A Hybrid Observer with a Continuous Intersample Injection in the Presence of Sporadic Measurements*

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Abstract— **This paper deals with the problem of estimating the state of a linear time-invariant system in the presence of sporadically available measurements. An observer with a continuous intersample injection term is proposed. Such an intersample injection is provided by a linear dynamical system, whose state is reset to the measured output estimation error at each sampling time. The resulting system is augmented with a timer triggering the arrival of a new measurement and analyzed in a hybrid system framework. The design of the observer is performed to achieve global exponential stability of a set wherein the estimation error is equal to zero. Moreover, three computationally efficient procedures are proposed to design the observer. Finally, the effectiveness of the proposed methodology is shown in two examples.**

I. Introduction

State estimation of continuous-time dynamical systems in the presence of discrete-time measurements has been an interesting and appealing issue addressed by researchers over the last decades. Indeed, in real-world engineering applications, assuming to continuously measuring the output of a given plant is undoubtedly unrealistic. This practical needed has brought to life a new research area aimed at developing observer schemes accounting the discrete nature of the available measurements; see, *e.g*, [1], [3], [4], [14], [15], just to cite a few. In these works, by assuming a periodical availability of the measured output, the authors propose a discrete-time approach to the state estimation problem, which consists of designing a discrete-time observer for the discretized version of the plant. However, this approach entails two main drawbacks. The first drawback stems from the fact that the intersample behavior is completely lost due only studying the evolution of the estimation error at sampling times. In fact, with such a discrete-time approach, no explicit bounds on the estimation error in between consecutive samples are available. The second drawback is that any mismatch between the actual sampling time and the one used to discretize the plant model induces an error in the discrete-time description of the state estimation problem. The third drawback is that in many modern applications, such as networked control systems, the output of the plant is often accessible only sporadically, making the fundamental assumption of periodically measuring unrealistic; see, *e.g*, [8], [10], [24].

To address these issues, several strategies are presented in the literature. Such strategies essentially belong to two main families. The first one pertains to observers whose state is entirely reset, according to a suitable law, whenever a new measurement is available, and that run open-loop in between such events (continuous-discrete observers). This approach is, for instance, pursued in [2], [5], [16]. The second family of strategies considers instead continuous-time observers, for which the output injection error in between consecutive samples is estimated via a continuous-time processing of the last received measurement. This approach is pursed, *e.g.*, in [13], [19], [21].

In this paper, to exponentially estimate the state of a continuous-time linear system in the presence of sporadic measurements, we propose an observer with a continuous intersample injection and state resets. Such an intersample injection is provided by a linear system, whose state is reset to the measured output estimation error at each sampling time. Differently from existing works, as those in [13], [18], [21], we base our methodology on the Lyapunov results for hybrid systems presented in [7]. Pursuing this approach, we first propose a hybrid model of the observer interconnected with the plant, that captures all the possible occurrences of the output sampling events. Then, building on this model, we provide a general result to guarantee global exponential stability of a set of points in which the estimation error is equal to zero. As a second step, our condition guaranteeing global exponential stability is exploited to derive three efficient design algorithms, based on the solution of linear matrix inequalities, for the proposed observer. The first one gives rise to a computationally tractable design for the scheme proposed in [13]. The second one yields an alternative design algorithm, though for the case of linear systems, for the zero order sample-and-hold scheme proposed in [21]. The third design procedure provides a completely novel scheme. The contribution of this paper is twofold. On the one hand, we provide a unified approach to analyze the observer scheme proposed by [13], [21]. On the other hand, for the first time, we propose a design procedure for the observer proposed in [13], as well as a completely novel observer scheme.

The remainder of the paper is organized as follows. Section II presents the system under consideration, the state estimation problem we solve, and the hybrid modeling of the proposed observer. Section III is dedicated to the main results. Section IV is devoted to design procedures for the proposed observer. Finally, in two numerical examples, Section V shows the effectiveness of the results presented. Due

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to space limitations, proofs of the results will be published elsewhere.

