# On Necessary and Sufficient Conditions for Incremental Stability of Hybrid Systems Using the Graphical Distance Between Solutions

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*Abstract*— This paper introduces new incremental stability notions for a class of hybrid dynamical systems given in terms of differential equations and difference equations with state constraints. Incremental stability is defined as the property that the distance between every pair of solutions to the system has stable behavior (incremental stability) and approaches zero asymptotically (incremental attractivity) in terms of graphical convergence. Basic properties of the class of graphically incrementally stable systems are considered as well as those implied by the new notions are revealed. Moreover, several sufficient and necessary conditions for a hybrid system with such a property are established. Examples are presented throughout the paper to illustrate the notions and results.

#### I. INTRODUCTION

In recent years, incremental stability-like properties have been used in the study of synchronization [1], [2], [3], observer design [4], [5], control design [6], as well as the study of convergent systems [7]. Unfortunately, the incremental stability notions based on Euclidean and Riemannian distance cannot be applied directly to systems with variables that can change continuously and, at times, jump discretely. These systems, known as *hybrid systems*, are capable of modeling a wide range of complex dynamical systems, including robotic, automotive, and power systems as well as natural processes. Although set stability theory in terms of Lyapunov functions is available (see [8], [9]), incremental stability notions for such systems would enable the study of properties similar to the current notion for continuoustime systems. While the initial effort in [10] defines an incremental stability notion that prioritizes ordinary time, the notion therein might be difficult to guarantee for hybrid systems that contain the so called "peaking phenomenon" as recognized in, e.g., [11], [12].

In this paper, we introduce concepts for the study of incremental graphical stability of hybrid systems that may have the "peaking phenomenon." In particular, for a class of hybrid systems in the framework of [8], we introduce new notions of incremental stability and incremental attractivity based on the graphical distance between solutions. Moreover, sufficient conditions based on those found in contraction theory and finite-time stability theory within a neighborhood of the jump set are established. Furthermore, it is shown that finite-time stability within a neighborhood of the jump set is also a necessary condition for a hybrid system to be graphically incrementally stable or attractive. To the best of our knowledge, the notion of incremental stability and its properties for hybrid systems have not been thoroughly studied, only discussed briefly in [13] for a class of transition systems in the context of bisimulations, and in [10] for a particular class of hybrid systems prioritizing ordinary time.

The remainder of this paper is organized as follows. Section II briefly reviews the hybrid system framework used and its basic properties. Section III introduces the notion of graphical incremental stability and attractivity, and illustrates them in two examples. Section IV presents the main results, which consist of several sufficient and necessary conditions. Examples are discussed throughout the paper to illustrate the results. Due to the space limitations, complete proofs will be published elsewhere.

## II. PRELIMINARIES ON HYBRID SYSTEMS AND GRAPH NOTIONS

## A. Notation

Given a set  $S \subset \mathbb{R}^n$ , the closure of S is the intersection of all closed sets containing S, denoted by  $\overline{S}$ ; S is said to be discrete if there exists  $\delta > 0$  such that for each  $x \in S$ ,  $(x+\delta \mathbb{B}) \cap S = \{x\}; \overline{\operatorname{con}}S$  is the closure of the convex hull of the set S.  $\mathbb{R}_{>0} := [0, \infty)$  and  $\mathbb{N} := \{0, 1, \dots\}$ . Given vectors  $\nu \in \mathbb{R}^n, \; w \in \mathbb{R}^m, \; |\nu|$  defines the Euclidean vector norm  $|\nu| = \sqrt{\nu^{\top} \nu}$  and  $[\nu^{\top} \ w^{\top}]^{\top}$  is equivalent to  $(\nu, w)$ . Given a function  $f : \mathbb{R}^m \to \mathbb{R}^n$ , its domain of definition is denoted by dom f, i.e., dom  $f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}.$ The range of f is denoted by rge f, i.e., rge  $f := \{f(x) :$  $x \in \text{dom } f$ . The right limit of the function f is defined as  $f^+(x) := \lim_{\nu \to 0^+} f(x+\nu)$  if it exists. Given a point  $y \in \mathbb{R}^n$  and a closed set  $\mathcal{A} \subset \mathbb{R}^n$ ,  $|y|_{\mathcal{A}} := \inf_{x \in \mathcal{A}} |x - y|$ . A function  $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$  is a class- $\mathcal{K}_{\infty}$  function, also written  $\alpha \in \mathcal{K}_{\infty}$ , if  $\alpha$  is zero at zero, continuous, strictly increasing, and unbounded;  $\alpha$  is positive definite, also written  $\alpha \in \mathcal{PD}$ , if  $\alpha(s) > 0$  for all s > 0 and  $\alpha(0) = 0$ . Given a real number  $x \in \mathbb{R}$ , floor(x) is the closest integer to x from below. A function  $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$  is called a Lyapunov function with respect to a set A if V is continuously differentiable and such that  $c_1(|x|_{\mathcal{A}}) \leq V(x) \leq c_2(|x|_{\mathcal{A}})$  for all  $x \in \mathbb{R}^n$  and some functions  $c_1, c_2 \in \mathcal{K}_{\infty}$ .

## B. Preliminaries

In this paper, a hybrid system  $\mathcal{H}$  has data (C, f, D, g) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &= g(z) \quad z \in D, \end{aligned}$$
 (1)

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where  $z \in \mathbb{R}^n$  is the state, the map f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map q defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution  $\phi$  to  $\mathcal{H}$  is parametrized by  $(t, j) \in \mathbb{R}_{>0} \times \mathbb{N}$ , where t denotes ordinary time and j denotes jump time. The domain dom  $\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$  is a hybrid time domain when it satisfies [8, Definition 2.3]. A solution to  $\mathcal{H}$  is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set  $\mathcal{S}_{\mathcal{H}}$ contains all maximal solutions to  $\mathcal{H}$ , and the set  $\mathcal{S}_{\mathcal{H}}(\xi)$ contains all maximal solutions to  $\mathcal{H}$  from  $\xi$ . A hybrid system  $\mathcal{H}$  is said to satisfy the hybrid basic conditions if it satisfies [8, Definition 6.5]. The property of pre-forward completeness of a hybrid system  $\mathcal{H}$  is characterized in [8, Definition 6.12]. The graph of a hybrid arc is defined in [8, Definition 5.20], and the distance of graphs of two hybrid arcs is measured by  $\varepsilon$ -closeness notion in [8, Definition 4.11]. In order to characterize the property of hybrid arcs graphically converging to each other, we introduce the following notion.

