How well-posedness of hybrid systems can extend beyond Zeno times

Rafal Goebel and Ricardo Sanfelice

Abstract— **The extension of a solution to a hybrid system beyond its Zeno time is a simple exercise in modeling. However, assuring that the extended system is well-posed in a certain sense, in particular, that the extension of a solution depends reasonably on initial, pre-Zeno, conditions, has not been addressed. In this paper it is shown that these results hold for hybrid systems that exhibit Zeno behavior when the set of Zeno equilibria forms a continuum that has certain stability properties. Several scenarios of going past Zeno are presented. Dependence of limits of Zeno solutions, of Zeno times, and of reachable sets on initial conditions is also discussed.**

I. Introduction

Hybrid dynamical systems are dynamical systems that exhibit features characteristic of continuous-time dynamical systems and features characteristic of discrete-time systems. In this paper, hybrid systems are modeled as hybrid inclusions [14]. This framework relies on generalized time domains that go back to [24], allows for multivalued dynamics, as pioneered in this setting by [3], [4], admits nonunique solutions, and, despite such generality, allows for a satisfying robust asymptotic stability theory [14]. What enables much of the stability and robustness analysis is reasonable dependence of solutions on initial conditions and perturbations, referred to in [14] as well-posedness, or nominal well-posedness when perturbations are not considered.

The Zeno phenomenon in a hybrid system is the occurrence of infinitely many jumps (or events, or switches, or discrete transitions) in a finite amount of (ordinary) time. It may arise from modeling abstractions and provide challenge to simulations [20]. There exist works on sufficient conditions for the Zeno behavior [26], [1] and on stability of isolated Zeno equilibria [22], [23], [15], [16]. Continuation of solutions beyond Zeno times is not possible in the hybrid inclusions framework, as the hybrid time domains representing Zeno behavior are already unbounded. Modeling frameworks, for example dynamical systems on time scales [7] or or measure-driven differential equations [8], that allow for continuation of solutions past (multiple) Zeno behaviors are not discussed in this paper. Some work on continuation of solutions past Zeno time has been done by [9], [27], [2]. Usually, it involves formal extension past Zeno and is not concerned with dependence of past-Zeno solution on pre-Zeno initial conditions.

Department of Mathematics and Statistics, Loyola University Chicago, 1032 W. Sheridan Road, Chicago, IL 60660. Email: *rgoebel1@luc.edu*. This work was partially supported by the Simons Foundation Grant 315326.

Department of Computer Engineering, University of California, Santa Cruz, CA, 95064, USA. E-mail: *ricardo@ucsc.edu*. This work was partially supported by the National Science Foundation CAREER Grant ECS-1150306 and by the Air Force Office of Scientific Research YIP Grant FA9550-12-1-0366.

The main goal of this paper is to show a situation where nominal well-posedness results can be extended to post-Zeno behaviors. The situation involves a nominally well-posed hybrid system which has a continuum of Zeno equilibria. Zeno solutions converge to the equilibria and their post-Zeno continuations, described by the same or a different nominally well-posed system, depend on which equilibirum the solutions converge to. Two properties are required from the Zeno equilibria: pointwise asymptotic stability, sometimes referred to as semistability, and a small ordinary time property. These properties ensure that limits and Zeno times of Zeno solutions depend regularly on the pre-Zeno initial conditions, which in turn lets one extend the well-posedness to post-Zeno behaviors.

Pointwise asymptotic stability is a property of a set of equilibria in a dynamical system, where every equilibrium is Lyapunov stable and from a neighborhood of it, every solution converges to possibly another equilibrium. This stability concept has been analyzed in the setting of differential equations [5], [6] and differential inclusions [19]. Necessary and sufficient conditions, for difference inclusions, were given by [10], [11] in terms of set-valued Lyapunov functions. This approach was generalized to hybrid inclusions in [13]. Some other results on semistability has appeared for switching [18] and hybrid systems [17]. The small ordinary time property of a Zeno equilibrium [22] or of a compact attractor [15] essentially requires that solutions that start near the equilibrium or attractor have uniformly small Zeno times.