Notation: The set $\mathbb{N}_0$ is the set of positive integers including zero, the set $\mathbb{N}$ is the set of strictly positive integers, and $\mathbb{R}_{\geq 0}$ represents the set of nonnegative real scalars. The identity matrix is denoted by $\mathbf{I}$, whereas the null matrix is denoted by $\mathbf{0}$. For a matrix $A \in \mathbb{R}^{n\times m}$, A' denotes the transpose of A, $\|A\|$ denotes the induced 2-norm, and $\operatorname{He}(A) = A + A'$. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. In partitioned symmetric matrices, the symbol $\bullet$ stands for symmetric blocks. The matrix $\operatorname{diag}\{A_1, A_2, \dots, A_n\}$ is the block-diagonal matrix having $A_1, A_2, \dots, A_n$ as diagonal blocks. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. Given two vectors x, y, we denote $(x, y) = [x' \; y']'$. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A}$, the distance of x from $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y\in\mathcal{A}} \|x - y\|$. For any function $z : \mathbb{R} \to \mathbb{R}^n$, we denote $z(t^+) := \lim_{s\to t^+} z(s)$.

II. Problem statement

A. System description

We consider continuous-time linear time-invariant systems of the form

$$\begin{aligned} \dot{z} &= Az \\ y &= Mz \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^q$ are, respectively, the state and the measured output of the system, while A and M are constant matrices of appropriate dimensions. Assume that the initial time $t_0 = 0$, our goal is to design an observer providing an asymptotic estimate $\hat{z}$ of the state z with sporadic measurements of y. Namely, we assume that the whole output y is available only at some time instances t_k, $k \in \mathbb{N}$, not known *a priori*. We assume that the sequence $\{t_k\}_1^\infty$ is strictly increasing and unbounded, and that for such a sequence there exist two positive real scalars $T_1 \leq T_2$ such that

$$\begin{aligned} &0 \leq t_1 \leq T_2 \\ &T_1 \leq t_{k+1} - t_k \leq T_2 \qquad \forall k \in \mathbb{N} \end{aligned} \tag{2}$$

As also pointed out in [9], the lower bound in condition (2) prevents the existence of accumulation points in the sequence $\{t_k\}_1^\infty$, and, hence, avoids the existence of Zeno behavior, which is typically undesired in practice. In fact, T_1 defines a strictly positive minimum time in between consecutive transmissions. Furthermore, T_2 defines the maximum time in between consecutive transmissions. For this reason, in the sequel we will refer to T_2 as the maximum transmission interval.

Since measurements of the output y are available in an impulsive fashion, assuming that the arrival of a new measurement can be instantaneously detected, to solve the considered estimation problem, building from [13], [18], [21], we propose the following observer with jumps

$$\begin{cases} \left.\begin{aligned} \dot{\hat{z}}(t) &= A\hat{z}(t) + L\theta(t) \\ \dot{\theta}(t) &= H\theta(t) \end{aligned}\right\} & \text{when } t \notin \{t_k\}_1^\infty \\[2ex] \left.\begin{aligned} \hat{z}(t^+) &= \hat{z}(t) \\ \theta(t^+) &= y(t) - M\hat{z}(t) \end{aligned}\right\} & \text{when } t \in \{t_k\}_1^\infty \end{cases} \tag{3}$$

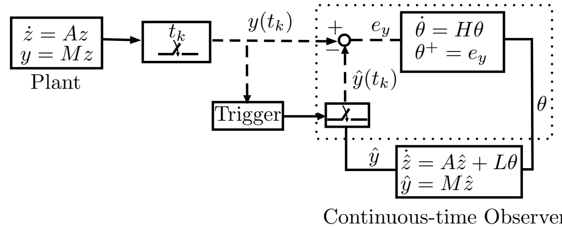

Fig. 1: The proposed observer scheme. The dashed arrows denote impulsive data streams, while the solid arrows denote continuous data streams.

where L and H are real matrices of appropriate dimensions to be designed.

The operating principle of the observer in (3) is as follows. The arrival of a new measurement triggers an instantaneous jump in the observer state. Specifically, at each jump, the measured output estimation error, *i.e.*, $y - M\hat{z}$, is instantaneously stored in θ. Then, in between consecutive measurements, θ is continuously updated according to linear continuous-time dynamics, and its value is continuously used as an intersample injection to feed a continuous-time observer; see Figure 1.