Definition 2.1: Given  $\varepsilon > 0$ , two hybrid arcs  $\phi_1$  and  $\phi_2$  are eventually  $\varepsilon$ -close if there exists T > 0 such that

(a) for each  $(t, j) \in \text{dom } \phi_1$  such that t + j > T, there exists  $(s, j) \in \text{dom } \phi_2$  satisfying  $|t - s| < \varepsilon$  and

$$|\phi_1(t,j) - \phi_2(s,j)| < \varepsilon, \tag{2}$$

(b) for each (t, j) ∈ dom φ<sub>2</sub> such that t + j > T, there exists (s, j) ∈ dom φ<sub>1</sub> satisfying |t − s| < ε and</p>

$$|\phi_2(t,j) - \phi_1(s,j)| < \varepsilon. \tag{3}$$

We refer the reader to [8] and [9] for more details on these notions and the hybrid systems framework.

#### III. INCREMENTAL GRAPHICAL STABILITY

In this paper, for hybrid systems  $\mathcal{H}$  as in (1), we are interested in characterizing the incremental stability property, namely, the notion that the graphical distance between every pair of maximal solutions to the system has stable behavior and approaches zero asymptotically. To highlight the intricacies of this property in the hybrid setting, a canonical example of hybrid systems, the so-called bouncing ball system, is considered.

Every maximal solution to the bouncing ball system is Zeno<sup>1</sup> and converges to the origin; see [8, Example 1.1 and 2.12] for more details. Consider two solutions to this system, given by  $\phi_1$  and  $\phi_2$ , from initial conditions  $\phi_1(0,0) = (5,0)$ (ball initialized at a positive height with zero velocity) and  $\phi_2(0,0) = (0,3)$  (ball initialized at the ground with a positive velocity). Figure 1(a) shows the position (first) component ( $\phi_i^1$  for  $i \in \{1,2\}$ ) of these two solutions, and Figure 1(b) shows the velocity (second) component ( $\phi_i^2$  for  $i \in \{1,2\}$ ) of them. The Zeno behavior of the solutions makes it

 ${}^{1}A$  solution is Zeno if it is complete and its domain is bounded in the *t* direction.



(a) The first component (height  $\phi_i^1$ ) of solutions from  $\phi_1(0,0) = (5,0)$  and  $\phi_2(0,0) = (0,3)$ .

(b) The projection of the second component (velocity  $\bar{\phi}_i^2$ ) of solutions from  $\bar{\phi}_1(0,0) = (3,3)$  and  $\bar{\phi}_2(0,0) = (3,3.1)$  on the *t* direction.

Fig. 1. Solutions to the bouncing ball system. The Euclidean distance, which is  $d(\bar{\phi}_1^2(t, j_1), \bar{\phi}_2^2(t, j_2)) := |\bar{\phi}_1^2(t, j_1) - \bar{\phi}_2^2(t, j_2)|$  for all  $(t, j_1) \in \text{dom } \bar{\phi}_1$  and  $(t, j_2) \in \text{dom } \bar{\phi}_2$ , has repetitive large peaks.

extremely difficult to analyze incremental property of these two solutions. In fact, since graphical incremental stability requires comparing the graphs between two solutions, if one solution ( $\phi_1$ ) reaches the Zeno time sooner than the second solution ( $\phi_2$ ), then the graphical distance between these two solutions cannot be evaluated after one of them has approached the Zeno time. This situation is shown in Figure 1(a), where  $\phi_2$  approaches Zeno at about  $t = 6 \sec$ while  $\phi_1$  is still describing the motion of the ball bouncing. Such extreme difference in the domains of the solutions makes it difficult (if not impossible) to compare  $\phi_1$  and  $\phi_2$ .

Furthermore, a similar situation is encountered if, instead, the pointwise distance is used to measure the distance between solutions. As shown in Figure 1(b), the pointwise distance of velocity (second) components of two solutions  $(\bar{\phi}_i^2 \text{ for } i \in \{1, 2\})$  has repetitive large peaks, even though they are initialized very close to each other.

To avoid such possible dramatic differences on the hybrid time domains and distances between solutions to a hybrid system, throughout the paper, we consider a class of hybrid systems that satisfies the following assumption.

Assumption 3.1: The hybrid system  $\mathcal{H} = (C, f, D, g)$  is such that

- 1) each maximal solution to  $\mathcal{H}$  has a hybrid time domain that is unbounded in the t direction;
- 2) the flow map f is continuously differentiable;
- there exists γ > 0 such that for each maximal solution to H, the flow time between two consecutive jumps is lower bounded by γ.

Lemma 2.7 in [14] provides a sufficient condition for condition 3). As we show later on, these assumptions are not restrictive, in fact, some of them are necessary for establishing sufficient conditions for a hybrid system to be graphically incrementally stable.

The notion of incremental stability used in this work measures the graph distance between solutions to hybrid systems. It is defined as follows.

