\infty)$ and $u_D^{-1}(0) = \mathbb{R} \times (-\infty, 0]$. Furthermore, we have $g(u_D^{-1}(0)) = [0, +\infty) \times \mathbb{R}$. Thus, [13, Theorem 8.2] implies that every maximal solution to $\mathcal{H}$ approaches the largest weakly invariant set given by $W^{-1}(r) \cap \mathbb{R}^2 \cap [(\mathbb{R} \times [0, +\infty)) \cup ((-\infty, 0] \times (-\infty, 0])]$. Then, given an arbitrary choice of r, this set can be rewritten as $K = \{x \in \mathbb{R}^2 : |x| = r, x_1 \geq 0\} \cup \{x \in \mathbb{R}^2 : |x| = r, x_2 \geq 0\}$. The set K is weakly forward invariant according to Proposition 3.4. In fact, condition 3.1) holds since for every point in $K \cap D$, the jump map returns a point in K, and for every point in $K \cap (\overline{C} \setminus D)$ the linear oscillator dynamics permits flowing within the flow set. Condition 3.2) holds due to the properties of the flow map. △

The next example illustrates Proposition 4.1.

Example 5.5 (Forward pre-Invariance of K): Consider the hybrid system $\mathcal{H} = (C, f, D, g)$ in $\mathbb{R}^2$ given by

$$f(x) := Ax := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x \qquad \forall x \in C := \mathbb{B},$$

$$g(x) := \begin{cases} 2x & \text{if } x \in D_1 \\ -x & \text{if } x \in D_2, \end{cases} \qquad \forall x \in D := D_1 \cup D_2,$$

where $D_1 := \{x \in \mathbb{R}^2 : x \notin \mathbb{B}\}$ and $D_2 := \{x \in \mathbb{R}^2 : x_2 = 0, |x| \leq 1\}$. First, we note that the matrix A is Hurwitz, so the origin is a stable focus, i.e., solutions to $\dot{x} = f(x)$ spiral toward the origin. We consider the function $W_1(x) = x^\top P_1 x$, where $P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then, we have, for each $x \in C$, $\langle \nabla W_1(x), f(x) \rangle = -\frac{23}{4}x_1^2 - 4\left(\frac{3}{4}x_1 - x_2\right)^2$, which is guaranteed to be less than or equal to zero for every $x \in \mathbb{R}^2$. We consider the largest sublevel set of W_1 within $C = \mathbb{B}$, which is $L_{W_1}(c_1)$ with $c_1 = 1$. In addition, $g(x) = -x$ gives $W_1(g(x)) - W_1(x) = 0$ for every $x \in L_{W_1}(c_1) \cap D$. Thus, according to Proposition 4.1, $K_1 = L_{W_1}(c_1)$ is forward pre-invariant for $\widetilde{\mathcal{H}}_1 = (K_1 \cap C, f, K_1 \cap D, g)$.

Similarly, we consider the function $W_2(x) = x^\top P_2 x$, where $P_2 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. Then, we have, for each $x \in C$,

$\langle \nabla W_2(x), f(x)\rangle = -6x_1^2 - 2x_2^2 - (x_1 - x_2)^2$, which is guaranteed to be less than or equal to zero for all points on $\mathbb{R}^2$. Again, we consider the largest sublevel set for W_2 within $C = \mathbb{B}$, which is $L_{W_2}(c_2)$ with $c_2 \approx 0.793$. Similar to the case for W_1, $g(x) = -x$ gives $W_2(g(x)) - W_2(x) = 0$ for every $x \in L_{W_2}(c_2) \cap D$. Thus, according to Proposition 4.1, $K_2 = L_{W_2}(c_2)$ is forward pre-invariant for $\widetilde{\mathcal{H}}_2 = (K_2 \cap C, f, K_2 \cap D, g)$. △

An example illustrating Theorem 4.3 is presented next.