Along the lines of [22], we formulate the state estimation problem as a set stabilization problem. Namely, our goal is to design the matrices L and H such that the set wherein the plant state z and its estimate $\hat{z}$ coincide is globally exponentially stable for the plant (1) interconnected with the observer in (3). At this stage, we define the following change of variables

$$\begin{aligned} \varepsilon &:= z - \hat{z} \\ \tilde{\theta} &:= M(z - \hat{z}) - \theta \end{aligned}$$

which defines, respectively, the estimation error and the difference between the output estimation error and θ. Hence, the two error dynamics are given by the following dynamical system with jumps:

$$\begin{cases} \left.\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{\tilde{\theta}}(t) \end{bmatrix} = \mathcal{F} \begin{bmatrix} \varepsilon(t) \\ \tilde{\theta}(t) \end{bmatrix} \right\} & \text{when } t \notin \{t_k\}_1^\infty \\[2ex] \left.\begin{bmatrix} \varepsilon(t^+) \\ \tilde{\theta}(t^+) \end{bmatrix} = \mathcal{G} \begin{bmatrix} \varepsilon(t) \\ \tilde{\theta}(t) \end{bmatrix} \right\} & \text{when } t \in \{t_k\}_1^\infty \end{cases} \tag{4}$$

where

$$\mathcal{F} := \begin{bmatrix} A - LM & L \\ MA - MLM - HM & ML + H \end{bmatrix}, \mathcal{G} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{5}$$

Notice that, in view of the linearity of the plant (1), the error dynamics are decoupled from the plant dynamics. Then, for the purpose of estimation, one can effectively only consider system (4).

B. Hybrid modeling

The fact that the observer experiences jumps when a new measurement is available and evolves according to a differential equation in between updates suggests that the updating process of the error dynamics can be described via a hybrid system. Due to this, we represent the whole system composed by the plant (1), the observer (3), and the logic triggering jumps as a hybrid system. The proposed hybrid systems approach requires to model the hidden time-driven mechanism triggering the jumps of the observer.

To this end, in this work, and in a similar manner as in [5], we augment the state of the system with an auxiliary timer variable τ that keeps track of the duration of flows and triggers a jump whenever a certain condition is verified. This additional state allows to describe the time-driven triggering mechanism as a state-driven triggering mechanism, which leads to a model that can be efficiently represented by relying on the framework for hybrid systems proposed in [7]. More precisely, we make τ to decrease as ordinary time t increases and, whenever $\tau = 0$, reset it to any point in $[T_1, T_2]$, so as to enforce (2). After each jump, we require the system to flow again. The whole system composed by the error states ε and $\tilde{\theta}$, and the timer variable τ can be represented by the following hybrid system, which we denote $\mathcal{H}$:

$$\mathcal{H}\begin{cases}\left.\begin{array}{rcl}\begin{bmatrix}\dot{\varepsilon}\\ \dot{\tilde{\theta}}\end{bmatrix} & = & \mathcal{F}\begin{bmatrix}\varepsilon\\ \tilde{\theta}\end{bmatrix}\\ \dot{\tau} & = & -1\end{array}\right\} & (\varepsilon,\tilde{\theta},\tau)\in C\\ \left.\begin{array}{rcl}\begin{bmatrix}\varepsilon^+\\ \tilde{\theta}^+\end{bmatrix} & = & \mathcal{G}\begin{bmatrix}\varepsilon\\ \tilde{\theta}\end{bmatrix}\\ \tau^+ & \in & [T_1,T_2]\end{array}\right\} & (\varepsilon,\tilde{\theta},\tau)\in D\end{cases} \tag{6a}$$

where the flow set and the jump set are defined as

$$C = \mathbb{R}^{n+q}\times[0,T_2],\quad D=\mathbb{R}^{n+q}\times\{0\}. \tag{6b}$$

The set-valued jump map allows to capture all possible transmission events occurring within T_1 or T_2 units of time from each other. Specifically, the hybrid model in (6) is able to characterize not only the behavior of the analyzed system for a given sequence $\{t_k\}_1^\infty$, but also for any sequence satisfying (2). We denote the state of $\mathcal{H}$ by

$$x = (\varepsilon, \tilde{\theta}, \tau)$$

and by f and G, respectively, the flow map and the jump map, *i.e.*,

$$f(x) = \begin{bmatrix}\mathcal{F}\begin{bmatrix}\varepsilon\\ \tilde{\theta}\end{bmatrix}\\ -1\end{bmatrix}\qquad \forall x\in C \tag{7a}$$

$$G(x) = \begin{bmatrix}\mathcal{G}\begin{bmatrix}\varepsilon\\ \tilde{\theta}\end{bmatrix}\\ [T_1,T_2]\end{bmatrix}\qquad \forall x\in D. \tag{7b}$$

We consider the following notions for a general hybrid system $\mathcal{H}$ in $\mathbb{R}^\ell$.