Definition 3.2: Consider a hybrid system  $\mathcal{H}$  with state  $z \in \mathbb{R}^n$ . The hybrid system  $\mathcal{H}$  is said to be

1) *incrementally graphically stable* ( $\delta$ S) if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for any two maximal

solutions  $\phi_1, \phi_2$  to  $\mathcal{H}, |\phi_1(0,0) - \phi_2(0,0)| \le \delta$  implies  $\phi_1$  and  $\phi_2$  are  $\varepsilon$ -close;

- 2) incrementally graphically locally attractive ( $\delta$ LA) if there exists  $\mu > 0$  such that for every  $\varepsilon > 0$ , for any two maximal solutions  $\phi_1, \phi_2$  to  $\mathcal{H}, |\phi_1(0,0) - \phi_2(0,0)| \le \mu$ implies that  $\phi_1$  and  $\phi_2$  are eventually  $\varepsilon$ -close;
- incrementally graphically globally attractive (δGA) if for every ε > 0, and for any two maximal solutions φ<sub>1</sub>, φ<sub>2</sub> to H, φ<sub>1</sub> and φ<sub>2</sub> are eventually ε-close;
- incrementally graphically locally asymptotically stable (δLAS) if it is both δS and δLA.

*Remark 3.3:* Note that the  $\delta$ LA notion is different from the  $\delta$ S notion. The former requires that every pair of maximal solutions to  $\mathcal{H}$  initialized close converge to each other graphically when complete, while the latter requires that every pair of two maximal solutions to  $\mathcal{H}$  initialized close stay close graphically.

The following examples illustrate some of the incremental stability notions ( $\delta$ S and  $\delta$ LAS) in Definition 3.2.

*Example 3.4:* (A timer) Consider the hybrid system  $\mathcal{H}$ 

$$\dot{z} = 1$$
  $z \in [0, 1],$  (4)

$$z^+ = 0$$
  $z = 1.$  (5)

Note that every maximal solution to  $\mathcal{H}$  is complete. Consider two maximal solutions  $\phi_1$ ,  $\phi_2$  to the system. To show  $\delta S$ , for a given  $\varepsilon > 0$ , let  $0 < \delta < \varepsilon$  and assume  $|\phi_1(0,0) - \phi_2(0,0)| < \delta$ . Without loss of generality, we further suppose  $\phi_1(0,0) > \phi_2(0,0)$ . Then, solution  $\phi_1$  jumps before  $\phi_2$ . For each  $j \in \mathbb{N} \setminus \{0\}$ , let  $\overline{t}_j = \max_{(t,j-1)\in \text{dom } \phi_1 \cap \text{dom } \phi_2 t}$  and  $\overline{t}'_j = \min_{(t,j)\in \text{dom } \phi_1 \cap \text{dom } \phi_2 t}$ . Then, we have that for each  $t \in [0, \overline{t}_1]$ , there exists  $(s, 0) \in \text{dom } \phi_2$  such that s = t and  $|\phi_1(t, 0) - \phi_2(t, 0)| = |\phi_1(0, 0) + t - \phi_2(0, 0) - t| \le \delta < \varepsilon$ . (6) For each  $t \in [\overline{t}_1, \overline{t}'_1]$ ,

 $\begin{aligned} |\phi_1(\bar{t}_1, 0) - \phi_2(t, 0)| &= |1 - \phi_2(0, 0) - t| \\ &\leq |\phi_1(0, 0) - \phi_2(0, 0)| \leq \delta < \varepsilon, \end{aligned}$ (7)

where we used the fact that  $\phi_1(0,0) + \bar{t}_1 = 1$ . Moreover, for each  $t \in [\bar{t}_1, \bar{t}'_1]$ ,

$$|\phi_1(t,1) - \phi_2(\bar{t}'_1,1)| = |t - \bar{t}_1| \le \delta < \varepsilon.$$
(8)

Note that  $|t - \bar{t}_1| \leq \delta < \varepsilon$  holds for all  $t \in [\bar{t}_1, \bar{t}_1']$ . In fact, for each  $t \in [\bar{t}'_{i-1}, \bar{t}_i]$ , where  $i \in \mathbb{N} \setminus \{0, 1\}$ ,

 $\begin{aligned} |\phi_1(t, i-1) - \phi_2(t, i-1)| \\ &= |(\phi_1(\vec{t}'_{i-1}, i-1) - \phi_2(\vec{t}'_{i-1}, i-1))| \le \delta < \varepsilon. \end{aligned}$ Moreover, for each  $t \in [\vec{t}_i, \vec{t}'_i]$ , where  $i \in \mathbb{N} \setminus \{0, 1\}$ .

$$\begin{aligned} |\phi_1(\bar{t}_i, i-1) - \phi_2(t, i-1)| \\ &= |\phi_1(\bar{t}_i, i-1) - \phi_2(\bar{t}_i, i-1) - (t-\bar{t}_i)| < \varepsilon, \end{aligned}$$

and  $|\phi_1(t,i) - \phi_2(\bar{t}'_i,i)| = |t - \bar{t}_i| \le \delta < \varepsilon$ . Therefore, the system is  $\delta S$ . Moreover, since the distance between  $\phi_1$ and  $\phi_2$  does not converge to zero,  $\phi_1, \phi_2$  are not eventually  $\varepsilon$ -close and thus the system is neither  $\delta LA$  nor  $\delta GA$ .

As shown in Figure 2(a), the domains of two solutions to the timer system are different from each other. In such case, the Euclidean distance may not be a good candidate of a distance function for the study of incremental properties,



(a) The projections of two solutions (b) Comparison between Euclidean from  $\phi_1(0,0) = 0$  and  $\phi_2(0,0) =$  distance (top) and graphical distance (bottom) between  $\phi_1$  and  $\phi_2$ .