The contribution of this paper is that if a hybrid system has a closed pointwise asymptotically stable set, then reasonable dependence of solutions on initial conditions extends from finite-time horizons to infinite-time horizons; the sets of limits of solutions depend reasonably on initial conditions; and under an additional assumption of small ordinary time, Zeno times depend reasonably on initial conditions too. These new results are in Section IV. Then, in sections V and VI, the main results are used to show how well-posedness, and also some characterizations of asymptotic stability, can be carried over to scenarios involving continuation of solutions past their Zeno times by a potentially different hybrid system. Different scenarios can be considered as well, including for example partial pointwise asymptotic stability, or repeated reinitialization of Zeno solutions in one hybrid system [12]. Background on hybrid systems and nominal well-posedness is in Section II below. The required stability concepts are presented in III, along with sufficient conditions for them. Examples in Section VII conclude the paper.

II. Hybrid Systems and Well-Posedness

A. Hybrid inclusions

Hybrid systems are modeled in this paper by hybrid inclusions [14]. Below, $C, D \subset \mathbb{R}^n$ are sets, called, respectively, the *flow set* and the *jump set* and $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings, called, respectively, the *flow map* and *jump map*. A hybrid inclusion is represented by

$$\begin{array}{lll} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x). \end{array} \tag{1}$$

A set $E \subset \mathbb{R}^2$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J} I_j \times \{j\}$, where $J \in \{0, 1, 2, \dots\}$ and $I_j = [t_j, t_{j+1}]$, $j = 0, 1, \dots, J$, for some $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1}$. A set E is a *hybrid time domain* if, for each $(T, J) \in E$, the set $\{(t, j) \in E \,|\, t \leq T, j \leq J\}$ is a compact hybrid time domain. Equivalently, a hybrid time domain is a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_1 \leq t_2 \leq \dots$, with the last interval, if it exists, possibly of the form $[t_j, t_{j+1})$ or $[t_j, \infty)$.

A function $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$, where $\operatorname{dom} \phi$ represents the domain of ϕ, is a *solution to the hybrid system* (1) if $\operatorname{dom} \phi$ is a hybrid time domain, $\phi(0, 0) \in \overline{C} \cup D$, and

- if $I_j := \{t \,|\, (t, j) \in \operatorname{dom} \phi\}$ has nonempty interior, then $t \mapsto \phi(t, j)$ is locally absolutely continuous on I_j and
$$\phi(t, j) \in C \quad \text{for all } t \in \operatorname{int} I_j \quad \text{and}$$
$$\frac{d}{dt}\phi(t, j) \in F(\phi(t, j)) \quad \text{for almost all } t \in I_j;$$
- if $(t, j) \in \operatorname{dom} \phi$ and $(t, j + 1) \in \operatorname{dom} \phi$ then
$$\phi(t, j) \in D \quad \text{and} \quad \phi(t, j + 1) \in G(\phi(t, j)).$$

A solution $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is *maximal* if it cannot be extended, and *complete* if $\operatorname{dom} \phi$ is unbounded.

Definition 2.1: *The hybrid system* (1) *is* forward complete *if every maximal solution to* (1) *is complete. The hybrid system* (1) *is* pre-forward complete *if every solution to* (1) *is bounded or complete.*

For conditions guaranteeing that maximal solutions are complete, see [14, Proposition 2.10 and Proposition 6.10]. In what follows, $\mathcal{S}$ denotes the set of all maximal solutions to (1), $\mathcal{S}(x)$ denotes the set of maximal solutions to (1) that start from x, and for a set $K \subset \mathbb{R}^n$, $\mathcal{S}(K) := \bigcup_{x \in K} \mathcal{S}(x)$.

B. Well-posedness

For standard differential equations, dependence of solutions on initial conditions and on parameters is expressed in terms of the uniform distance and uniform convergence. An essential tool for proving results in such setting is the Arzela-Ascoli lemma about extracting uniformly convergent subsequences from sequences of uniformly continuous functions. For sequences of solutions to hybrid systems, the right notion of convergence is graphical convergence. For details, see [25] or [14]. The following is a consequence of [25, Theorem 4.18], restated for hybrid arcs in [14, Theorem 6.1].