Definition 1: (Maximal solutions [7, Definition 2.7]) A solution ϕ to $\mathcal{H}$ is maximal if there does not exist another solution ψ to $\mathcal{H}$ such that $\operatorname{dom}\phi$ is a proper subset of $\operatorname{dom}\psi$ and $\phi(t,j)=\psi(t,j)$ for all $(t,j)\in\operatorname{dom}\phi$.

Definition 2: (Global exponential stability [23]) Let $\mathcal{A}\subset\mathbb{R}^\ell$ be closed. The set $\mathcal{A}$ is said to be *globally exponentially stable* (GES) for the hybrid system $\mathcal{H}$ if every maximal solution to $\mathcal{H}$ is complete, *i.e.*, its domain is unbounded and there exist strictly positive real numbers λ,ω such that every solution ϕ to $\mathcal{H}$ satisfies for all $(t,j)\in\operatorname{dom}\phi$

$$|\phi(t,j)|_\mathcal{A}\le\omega e^{-\lambda(t+j)}|\phi(0,0)|_\mathcal{A}. \tag{8}$$

Then, by introducing the set[1]

$$\mathcal{A}=\{0\}\times\{0\}\times[0,T_2] \tag{9}$$

the problem to solve is formulated as follows:

Problem 1: Given the matrices A and M of appropriate dimensions, and two positive scalars $T_1\le T_2$, design the matrices $L\in\mathbb{R}^{n\times q}$ and $H\in\mathbb{R}^{q\times q}$ such that the set $\mathcal{A}$ defined in (9) is GES for the hybrid system (6).

Concerning the existence of solutions to system (6), by relying on the concept of solution proposed in [7, Definition 2.6], it is straightforward to check that, for every initial condition $\phi(0,0)\in C\cup D$, every maximal solution to (6) is complete. In addition, we can characterize the domain of these solutions. Indeed, as in [5], for every initial condition $\phi(0,0)\in C\cup D$, the domain of every maximal solution ϕ to (6) can be written as follows:

$$\operatorname{dom}\phi=\bigcup_{j\in\mathbb{N}_0}([t_j,t_{j+1}])\times\{j\} \tag{10}$$

with

$$\begin{array}{ll}T_1\le t_{j+1}-t_j\le T_2 & \forall j\in\mathbb{N}\\ 0\le t_1\le T_2 & \end{array} \tag{11}$$

where $\operatorname{dom}\phi$ is the domain of the solution ϕ, which is a hybrid time domain; see [7] for more details about solutions to hybrid systems. Furthermore, the structure of the above hybrid time domain implies that for each $(t,j)\in\operatorname{dom}\phi$, we have

$$t\ge T_1 j - T_1 \tag{12}$$

Such a relation will exploited later to assess GES of the set $\mathcal{A}$ in (9) for $\mathcal{H}$.

III. Main results

A. Conditions for GES

The following result provides conditions for GES of the set $\mathcal{A}$ defined in (9) for system (6).

Theorem 1: Let T_2 be a given positive real scalar. If there exist two symmetric positive definite matrices $P_1\in\mathbb{R}^{n\times n}, P_2\in\mathbb{R}^{q\times q}$, a positive real scalar σ, and two matrices $L\in\mathbb{R}^{n\times q}, H\in\mathbb{R}^{q\times q}$, such that

$$\mathcal{M}_1<\mathbf{0},\mathcal{M}_2<\mathbf{0} \tag{14}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are symmetric matrices defined in (13) (at the top of the next page), then the set $\mathcal{A}$ in (9) is GES for the hybrid system (6). □

Sketch of the proof: Inspired by [6, Example 27], consider the following Lyapunov function candidate for the hybrid system (6) defined for every $x\in\mathbb{R}^{n+q}\times\mathbb{R}_{\ge0}$:

$$V(x)=\varepsilon' P_1\varepsilon+e^{\sigma\tau}\tilde{\theta}' P_2\tilde{\theta}, \tag{15}$$