Fig. 2. Two solutions  $\phi_1$  and  $\phi_2$  to the timer system in Example 3.4. Unlike the Euclidean distance, which is  $d(\phi_1(t, j_1), \phi_2(t, j_2)) := |\phi_1(t, j_1) - \phi_2(t, j_2)|$  for all  $(t, j_1) \in \operatorname{dom} \phi_1$  and  $(t, j_2) \in \operatorname{dom} \phi_2$ , which assumes the value 0.8 for 0.2 seconds after every 0.8 seconds, the graphical distance satisfies  $d_g(\phi_1(t, j), \phi_2) := \min\{|\phi_1(t, j) - \phi_2(s, j)| : |t - s| \leq 0.2, (t, j) \in \operatorname{dom} \phi_1, (s, j) \in \operatorname{dom} \phi_2\}$ , which is upper bounded by 0.2.

as shown in the top subfigure of Figure 2(b). Note that no matter how close the two maximal solutions are initialized, the peak always exists for the Euclidean distance between them. However, the graphical distance between the solutions, as shown in the bottom subfigure of Figure 2(b), is bounded by 0.2 for all hybrid time in the respective domain of the solutions. We will present a systematic way to check the property later on in this paper.<sup>2</sup>

*Example 3.5:* Consider a hybrid system  $\mathcal{H}_1$  with data

$$f(z) := -z \qquad \forall \ z \in C := \bigcup_{i \in \{2k:k \in \mathbb{N}\}} [i, i+1]$$
  
$$g(z) := z - 1 \qquad \forall \ z \in D := \{2k:k \in \mathbb{N} \setminus \{0\}\}.$$
(9)

Note that every maximal solution to  $\mathcal{H}_1$  is complete. Given  $\varepsilon > 0$ , consider two maximal solutions  $\phi_1$  and  $\phi_2$  such that  $|\phi_1(0,0) - \phi_2(0,0)| < \delta$ , where  $\delta = \min\{1,\varepsilon\}$ . Then, it is guaranteed that  $\overline{J} := \sup_{(t,j)\in \text{dom }\phi_1} j = \sup_{(t,j)\in \text{dom }\phi_2} j < \infty$ . For each  $j \in \mathbb{N} \setminus \{0\}$ , let  $\overline{t}_j = \max_{(t,j-1)\in \text{dom }\phi_1\cap \text{dom }\phi_2} t$  and  $\overline{t}'_j = \min_{(t,j)\in \text{dom }\phi_1\cap \text{dom }\phi_2} t$ . Without loss of generality, assume  $\phi_2(0,0) > \phi_1(0,0) \ge 2$ , then  $\phi_1$  jumps first. Then, we have that for each  $t \in [0, \overline{t}_1]$ , there exists  $(s,0) \in \text{dom }\phi_2$  such that s = t and

$$|\phi_1(t,0) - \phi_2(t,0)| = |\phi_1(0,0)e^{-t} - \phi_2(0,0)e^{-t}| \le \delta < \varepsilon.$$
(10)  
For each  $t \in [\bar{t}, \bar{t}'_{-}]$ 

For each 
$$t \in [t_1, t'_1]$$
,

$$\begin{aligned} |\phi_1(\bar{t}_1, 0) - \phi_2(t, 0)| &= |e^{-t_1}\phi_1(0, 0) - e^{-t}\phi_2(0, 0)| \\ &\leq |e^{-\bar{t}_1}\phi_1(0, 0) - e^{-\bar{t}_1}\phi_2(0, 0)| \leq \delta < \varepsilon, \end{aligned}$$
(11)

where we used the property  $e^{-\bar{t}_1}\phi_1(0,0) = \text{floor}(\phi_1(0,0)) = \text{floor}(\phi_2(0,0)) = e^{-\bar{t}_1'}\phi_2(0,0)$ . Note that  $\bar{t}_1 = \ln(\phi_1(0,0)) - \ln(\text{floor}(\phi_1(0,0)))$  and  $\bar{t}_1' = \ln(\phi_2(0,0)) - \ln(\text{floor}(\phi_2(0,0)))$ , therefore,  $\bar{t}_1' - \bar{t}_1 = \ln(\phi_2(0,0)) - \ln(\phi_1(0,0))$ . Furthermore, by mean value theorem, there exists  $\phi_0^* \in [\phi_1(0,0), \phi_2(0,0)]$  such that  $|\bar{t}_1' - \bar{t}_1| = \frac{1}{\phi_0^*}|\phi_1(0,0) - \phi_2(0,0)| \le |\phi_1(0,0) - \phi_2(0,0)| \le \delta < \varepsilon$ . Similarly, for each  $t \in [\bar{t}_1, \bar{t}_1']$ ,

$$|\phi_1(t,1) - \phi_2(\bar{t}'_1,1)| \le \phi_2(0,0) - \phi_1(0,0) \le \delta.$$
 (12)

If 
$$\phi_1(0,0), \phi_2(0,0) \leq 1$$
, we have that  $|\phi_1(t,0) - \phi_2(0,0)| \leq 1$ 

 $^{2}$ It may be possible to construct an alternative distance function that is decreasing along solutions using the ideas in [15].



(a) The projections of two solutions (b) Comparison between graphical from  $\phi_1(0,0) = 4.5$  and  $\phi_2(0,0) =$ 4.2 on the t direction.

distance and Euclidean distance between  $\phi_1$  and  $\phi_2$ .

Fig. 3. Two solutions  $\phi_1$  and  $\phi_2$  to the system  $\mathcal{H}_1$ . Unlike the Euclidean distance, which is  $d(\phi_1(t, j_1), \phi_2(t, j_2)) := |\phi_1(t, j_1) - \phi_2(t, j_2)|$  for all  $(t, j_1) \in \operatorname{dom} \phi_1$  and  $(t, j_2) \in \operatorname{dom} \phi_2$ , which does not decrease along solutions, the graphical distance satisfies  $d_g(\phi_1(t, j), \phi_2) :=$  $\min\{|\phi_1(t,j) - \phi_2(s,j)| : |t-s| \le 0.2, (t,j) \in \operatorname{dom} \phi_1, (s,j) \in$ dom  $\phi_2$ }, which decreases along solutions.