Theorem 2.2: *For every sequence $\phi_i \in \mathcal{S}$ with convergent initial conditions $\phi_i(0, 0)$ there exists a graphically convergent subsequence, the graphical limit ϕ of which satisfies $\phi(0, 0) = \lim_{i \to \infty} \phi_i(0, 0)$.*

To ensure that the graphical limit ϕ above is a solution, further conditions need to be placed on (1). Systems where this property holds have been named *nominally well-posed* in [14]. The definition of *well-posed* systems considers vanishing perturbations, and is not considered here.

Definition 2.3: *The hybrid system* (1) *is* nominally well-posed *if for every graphically convergent sequence ϕ_i of solutions with convergent initial conditions $\phi_i(0, 0)$, either*

(a) *the sequence ϕ_i is locally eventually bounded and its graphical limit ϕ is a solution to* (1) *with $\phi(0, 0) = \lim_{i \to \infty} \phi_i(0, 0)$ and with $\lim_{i \to \infty} \operatorname{length} \operatorname{dom} \phi_i = \operatorname{length} \operatorname{dom} \phi$, or*
(b) *there exists an unbounded and not complete solution from $\lim_{i \to \infty} \phi_i(0, 0)$.*

Above, $\operatorname{length} \operatorname{dom} \phi := \sup \{t + j \,|\, (t, j) \in \operatorname{dom} \phi\}$. Below, $\operatorname{length}_t \operatorname{dom} \phi := \sup \{t \,|\, (t, j) \in \operatorname{dom} \phi\}$ is also used.

If a nominally well-posed system is pre-forward complete, then it does not *exhibit finite-time blow-up*, i.e., there is no unbounded and not complete solutions to it, and so every graphically convergent subsequence ϕ_i as above is locally eventually bounded and (a) above holds.

A sufficient condition for nominal well-posedness, in fact for well-posedness, of (1) is given by [14, Theorem 6.8]. The result says that (1) is well-posed if the data satisfies some regularity conditions, termed hybrid basic assumptions. The set-valued analysis terminology below follows [25]: a set-valued map $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for every convergent sequence of x_i's and every convergent sequence of $y_i \in M(x_i)$, $\lim y_i \in M(\lim x_i)$, and *locally bounded* if for every compact $K \subset \mathbb{R}^n$ there exists a compact $K' \subset \mathbb{R}^n$ such that $M(K) \subset K'$. When dynamics are single valued, the conditions on F and G below reduce to $F : C \to \mathbb{R}^n$, $G : D \to \mathbb{R}^n$ being continuous functions.

Definition 2.4: *The hybrid system* (1) *satisfies the* hybrid basic assumptions *if its data, C, F, D, G, satisfies the following conditions: $C, D \subset \mathbb{R}^n$ are closed; the set-valued mappings $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are locally bounded and outer semicontinuous; for every $x \in C$, $F(x)$ is nonempty, closed, and convex; for every $x \in D$, $G(x)$ is nonempty and closed.*

Select consequences of (1) being well-posed include:

(a) Solutions depend on initial conditions in an outer-semicontinuous way (roughly, limits of solutions are solutions) and this can be characterized in terms of distances between graphs of solutions.
(b) Solutions depend on initial conditions continuously when uniqueness of solutions can be ensured.
(c) The Krasovskii-LaSalle invariance principle, and other arguments relying on invariance, can be extended to (1).
(d) For a compact asymptotically stable set in (1), the basin of attraction is open and from it, the convergence to the set is uniform and it admits a $\mathcal{KL}$ bound.

For details, consult [14]. Further meaning to (a) and (b) above can be found in [14, Theorem 5.25] which says that a locally eventually bounded sequence ϕ_i of hybrid arcs graphically converges to a hybrid arc ϕ if and only if for every $\varepsilon > 0$, $\tau > 0$, the arcs ϕ_i and ϕ are (τ, ε)-close for all large enough i. The concept of (τ, ε)-closeness is defined below, along with other closeness concepts that apply to solutions under assumptions involving pointwise asymptotic stability and small ordinary time property.