[1]By the definition of system (6) and of the set $\mathcal{A}$, for every $x\in C\cup D\cup G(D)$, $|x|_\mathcal{A}=\|(\varepsilon,\tilde{\theta})\|$.

$$\mathcal{M}_1 = \begin{bmatrix} \mathrm{He}(P_1(A-LM)) & P_1L + (MA - MLM - HM)'P_2 \\ \bullet & \mathrm{He}(P_2(ML+H)) - \sigma P_2 \end{bmatrix}$$
$$\mathcal{M}_2 = \begin{bmatrix} \mathrm{He}(P_1(A-LM)) & P_1L + e^{\sigma T_2}(MA - MLM - HM)'P_2 \\ \bullet & e^{\sigma T_2}\left(\mathrm{He}(P_2(ML+H)) - \sigma P_2\right) \end{bmatrix} \tag{13}$$

where σ is a positive real scalar. Now, let ϕ be a maximal solution to (6), it can be shown that the satisfaction of (14) implies that there exist two positive real scalars β, α_2 such that for every $(t,j) \in \operatorname{dom}\phi$

$$V(\phi(t,j)) \leq e^{-\frac{\beta}{\alpha_2}t} V(\phi(0,0)). \tag{16}$$

Finally by exploiting (12), via standard arguments, one can show that the set $\mathcal{A}$ defined in (9) is GES for system (6).

Remark 1: Notice that, for them to be feasible, the conditions in (14) require a detectable pair (A, M) (though this condition is in general only necessary). It is worthwhile to remark that, differently from [5], [20], *a priori*, we do not require the detectability of the pair (e^{Av}, Me^{Av}) for each v belonging to $[T_1, T_2]$, which would be a more restrictive condition.

IV. Observer design

In the previous section, a condition to guarantee GES of the set $\mathcal{A}$ for system (6) was provided. However, due to its form, such a condition is in general not computationally tractable to provide a viable solution to Problem 1. Indeed, condition (14) is nonlinear in the design variables P_1, P_2, σ, H and L, so further work is needed to derive a design procedure for the proposed observer. Specifically, the nonlinearities present in (14) are due to both the bilinear terms involving the matrices P_1, P_2, L, H, and the scalar σ, as well as the fact that σ also appears in a nonlinear fashion via the exponential function. Nevertheless, from a numerical standpoint, the nonlinearities involving the scalar σ are easily manageable. Indeed, σ can be treated as a tuning parameter or being selected via an iterative search. Thus, the main issue to tackle concerns with the other nonlinearities present in (14). To address these, in the sequel, we provide three constructive sufficient conditions to solve Problem 1 via the solution of sets of linear matrix inequalities.

A. Intersample predictor observer scheme

Proposition 1: Let T_2 be a given positive real scalar. If there exist two symmetric positive definite matrices $P_1 \in \mathbb{R}^{n\times n}, P_2 \in \mathbb{R}^{q\times q}$, a positive real scalar σ, and a matrix $J \in \mathbb{R}^{n\times q}$ such that

$$\begin{bmatrix} \mathrm{He}(P_1A - JM) & J + A'M'P_2 \\ \bullet & -\sigma P_2 \end{bmatrix} < \mathbf{0}$$
$$\begin{bmatrix} \mathrm{He}(P_1A - JM) & J + e^{\sigma T_2}A'M'P_2 \\ \bullet & -e^{\sigma T_2}\sigma P_2 \end{bmatrix} < \mathbf{0} \tag{17}$$

then $L = P_1^{-1}J, H = -ML, P_1, P_2$ and σ satisfy (14). □

Remark 2: It is worthwhile to notice that the proposed choice for the gain H leads to the predictor-based observer scheme proposed in [13], though written in different coordinates. Indeed, whenever $H = -ML$, by rewriting (3) via the following invertible change of variables $(\hat{z}, w) = (\hat{z}, \theta + M\hat{z})$, yields the same observer in [13].

In the next sections, we present two other design procedures, whose derivation is based on an equivalent condition to Theorem 1, that is formulated introducing slack variables via the use of the projection lemma; see [17].

B. Slack variables-based design

The following result provides an equivalent condition to condition (14) in Theorem 1, in which the term MLM no longer appears.