 $|\phi_2(t,0)| \leq e^{-t} |\phi_1(0,0) - \phi_2(0,0)| \leq \delta$  for all  $(t,0) \in$ dom  $\phi_1 = \operatorname{dom} \phi_2$ . In fact,  $\lim_{t \to \infty, (t,0) \in \operatorname{dom} \phi_1} |\phi_1(t,0) - \phi_1| = \operatorname{dom} \phi_2$ .  $\phi_2(t,0) = 0$ . Therefore, the system  $\mathcal{H}_1$  is  $\delta$ LAS.

As shown in Figure 3(a), the domains of two solutions to the system  $\mathcal{H}_1$  are different from each other. The Euclidean distance between  $\phi_1$  and  $\phi_2$  is shown in the top subfigure of Figure 3(b), while the graphical distance between them, as shown in the bottom subfigure of Figure 3(b), converges to zero asymptotically. Δ

## **IV. SUFFICIENT AND NECESSARY CONDITIONS** FOR GRAPHICAL INCREMENTAL STABILITY NOTIONS

In this section, we present several sufficient and necessary conditions of incremental graphical stability properties for hybrid systems that satisfy Assumption 3.1.

The following result highlights a necessary property of the hybrid time domains of two hybrid arcs that are graphically close. In particular, it holds for solutions to a hybrid system  $\mathcal{H}$  that is  $\delta S$  or  $\delta GA$ .

*Lemma 4.1: Given*  $\varepsilon > 0$  *and two hybrid arcs*  $\phi_1$  *and*  $\phi_2$ *,* the following hold:

1) If  $\phi_1$  and  $\phi_2$  are  $\varepsilon$ -close, then

$$\sup_{(t,j)\in\operatorname{dom}\phi_1} j = \sup_{(t,j)\in\operatorname{dom}\phi_2} j.$$
(13)

2) If  $\phi_1$  and  $\phi_2$  are complete and eventually  $\varepsilon$ -close, then (13) holds.

Example 4.2: Consider the Example 3.5, and two solutions  $\phi_1$  and  $\phi_2$  to  $\mathcal{H}_1$  from  $\phi_1(0,0) = 4.5$  and  $\phi_2(0,0) = 1$ , respectively. Solution  $\phi_1$  jumps twice while solution  $\phi_2$  never jumps. Therefore,  $\phi_1$  and  $\phi_2$  is not eventually  $\varepsilon$ -close which prevents the system from being  $\delta GA$  even though it is shown to be  $\delta$ LAS in Example 3.5.  $\triangle$ 

The following result establishes that uniqueness is a necessary condition for  $\delta S$ . In turn, in particular, it justifies the choice of single-valued flow and jump maps in the definition of  $\mathcal{H}$  in (1).

Proposition 4.3: (uniqueness of solutions under  $\delta S$ ) Consider a hybrid system  $\mathcal{H}$  with state  $z \in \mathbb{R}^n$ . Suppose  $\mathcal{H}$ satisfies Assumption 3.1 and  $\mathcal{H}$  is  $\delta S$ . Then, every maximal solution to  $\mathcal{H}$  is unique.

Based on Proposition 4.3, assuming uniqueness of solutions to  $\mathcal{H}$  is not at all restrictive when studying incremental graphical stability. Hence, in the following results we impose the following uniqueness of solutions assumption.

Assumption 4.4: The hybrid system  $\mathcal{H} = (C, f, D, q)$  is such that each maximal solution  $\phi$  to  $\mathcal{H}$  is unique.

A sufficient condition for guaranteeing uniqueness of maximal solutions requires f to be locally Lipschitz and no flow from  $C \cap D$ , see [8, Proposition 2.11].

When the jump set D is a discrete set, the following sufficient conditions for a hybrid system  $\mathcal{H}$  to be  $\delta LAS$ are established. In particular, the graphical distance between any two maximal solutions to a hybrid system  $\mathcal{H}$  strictly decreases during flows.

Theorem 4.5: ( $\delta LAS$  through flow with D being a discrete set) Consider a hybrid system  $\mathcal{H} = (C, f, D, g)$  with state  $z \in \mathbb{R}^n$ . Suppose  $\mathcal{H}$  satisfies Assumption 3.1, Assumption 4.4, and the hybrid basic conditions. Moreover, suppose D is a discrete set. If there exist  $\beta > 0$  and  $\delta_0 > 0$  such that  $\mathcal{H}$ satisfies

1)  $\nabla f(z) + \nabla f^{\top}(z) \leq -2\beta I$  for all  $z \in \overline{\operatorname{con}}C$ ;

2) for each  $\delta \in [0, \delta_0]$ , each maximal solution  $\phi$  to  $\mathcal{H}$  from  $\phi(0,0)$  satisfying

$$\phi(0,0) \in C, \quad |\phi(0,0)|_D = \delta$$
 (14)

is such that there exists  $s \in [0, \delta]$  satisfying

$$|\phi(s,0)|_D = 0, \quad |\phi(t,0)|_D \le \delta \quad \forall t \in [0,s], \quad (15)$$

and the maximal solution  $\overline{\phi}$  to  $\mathcal{H}$  from  $q(\phi(s,0))$ satisfies

$$|\bar{\phi}(t,0) - g(\phi(s,0))| \le \delta \quad \forall t \in [0,s];$$
(16)

then,  $\mathcal{H}$  is  $\delta LAS$ .

Sketch of the Proof: Given  $\varepsilon > 0$ , and using  $\delta_0$  as in the item 2) of assumption and  $\gamma$  as in Assumption 3.1, consider two maximal solutions  $\phi_1$ ,  $\phi_2$  to  $\mathcal{H}$  such that  $|\phi_1(0,0) - \phi_2(0,0)| < \delta$ , where  $\delta$  is chosen such that  $0 < \delta$  $\delta \leq \min\{\varepsilon, \delta_0, \gamma\}$  and for each  $z \in D$ ,  $(z + \delta \mathbb{B}) \cap D = \{z\}$ .