Definition 2.5: *For a given $\varepsilon > 0$, $\tau > 0$, two hybrid arcs $\phi, \phi' \in \mathcal{S}$ are*

(a) (τ, ε)-close *if*
  - (i) $\forall\ (t,j) \in \operatorname{dom}\phi$ *with* $t+j < \tau\ \exists\ (t',j) \in \operatorname{dom}\phi'$ *with* $|t-t'| < \varepsilon$, $\|\phi(t,j) - \phi'(t',j)\| < \varepsilon$;
  - (ii) $\forall\ (t',j') \in \operatorname{dom}\phi'$ *with* $t'+j' < \tau\ \exists\ (t,j') \in \operatorname{dom}\phi$ *with* $|t-t'| < \varepsilon$, $\|\phi'(t',j') - \phi(t,j')\| < \varepsilon$.

(b) ε-close to τ-truncations of one another *if*
  - (i) $\forall\ (t,j) \in \operatorname{dom}\phi\ \exists\ (t',j') \in \operatorname{dom}\phi'$, $t'+j' < \tau$ *with* $\|\phi(t,j) - \phi'(t',j')\| < \varepsilon$;
  - (ii) $\forall\ (t',j') \in \operatorname{dom}\phi'\ \exists\ (t,j) \in \operatorname{dom}\phi$, $t+j < \tau$ *with* $\|\phi'(t',j') - \phi(t,j)\| < \varepsilon$.

(c) ε-close *if*
  - (i) $\forall\ (t,j) \in \operatorname{dom}\phi\ \exists\ (t',j) \in \operatorname{dom}\phi'$ *with* $|t-t'| < \varepsilon$ *and* $\|\phi(t,j) - \phi'(t',j)\| < \varepsilon$
  - (ii) $\forall\ (t',j') \in \operatorname{dom}\phi'$ *there exists* $(t,j') \in \operatorname{dom}\phi$ *with* $|t-t'| < \varepsilon$ *and* $\|\phi'(t',j') - \phi(t,j')\| < \varepsilon$.

III. Stability concepts

This section presents the pointwise asymptotic stability and small ordinary time stability concepts.

A. Pointwise asymptotic stability

Definition 3.1: *A set $A \subset \mathbb{R}^n$ is* pointwise asymptotically stable (PAS) *if:*

(a) *every $a \in A$ is* Lyapunov stable*, that is, for every $a \in A$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(t,j) \in a + \varepsilon\mathbb{B}$ for every $\phi \in \mathcal{S}(a+\delta\mathbb{B})$, every $(t,j) \in \operatorname{dom}\phi$, and*

(b) *there exists a neighborhood U of A such that every $\phi \in \mathcal{S}(U)$ is bounded, and if it is complete, then $\lim_{t+j\to\infty}\phi(t,j)$ exists and belongs to A.*

The basin of pointwise attraction *of a pointwise asymptotically stable set A, denoted $\mathcal{B}(A)$, is the set of $x \in \mathbb{R}^n$ such that every $\phi \in \mathcal{S}(x)$ is bounded and if it is complete, then $\lim_{t+j\to\infty}\phi(t,j)$ exists, and belongs to A.*

Sufficient conditions for pointwise asymptotic stability in hybrid systems, using strict or weak set-valued Lyapunov functions or relying on so-called "finite-length Lyapunov functions" are in [13].

B. Small ordinary time property

Definition 3.2:

(a) *A point $a \in \mathbb{R}^n$ is* small ordinary time stable (SOT stable) *if it is Lyapunov stable and for every $\varepsilon > 0$ there exists $\delta > 0$ such that* $\operatorname{length}_t \operatorname{dom}\phi < \varepsilon$ *for every $\phi \in \mathcal{S}(a+\delta\mathbb{B})$.*

(b) *A set $A \subset \mathbb{R}^n$ is* pointwise small ordinary time asymptotically stable (PSOTAS) *if it is pointwise asymptotically stable and every $a \in A$ is SOT stable.*

Necessary or sufficient conditions for related properties, either of an equilibrium or of a compact attractor, are in [21], [23], [15], and [16]. The following sufficient condition for PSOTAS is motivated by [15, Proposition 3.2].