Corollary 1: Let T_2 be a given positive real scalar. The symmetric positive matrices P_1, P_2, the matrices H, L, and the scalar σ satisfy (14) if and only if there exist matrices $X_1, Y_1, X_3, Y_3 \in \mathbb{R}^{n\times n}, X_2, Y_2 \in \mathbb{R}^{n\times q}, X_4, Y_4, X_6, Y_6 \in \mathbb{R}^{q\times n}, X_5, Y_5 \in \mathbb{R}^{q\times q}$ such that

$$\begin{bmatrix} \mathrm{He}(S_1^X) & S_2^X + \hat{P} \\ \bullet & N_1 + \mathrm{He}(S_3^X) \end{bmatrix} < \mathbf{0}$$
$$\begin{bmatrix} \mathrm{He}(S_1^Y) & S_2^Y + \hat{P}_{T_2} \\ \bullet & N_2 + \mathrm{He}(S_3^Y) \end{bmatrix} < \mathbf{0} \tag{18}$$

where

$$\begin{matrix} \hat{P} = \mathrm{diag}\{P_1, P_2\} & \hat{P}_{T_2} = \mathrm{diag}\{P_1, P_2 e^{\sigma T_2}\} \\ N_1 = \mathrm{diag}\{\mathbf{0}, -\sigma P_2\} & N_2 = \mathrm{diag}\{\mathbf{0}, -\sigma e^{\sigma T_2} P_2\} \end{matrix} \tag{19}$$

where

$$\begin{aligned}
S_1^X &= \begin{bmatrix} -X_1 + M'X_4 & -X_2 + M'X_5 \\ -X_4 & -X_5 \end{bmatrix} \\
S_2^X &= \begin{bmatrix} X_1'(A-LM) - X_4'HM - X_3 + M'X_6 & X_1'L + X_4'H \\ X_2'(A-LM) - X_5'HM - X_6 & X_2'L + X_5'H \end{bmatrix} \\
S_3^X &= \begin{bmatrix} (A-LM)'X_3 - M'H'X_6 & \mathbf{0} \\ L'X_3 + H'X_6 & \mathbf{0} \end{bmatrix} \\
S_1^Y &= \begin{bmatrix} -Y_1 + M'Y_4 & -Y_2 + M'Y_5 \\ -Y_4 & -Y_5 \end{bmatrix} \\
S_2^Y &= \begin{bmatrix} Y_1'(A-LM) - Y_4'HM - Y_3 + M'Y_6 & Y_1'L + Y_4'H \\ Y_2'(A-LM) - Y_5'HM - Y_6 & Y_2'L + Y_5'H \end{bmatrix} \\
S_3^Y &= \begin{bmatrix} (A-LM)'Y_3 - M'H'Y_6 & \mathbf{0} \\ L'Y_3 + H'Y_6 & \mathbf{0} \end{bmatrix}.
\end{aligned} \tag{20}$$

□

The above result yields an equivalent condition to (14) that can be exploited to derive an efficient design procedure for the proposed observer. To this end, one needs to suitably manipulate (18) to obtain conditions that are linear in the decision variables. Specifically, the two results given in the next sections provide two possible approaches to derive LMI-based design procedures for the proposed observer.

B.1): Zero order sample-and-hold intersample scheme

Proposition 2: Let T_2 be a given positive real scalar. If there exist two symmetric positive definite matrices $P_1 \in \mathbb{R}^{n\times n}, P_2 \in \mathbb{R}^{q\times q}$, a positive real scalar σ, matrices $X \in \mathbb{R}^{n\times n}, X_4, Y_4, X_6, Y_6 \in \mathbb{R}^{q\times n}, X_5, Y_5 \in \mathbb{R}^{q\times q}, J \in \mathbb{R}^{n\times q}$ such that

$$\begin{bmatrix} \mathrm{He}(Q_1) & Q_2 + \hat{P} \\ \bullet & \mathrm{He}(Q_3) + N_1 \end{bmatrix} < \mathbf{0}$$
$$\begin{bmatrix} \mathrm{He}(R_1) & R_2 + \hat{P}_{T_2} \\ \bullet & \mathrm{He}(Q_3) + N_2 \end{bmatrix} < \mathbf{0} \tag{21}$$