Example 5.6 (Estimating Weakly Forward Invariant Set): Consider the hybrid system $\mathcal{H} = (C, F, D, g)$ in $\mathbb{R}^2$ given by

$$F(x) := \begin{cases} [-x_2 \;\; x_1 - 0.5]^\top & \text{if } x_1 > 0 \\ [0 \;\; -0.5]^\top & \text{if } x_1 = 0 \\ [-x_2 \;\; x_1 + 0.5]^\top & \text{if } x_1 < 0 \end{cases}$$
$$\forall x \in C := ((0, 0.5) + 0.5\mathbb{B}) \bigcup ((0, -0.5) + 0.5\mathbb{B})$$
$$g(x) := \begin{cases} [1 - x_1 \;\; 0]^\top & \text{if } x \in D_1 \\ [-1 - x_1 \;\; 0]^\top & \text{if } x \in D_2 \end{cases}$$
$$\forall x \in D := D_1 \cup D_2,$$

where $D_1 := \{x \in \mathbb{R}^2 : x_2 = 0, x_1 \geq 0.5\}$ and $D_2 := \{x \in \mathbb{R}^2 : x_2 = 0, x_1 \leq -0.5\}$.

It is not possible to include every point in C using a single sublevel set of a Lyapunov like function. However, it is possible to use two different functions, W_1 and W_2, such that every point within C is captured in the union of two sublevel sets. We propose two candidates $W_1(x) = (x_1 - 0.5)^2 + x_2^2$ and $W_2(x) = (x_1 + 0.5)^2 + x_2^2$. For each $x \in \{x \in C : x_1 > 0\}$, we have $\langle \nabla W_1(x), F(x)\rangle = 0$. For each $x \in \{x \in C : x_1 < 0\}$, we have $\langle \nabla W_2(x), F(x)\rangle = 0$. Then, at the origin, $\langle \nabla W_1(x), F(x)\rangle = \langle \nabla W_2(x), F(x)\rangle = 0$.

Then, we check W at jumps. For every point in D_1 we have $W_1(g(x)) - W_1(x) = 0$, and for every point in D_2, we have $W_2(g(x)) - W_2(x) = 0$. We choose $K_1 = L_{W_1}(c_1)$ and $K_2 = L_{W_2}(c_2)$, which are subsets of C, for $\mathcal{H}$ with $c_1 = c_2 = 1$. Then, according to Proposition 4.1, K_1 is forward pre-invariant for $\widetilde{\mathcal{H}}_1 = (K_1 \cap C, F, K_1 \cap D, g)$, and K_2 is forward pre-invariant for $\widetilde{\mathcal{H}}_2 = (K_2 \cap C, F, K_2 \cap D, g)$. We verify that K_1 and K_2 are forward invariant for $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_2$, respectively, by applying Proposition 4.2. According to the data of $\widetilde{\mathcal{H}}_i, i \in \{1, 2\}$, solutions to $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_2$ can always be extended, respectively, by either flowing or jumping on K_1 and K_2, respectively.

In addition, solutions starting from the origin ($x = 0$) can either flow into the left circle or the right circle according to F. Therefore, we know neither K_1 nor K_2 is forward invariant set for the given $\mathcal{H}$. On the other hand, Theorem 4.3 implies that the sets K_1, K_2, and $K = K_1 \cup K_2$ are weakly forward invariant for $\mathcal{H}$. △

VI. Conclusion

In this paper, notions characterizing forward invariance of sets for hybrid systems and associated sufficient conditions are presented. Each notion is defined based on the existence and completeness of maximal solutions. Sufficient conditions involving properties of the data of the system for each notion are derived. A Lyapunov-like function approach to estimate weakly forward invariant sets using the proposed tools in invariance principles is presented. Applications and examples illustrating the invariance notions and results are worked out in details.

Note that the tangent cone conditions and the Lyapunov-based tools used in this work can be computationally challenging. Future efforts will target such issues with the hope of deriving algorithms that automatically compute invariant sets for hybrid systems. Future research also includes deriving necessary conditions, studying forward invariance property for hybrid systems with input; and developing results for the design of invariance-based feedback controllers.

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