Proposition 3.3: *Suppose that a closed set $A \subset \mathbb{R}^n$ is PAS for* (1) *and there exists a continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}$ such that*

(a) *V is positive definite with respect to A;*

(b) *there exists $c > 0$ and $\rho \in [0,1)$ such that*

$$\nabla V(x) \cdot f \leq -c\,(V(x))^\rho \quad \forall x \in C,\ f \in F(x);$$

(c) *$V(g) \leq V(x) \quad \forall x \in D,\ g \in G(x);$*

and there exist no nontrivial flowing solutions ϕ satisfying $\phi(t,j) \subset A$ for all $(t,j) \in \operatorname{dom}\phi$. Then A is PSOTAS.

IV. Nominal well-posedness under PAS and SOT

The main result of the paper, describing well-posedness properties in presence of a PAS and PSOTAS closed set, is below. It provides a tool for establishing properties of the limit point mappings and of Zeno times, as given in the corollary.

Theorem 4.1: *Suppose that* (1) *is nominally well-posed and forward complete. For every sequence of $\phi_i \in \mathcal{S}$ with $\phi_i(0,0)$ convergent, there exists a graphically convergent subsequence, which is not relabeled, such that*

(a) *the graphical limit ϕ of the graphically convergent subsequence ϕ_i is a complete solution to* (1).

If, additionally, the closed set $A \subset \mathbb{R}^n$ is PAS and $\lim_{i\to\infty}\phi_i(0,0) \in \mathcal{B}(A)$, then

(b) *for all large enough i, $\lim_{t+j\to\infty}\phi_i(t,j)$ exists and belongs to A;*

(c) *$\lim_{i\to\infty}\lim_{t+j\to\infty}\phi_i(t,j) = \lim_{t+j\to\infty}\phi(t,j)$;*

(d) *convergence of ϕ_i to ϕ is uniform in the following sense: for every $\varepsilon > 0$ there exists $\tau > 0$ such that, for every large enough i, ϕ_i and ϕ are ε-close to τ-truncations of one another.*

If, additionally, A is PSOTAS, then

(e) *ϕ and ϕ_i for all large enough i are Zeno, and $\lim_{i\to\infty}\operatorname{length}_t\operatorname{dom}\phi_i = \operatorname{length}_t\operatorname{dom}\phi$;*

(f) *convergence of ϕ_i to ϕ is uniform in the following sense: for every $\varepsilon > 0$ and every large enough i, ϕ_i and ϕ are ε-close to one another.*

Corollary 4.2: *Suppose* (1) *is nominally well-posed and forward complete and the closed set $A \subset \mathbb{R}^n$ is PAS. Then:*

(a) *the basin of pointwise attraction of A, $\mathcal{B}(A)$, is an open neighborhood of A;*

(b) *the set-valued mapping $\mathcal{L} : \mathcal{B}(A) \rightrightarrows \mathbb{R}^n$ defined by*

$$\mathcal{L}(x) = \left\{ \lim_{t+j\to\infty}\phi(t,j) \,\middle|\, \phi \in \mathcal{S}(x) \right\} \tag{2}$$

is outer semicontinuous and locally bounded.

If, additionally, A is PSOTAS, then:

(d) *the function* $\operatorname{Length}_t : \mathcal{B}(A) \to [0, \infty]$ *defined by*

$$\operatorname*{Length}_t(x) = \sup \left\{ \operatorname*{length}_t \operatorname{dom} \phi \,\middle|\, \phi \in \mathcal{S}(x) \right\}$$

is upper semicontinuous, locally bounded (in particular, finite-valued), and the sup *defining* $\operatorname{Length}_t(x)$ *is attained for every* $x \in \mathcal{B}(A)$.