where $\hat{P}, \hat{P}_{T_2}, N_1, N_2$ are defined in (19) and

$$Q_1 = \begin{bmatrix} -X + M'X_4 & M'X_5 \\ -X_4 & -X_5 \end{bmatrix}, R_1 = \begin{bmatrix} -X + M'Y_4 & M'Y_5 \\ -Y_4 & -Y_5 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} -X + M'X_6 + X'A - JM & J \\ -X_6 & \mathbf{0} \end{bmatrix}$$
$$R_2 = \begin{bmatrix} -X + M'Y_6 + X'A - JM & J \\ -Y_6 & \mathbf{0} \end{bmatrix}$$
$$Q_3 = \begin{bmatrix} A'X - M'J' & \mathbf{0} \\ J' & \mathbf{0} \end{bmatrix}$$

then $L = X'^{-1}J$ and $H = \mathbf{0}$ are a solution to Problem 1. □

It should be noticed that the above design procedure leads to the well-known zero-order sample-and-hold scheme.

B.2): A novel observer scheme

Proposition 3: Let T_2 be a given positive real scalar. If there exist two symmetric positive definite matrices $P_1 \in \mathbb{R}^{n\times n}, P_2 \in \mathbb{R}^{q\times q}$, a positive real scalar σ, matrices $X \in \mathbb{R}^{n\times n}, U, W \in \mathbb{R}^{q\times q}, J \in \mathbb{R}^{n\times q}$ such that

$$\begin{bmatrix} \mathrm{He}(Z_1) & Z_2 + \hat{P} \\ \bullet & \mathrm{He}(Z_3) + N_1 \end{bmatrix} < \mathbf{0}$$
$$\begin{bmatrix} \mathrm{He}(Z_1) & Z_2 + \hat{P}_{T_2} \\ \bullet & \mathrm{He}(Z_3) + N_2 \end{bmatrix} < \mathbf{0} \tag{22}$$

where $\hat{P}, \hat{P}_{T_2}, N_1, N_2$ are defined in (19) and

$$Z_1 = \begin{bmatrix} -X & U \\ \mathbf{0} & -U \end{bmatrix}, Z_2 = \begin{bmatrix} -X + X'A - JM & J \\ -WM & W \end{bmatrix}$$
$$Z_3 = \begin{bmatrix} A'X - M'J' & \mathbf{0} \\ J' & \mathbf{0} \end{bmatrix}$$

then $L = X'^{-1}J$ and $H = U'^{-1}W$ are solution to Problem 1. □

Remark 3: The above result gives rise to a novel observer scheme. Indeed, as a difference to Proposition 1 and Proposition 2, Proposition 3 does not impose any structural constraint on the gain H. This is a worthwhile novelty introduced by our approach with respect to classical approaches as [13], [21] and alike, where the choice of the gain H is *a priori* constrained. Thus, in general, the use of Proposition 3 may lead to observation schemes that are not encompassed either by Proposition 1 and Proposition 2 or by existing approaches.

Remark 4: The derivation of Proposition 2 and Proposition 3 consists of some particular choices of the slack variables X and Y introduced in Corollary 1. Therefore, when one is interested in enlarging as much as possible the maximum allowable transmission interval T_2, the use of such results as a design tool may introduce conservatism. To overcome this problem, one can envision a two-stage procedure. Indeed, whenever L, H, σ and T_2 are fixed, condition (13) is linear in the decision variables. Thus, once the observer has been designed via either Proposition 2 or Proposition 3, by testing the feasibility of (13) with respect to P_1, P_2 over a selected grid for the variables σ and T_2, one may enable to enlarge the maximum allowable transmission interval T_2.

Design	# scalar variables
Prop. 1	$(n+1)/2n + nq + q(q+1)/2$
Prop. 2	$(n+1)/2n + q(q+1)/2 + n^2 + 4qn + 2q^2 + nq$
Prop. 3	$(n+1)/2n + q(q+1)/2 + n^2 + 2q^2 + nq$

TABLE I: Number number of scalar variables entailed by the different designs.

Remark 5: The proposed design procedures entail a different number of scalar variables in the associated LMIs. Table I reports such a number for each of the proposed designs. As it appears from the table, the LMIs related to Proposition 2 and Proposition 3, due to the introduction of additional slack variables, entail a greater number of scalar variables with respect to the LMIs issued from Proposition 1. Hence, the designs based on Proposition 2 and on Proposition 3 are in general more complex from a numerical standpoint.