If  $\phi_1(0,0), \phi_2(0,0) \in C$  and no jump occurs to either  $\phi_1$  or  $\phi_2$ , then, by the generalized mean value theorem for vectorvalued functions and item 1), for all  $t \in (0, \infty)$ , we have

$$\frac{d}{dt} |\phi_1(t,0) - \phi_2(t,0)|^2 \le -2\beta |\phi_1(t,0) - \phi_2(t,0)|^2.$$
(17)

Then, by the comparison lemma, for all  $t \in [0, \infty)$ ,

$$\phi_1(t,0) - \phi_2(t,0) | \le \exp(-\beta t) |\phi_1(0,0) - \phi_2(0,0)| \le \delta.$$
 (18)

Now consider the case when either  $\phi_1$  or  $\phi_2$  jump. Assume  $\phi_1$  jumps first and  $J = \infty$ . Furthermore, for each  $j \in \mathbb{N} \setminus \{0\}$ , let  $\overline{t}_j = \max_{(t,j-1) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$  and  $\overline{t}'_j = \min_{(t,j) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$ , and  $\overline{t}'_0 = 0$ .

**Case I:** When  $\phi_1(0,0), \phi_2(0,0) \in C$ , similarly as in (17), for all  $t \in [0, \bar{t}_1]$ , we have that

$$|\phi_1(t,0) - \phi_2(t,0)| \le \delta.$$
(19)

When  $t = \bar{t}_1$ , since  $\phi_1$  jumps first,  $\phi_1(\bar{t}_1, 0) \in D$  and  $\phi_1(\bar{t}_1, 1) = g(\phi_1(\bar{t}_1, 0))$ . Note that under item 3) of Assumption 3.1,  $g(D) \cap D = \emptyset$ . Therefore, by using (17) and item 2), we obtain

a) for each 
$$j \in \mathbb{N} \setminus \{0\}$$
 and each  $t \in [\vec{t}'_j, \bar{t}_{j+1}]$ :  
 $|\phi_1(t, j) - \phi_2(t, j)|$   
 $\leq \exp(-\beta(t - \vec{t}'_j + \Delta_j))|\phi_1(0, 0) - \phi_2(0, 0)| \leq \varepsilon,$ 
(20)

where  $\Delta_j := \sum_{k=1}^{j} (\bar{t}_k - \bar{t}'_{k-1}),$ b) for each  $j \in \mathbb{N} \setminus \{0\}$  and each  $t \in [\bar{t}_j, \bar{t}'_j]:$  $|\phi_2(t, j-1) - \phi_1(\bar{t}_j, j-1)|$ 

$$\sum_{j=1}^{j} |\phi_1(i_j, j-1)| \leq \varepsilon,$$

$$\sum_{j=1}^{j} |\phi_1(0, 0) - \phi_2(0, 0)| \leq \varepsilon,$$

$$i \in \mathbb{N} \setminus \{0\} \text{ and each } t \in [\overline{t}, \overline{t}.]:$$

$$(21)$$

c) for each  $j \in \mathbb{N} \setminus \{0\}$  and each  $t \in [t_j, t'_j]$ :

$$\begin{aligned} &|\phi_1(t,j) - \phi_2(t_j,j)| \\ &\leq \exp(-\beta \Delta_j) |\phi_1(0,0) - \phi_2(0,0)| \leq \varepsilon. \end{aligned} (22)$$

Therefore,  $\phi_1$  and  $\phi_2$  are  $\varepsilon$ -close.

**Case II** If  $\phi_1(0,0), \phi_2(0,0) \in D$ , since *D* is a discrete set and  $|\phi_1(0,0) - \phi_2(0,0)| \leq \delta$ ,  $\phi_1(0,0) = \phi_2(0,0)$ . By the uniqueness of solutions, we have that  $\phi_1(t,j) = \phi_2(t,j)$  for all  $(t,j) \in \text{dom } \phi_1$ . Thus,  $\phi_1$  and  $\phi_2$  are  $\varepsilon$ -close.

**Case III** If  $\phi_1(0,0) \in C, \phi_2(0,0) \in D$ , the arguments follows similarly as in Case I.

Therefore, by combining arguments in three cases, it is proved that  $\phi_1$  and  $\phi_2$  are  $\varepsilon$ -close which implies  $\mathcal{H}$  is  $\delta S$ . Note that the case when  $J < \infty$  follows similarly with  $[\tilde{t}'_J, \infty) \times \{J\} \subset \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2$  and above.

On the other hand, the  $\delta$ LA property follows from (20), (21) and (22).

*Remark 4.6:* The first condition in Theorem 4.5 guarantees strict decrease of the graphical distance between two maximal solutions on the intersections of their hybrid time domains. The second condition in Theorem 4.5 implies that on the mismatched parts of their hybrid time domains, the graphical distance between them does not grow.

The condition proposed in item 2) of Theorem 4.5 can be guaranteed by the following sufficient condition.

Proposition 4.7: Consider a hybrid system  $\mathcal{H} = (C, f, D, g)$  with state  $z \in \mathbb{R}^n$ . Suppose  $\mathcal{H}$  satisfies Assumption 3.1, Assumption 4.4, and the hybrid basic conditions. Moreover, suppose D is a discrete set. Then, item 2) of Theorem 4.5 holds if there exists  $\delta_0 > 0$  such that, for any  $z^* \in D$ , the following hold: there exist  $c_1, c_2 > 0$ ,  $c_2 \in (0, c_1]$ , and  $\alpha \in (0, 1)$  such that

- 1) for  $V_1(z) = |z z^*|^2$ , we have  $\langle \nabla V_1(z), f(z) \rangle + c_1 V_1^{\alpha}(z) < 0$  and  $|z z^*|^{1-2\alpha} \leq c_1(1-\alpha)$  for all  $z \in C \cap ((z^* + \delta_0 \mathbb{B}) \setminus D)$ ,
- 2) for  $V_2(z) = |z g(z^*)|^2$ , we have  $\langle \nabla V_2(z), f(z) \rangle c_2 V_2^{\alpha}(z) < 0$  for all  $z \in C \bigcap (g(z^*) + \delta_0 \mathbb{B})$ .

The following example illustrates the sufficient condition in Theorem 4.5.

*Example 4.8:* Consider the system in Example 3.5 and the set  $M = [0, \overline{M}]$ , where  $\overline{M} > 0$ . For the hybrid system  $\mathcal{H}_M = (C \cap M, f, D \cap M, g)$ , the conditions in Theorem 4.5 can be verified as follows. Each maximal solution to  $\mathcal{H}_M$  is complete and its domain is unbounded in the t direction. Moreover, the flow map is continuously differentiable on  $\overline{\operatorname{con}(C \cap M)}$ . Furthermore, for any maximal solution  $\phi$  to the system  $\mathcal{H}_M$  from  $\phi(0,0) \in (C \cup D) \cap M$ , denote  $\begin{array}{ll} d^{\star}:=\max\{x:x\in C,x\leq \phi(0,0)\}. \mbox{ If }\phi(0,0)\leq 1,\mbox{ then }\phi\mbox{ never jumps and the jump time between two consecutive jumps of }\phi\mbox{ is bounded below by }\infty. \mbox{ If }d^{\star}\geq 2,\mbox{ the flow time between two consecutive jumps of }\phi\mbox{ is bounded below by }\ln\frac{d^{\star}}{d^{\star}-1}\leq \ln\frac{\mathrm{floor}(\overline{M}+2)}{\mathrm{floor}(\overline{M}+1)}.\mbox{ For all }z\in\overline{\mathrm{con}}(C\cap M),\nbox{ }\nabla f(z)+\\ \nabla f(z)^{\top}=-2,\mbox{ so item 1})\mbox{ in Theorem 4.5 is satisfied with }\beta=1.\mbox{ Moreover, given }z^{\star}\in D\cap M,\mbox{ the Lyapunov function }V_1(z)=|z-z^{\star}|^2\mbox{ satisfies }\langle\nabla V_1(z),f(z)\rangle=\\ 2(z-z^{\star})(-z)\leq -2z^{\star}(z-z^{\star})=-2z^{\star}V_1^{\frac{1}{2}}(z)\mbox{ for }z\in (M\cap C)\bigcap((z^{\star}+\ln\frac{d^{\star}}{d^{\star}-1}\mathbb{B})\setminus(M\cap D)),\mbox{ where we used the property that }z\geq z^{\star}\mbox{ for all }z\in (M\cap C)\bigcap((z^{\star}+\ln\frac{d^{\star}}{d^{\star}-1}\mathbb{B})\setminus(M\cap D)).\mbox{ Furthermore, the Lyapunov function }V_2(z)=|z-g(z^{\star})|^2\mbox{ satisfies }\langle\nabla V_2(z),f(z)\rangle=2(z-g(z^{\star}))(-z)\leq 2z^{\star}(g(z^{\star})-z)=2g(z^{\star})V_2^{\frac{1}{2}}(z)\mbox{ for }z\in (g(z^{\star})+\ln\frac{d^{\star}}{d^{\star}-1}\mathbb{B})\bigcap(C\cap M),\mbox{ where we used the property that }z\leq g(z^{\star})\mbox{ for all }z\in (g(z^{\star})+\ln\frac{d^{\star}}{d^{\star}-1}\mathbb{B})\cap(C\cap M)\mbox{ and }g(z^{\star})=z^{\star}-1<<z^{\star}.\mbox{ Then, by Proposition 4.7 and Theorem 4.5, we have that }\mathcal{H}_M\mbox{ is }\delta\text{LAS}.\end{tabular}$ 

It follows that the finite-time convergence property in item 2) of Theorem 4.5 is a necessary condition for a hybrid system  $\mathcal{H}$  to be  $\delta S$  or  $\delta LA$ . Indeed, without the finite-time convergence property nearby D and g(D), the graphs of the solutions would not be close.

Theorem 4.9: (necessary condition for  $\delta S$  and  $\delta LA$ ) Consider a hybrid system  $\mathcal{H}$  with state  $z \in \mathbb{R}^n$ . Suppose  $\mathcal{H}$  satisfies Assumption 3.1, Assumption 4.4, and the hybrid basic conditions. Furthermore, suppose  $\mathcal{H}$  is  $\delta S$  or  $\delta LA$ . Then, there exists  $\delta_0 > 0$  such that each maximal solution  $\phi$  to  $\mathcal{H}$  from  $\phi(0,0)$  satisfying  $|\phi(0,0)|_D \leq \delta_0$  and  $\phi(0,0) \in C$  converges to D within finite time, i.e., there exists s > 0 such that  $|\phi(s,0)|_D = 0$ .

Under further assumptions, the above results for a discrete jump set can be extended to the case of a generic set D. For this purpose, we introduce the following forward invariant notion.

Definition 4.10: (forward invariance from away of D) A set  $\mathcal{A} \subset \mathbb{R}^n$  is said to be forward invariant for  $\mathcal{H}$  from away of D if for each solution  $\phi$  to  $\mathcal{H}$  from  $\phi(0,0) \in \mathcal{A} \setminus D$ ,  $\phi(t,0) \in \mathcal{A}$  for all  $(t,0) \in \operatorname{dom} \phi$ .

Now, we are ready to present sufficient conditions for a generic jump set.