V. Numerical examples

Example 1: In this first example, we want to show the improvement provided by our methodology with respect to existing results. Specifically, concerning the predictor-based scheme in Section IV-A, consider the example in [12] (where the same observer is considered), which is defined by the following data: $A = \left[\begin{smallmatrix} 0 & 1 \\ -4 & 0 \end{smallmatrix}\right], M = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right], L' = \left[\begin{smallmatrix} 4 & 0 \end{smallmatrix}\right]$. We want to exploit Theorem 1 to determine an estimate of the maximum allowable transmission time T_2 for the design proposed in [12]. In particular, it turns out that the conditions of Theorem 1 are feasible for T_2 up to 0.31. This bound on the maximum allowable transmission interval is about 3.48 times less conservative than the one in [12]. Furthermore, via Proposition 1, one can also design a new gain L_s to tentatively enlarge the maximum allowable transmission interval. Specifically, by designing the observer gain via Proposition 1, T_2 can be increased up to 0.41. The observer gain obtained for $T_2 = 0.41$ is $L'_s = \begin{bmatrix} 0.3648 & -0.4655 \end{bmatrix}$.

Example 2: Consider the model of the longitudinal dynamics of the F8 aircraft in [11], whose state-space model is given by

$$\dot{x} = \begin{bmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.6 & -32.2 \\ 1 & -10^{-4} & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x, \quad y = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{smallmatrix}\right] x.$$

The two outputs are, respectively, the pitch angle and the flight path angle.

In Table II, we report, for each design methodology, the values of the maximum T_2 for which conditions (13) are feasible along with the corresponding value of σ, and the two designed gains L and H. Concerning the design based on Proposition 2 and Proposition 3, as mentioned in Remark 4, to enlarge as much as possible the maximum transmission interval T_2 allowable, after a first design step, we performed a further analysis stage via Theorem 1. Concerning the design

Design	σ	T_2	L	H
Prop. 1	0.5	5	$\begin{bmatrix} 0.24 & 0.33 \\ -500 & -222 \\ 0.3 & 0.12 \\ 2.4 & 1 \end{bmatrix}$	$-ML$
Prop. 2	0.7	3.4	$\begin{bmatrix} -0.25 & 0.31 \\ -19 & -39 \\ -3\cdot 10^{-3} & 0.027 \\ 0.1 & 0.15 \end{bmatrix}$	$\mathbf{0}_{2\times 2}$
Prop. 3	0.7	5.6	$\begin{bmatrix} -0.049 & 0.098 \\ -36.1 & -47.4 \\ 0.019 & 0.012 \\ 0.16 & 0.19 \end{bmatrix}$	$\begin{bmatrix} -0.293 & 0.001 \\ -0.01 & -0.25 \end{bmatrix}$

TABLE II: Values of T_2 and σ and the designed observer gains L and H for the three design procedures.

based on Proposition 2, it is worthwhile to notice that, the design conditions for the same observer scheme given in [21] (when they are specialized to the linear systems case) are feasible for T_2 up to 0.4. Namely, the proposed design, in this specific case, enables to enlarge the maximum transmission interval allowable 8.5 times with respect to [21]. Moreover, it turns out that the design procedure 3, in this specific case, provides the largest value for T_2. Furthermore, it is interesting to notice that, in this specific case, the designed matrix H is Hurwitz. Therefore, in between consecutive transmissions the last measured value of the output estimation error is exponentially disregarded; in other words, H introduces a forgetting effect.

VI. Conclusion

Building from the general ideas in [13], this paper proposed a novel methodology to design, via linear matrix inequalities, an observer with intersample injection to exponentially estimate the state of a continuous-time linear system in the presence of sporadically available measurements. Specifically, pursuing a unified approach, we provided three design methodologies to design the observer, which are computationally efficient, *i.e.*, the design algorithm entails a time of computation which is polynomial with respect to the dimension of the data. Two of them lead back respectively to the observer scheme proposed in [13] and to the zero order sample-and-hold proposed in [21], while the remaining leads to a completely novel scheme. Notice that, although we recover some existing schemes, the design procedures we propose are novel and, in some cases, outperform the corresponding existing design techniques.

The results presented in this paper seem to be promising and the framework adopted quite flexible to envision interesting extensions of the results presented here. Among them, we mention the extension to the case of multi-output systems with asynchronous sampling, which is currently part of our work.

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