Theorem 4.11: ( $\delta$ LAS through flow for generic D) Consider a hybrid system  $\mathcal{H} = (C, f, D, g)$  with state  $z \in \mathbb{R}^n$ . Suppose  $\mathcal{H}$  satisfies Assumption 3.1, Assumption 4.4, and the hybrid basic conditions. If there exist  $\beta > 0$  and  $\delta_0 > 0$  such that  $\mathcal{H}$  satisfies

- 1)  $\nabla f(z) + \nabla f^{\top}(z) \leq -2\beta I$  for all  $z \in \overline{\operatorname{con}}C$ ;
- 2) for each  $\delta \in [0, \delta_0]$ , each maximal solution  $\phi$  to  $\mathcal{H}$  satisfying

$$\phi(0,0) \in C, \quad |\phi(0,0)|_D = \delta$$
 (23)

is such that there exists  $s \in [0, \delta]$  satisfying

$$|\phi(s,0)|_D = 0 \tag{24}$$

and the set  $\phi(s, 0) + \delta \mathbb{B}$  is forward invariant from away

of D, and each maximal solution  $\overline{\phi}$  to H from

$$\bar{\phi}(0,0) \in g(\phi(s,0)) + \delta \mathbb{B}$$
(25)

satisfies

$$\bar{\phi}(t,0) \in g(\phi(s,0)) + \delta \mathbb{B}$$
(26)

for all  $t \in [0, s]$ ;

3) the jump map g is locally Lipschitz on D with Lipschitz constant  $L_1 \in [0, 1]$ ,<sup>3</sup> i.e.,  $|g(z_1) - g(z_2)| \le L_1 |z_1 - z_2|$ for all  $z_1, z_2 \in D$  such that  $|z_1 - z_2| \le \delta_0$ ;

## then, $\mathcal{H}$ is $\delta LAS$ .

The following result establishes a sufficient condition for a hybrid system  $\mathcal{H}$  to be  $\delta LAS$  "through jumps." In particular, the graphical distance between any two maximal solutions to a hybrid system  $\mathcal{H}$  strictly decreases during jumps.

Theorem 4.12: ( $\delta$ LAS through jumps for generic D) Consider a hybrid system  $\mathcal{H} = (C, f, D, g)$  with state  $z \in \mathbb{R}^n$ . Suppose  $\mathcal{H}$  satisfies Assumption 3.1, Assumption 4.4, and the hybrid basic conditions. If there exist  $\delta_0, L_1, L_2 > 0$  such that

1)  $\nabla f(z) + \nabla f(z)^{\top} \leq 0$  for all  $z \in \overline{\operatorname{con}}C$ ;

2) for each  $\delta \in [0, \delta_0]$ , each maximal solution  $\phi$  to  $\mathcal{H}$  from  $\phi(0, 0)$  satisfying

 $\phi(0,0) \in C, \quad |\phi(0,0)|_D = \delta$ 

satisfies  $|\phi(s,0)|_D = 0$  for some  $s \in [0,\delta]$ ;

- 3) for each  $z \in D$  and each  $\delta \in [0, \delta_0]$ , the set  $z + \delta \mathbb{B}$  is forward invariant for  $\mathcal{H}$  from away of D;
- 4) the jump map g is locally Lipschitz on D with Lipschitz constant  $L_1$ , i.e.,  $|g(z_1) - g(z_2)| \le L_1|z_1 - z_2|$  for all  $z_1, z_2 \in D$  such that  $|z_1 - z_2| \le \delta_0$ ;
- 5) f is bounded on  $\overline{\operatorname{con}}C$  with bound  $L_2$ , i.e.,  $|f(z)| \leq L_2$ for all  $z \in \overline{\operatorname{con}}C$ ;
- 6)  $L_1 + L_2 \le 1$ ;

then,  $\mathcal{H}$  is  $\delta S$ . Furthermore, if the domain of each maximal solution to  $\mathcal{H}$  is unbounded in the *j* direction, and  $L_1$  and  $L_2$  can be chosen such that  $L_1 + L_2 < 1$ , then,  $\mathcal{H}$  is  $\delta LAS$ .

The following example illustrates the conditions in Theorem 4.12.

Example 4.13: Consider the timer system in Example 3.4. Each maximal solution  $\phi$  to it has a domain that is unbounded in the t and j direction. Moreover, the flow time between two consecutive jumps of  $\phi$  is lower bounded by 1. The condition in item 1) of Theorem 4.12 can be verified as  $\nabla f(z) + \nabla f(z)^{\top} = 0$  for all  $z \in \overline{\text{con}}C$ . The condition in item 2) can be verified according to Proposition 4.7. Consider  $\delta_0 \in (0,1)$  and the Lyapunov function  $V(z) = |z|_D^2$ . For each  $z \in (D + \delta_0 \mathbb{B}) \cap C \setminus D$ , we have  $V(z) = (z - 1)^2$  and  $\langle \nabla V(z), f(z) \rangle = -2(1-z) = -2V^{\frac{1}{2}}(z)$ , where we used the property that  $z \leq 1$  for all  $z \in (D + \delta_0 \mathbb{B}) \setminus D$ . Item 3) of Theorem 4.12 follows from the fact  $D = \{1\}$  is a singleton and  $\langle \nabla V(z), f(z) \rangle = -2(1-z) < 0$  for all  $z \in (D+\delta_0 \mathbb{B}) \cap C \setminus D$ . Item 4) of Theorem 4.12 is satisfied with  $c_1 = 0$ , and item 5) of Theorem 4.12 is satisfied with  $c_2 = 1$ . Therefore, the timer system in Example 3.4 is  $\delta S$ . Δ

#### V. CONCLUSION

In this paper, we show that graphical incremental stability is a key notion for the study of incremental stability for hybrid systems. Other notions based on pointwise Euclidean distance fall short when applied to systems that exhibit the "peaking phenomenon," which is a typical behavior in tracking and observer design for hybrid systems. Several sufficient and necessary conditions for a hybrid system to be graphically incrementally stable and graphically incrementally attractive were provided and illustrated in examples.

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<sup>&</sup>lt;sup>3</sup>Such g is also known as a weak contraction map.