V. Scenario 1: past Zeno to a compact attractor

The first scenario considers a hybrid system with a PSOTAS closed set and a continuation of solutions to it using a second hybrid system, with initial conditions for the continuations depending on the limits, which are Zeno equilibria, of the pre-Zeno solutions. The following assumption is in place throughout the section.

Assumption 5.1:

- (C_1, F_1, G_1, D_1) *satisfy hybrid basic assumptions and define a forward complete hybrid system in* $\mathbb{R}^{n_1}$*;*
- A_1 *is a nonempty and closed set that is PSOTAS for* (C_1, F_1, G_1, D_1)*;*
- (C_2, F_2, G_2, D_2) *satisfy hybrid basic assumptions and define a pre-forward complete hybrid system in* $\mathbb{R}^{n_2}$*;*
- $\Psi : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$ *is an outer semicontinuous and locally bounded set-valued mapping.*

System (C_1, F_1, G_1, D_1) models the dynamics of the hybrid system before Zeno while (C_2, F_2, D_2, G_2) models the post-Zeno dynamics. Let $\mathcal{B}_1(A_1) \subset \mathbb{R}^{n_1}$ be the basin of pointwise attraction of A_1 for (C_1, F_1, G_1, D_1).

Definition 5.2: *A solution to the system above is a pair* (ϕ, ψ) *such that*

- ϕ *is a complete solution to* (C_1, F_1, G_1, D_1)*;*
- $\phi(0, 0) \in \mathcal{B}_1(A_1)$
- $\psi(T, 0) \in \Psi \left(\lim_{t+j \to \infty} \phi(t, j) \right)$, *where* $T = \operatorname{length}_t \operatorname{dom} \phi$*;*
- ψ *is a maximal solution to* (C_2, F_2, G_2, D_2) *defined on a hybrid time domain from* $(T, 0)$.

Above, a *hybrid time domain from* $(T, 0)$ is a set S such that $S - (T, 0)$ is a hybrid time domain in the usual sense (in other words, a hybrid time domain from $(0, 0)$). Similarly, a *solution to* (C_2, F_2, G_2, D_2) *defined on a hybrid time domain from* $(T, 0)$ is a function ψ such that $(t, k) \mapsto \psi(t + T, k)$ is a solution to (C_2, F_2, G_2, D_2) on a hybrid time domain $S - (T, 0)$. In what follows, $\mathcal{S}$ is the set of solutions as defined above, $\mathcal{S}_1$ is the set of solutions to (C_1, F_1, G_1, D_1), and $\mathcal{S}_2$ is the set of solutions to (C_2, F_2, G_2, D_2). $\mathcal{S}(x)$ is the set of solutions (ϕ, ψ) as above such that $\phi(0, 0) = x$.

Theorem 5.3: *Suppose Assumption 5.1 holds. Let* $(\phi_i, \psi_i) \in \mathcal{S}(x_i)$ *with* $x_i \to x \in \mathcal{B}_1(A_1)$. *Then, there exists a graphically convergent subsequence of* (ϕ_i, ψ_i)*, the limit* (ϕ, ψ) *of which is a solution in* $\mathcal{S}(x)$. *Furthermore, for every* $\varepsilon > 0$, $\tau > 0$ *and all large enough* i, ψ_i *and* ψ *are* (τ, ε)-close*, in the sense of Definition 2.5 (a).*

Further results can be shown when the second hybrid system has a pre-asymptotically stable compact set [14].

Assumption 5.4: A_2 *is a compact pre-asymptotically stable set for* (C_2, F_2, G_2, D_2).

Definition 5.5: *Under Assumptions 5.1 and 5.4, the basin of attraction* $\mathcal{B}_1(A_2) \subset \mathbb{R}^{n_1}$ *is the set of all* $x \in \mathbb{R}^{n_1}$ *such that every* $(\phi, \psi) \in \mathcal{S}(x)$ *is bounded and if* ψ *is complete, then* $\lim_{t+k \to \infty} d_{A_2}(\psi(t, k)) = 0$.

Above, d_{A_2} is the distance from A_2, that is $d_{A_2}(x) = \inf_{a \in A_2} \|x - a\|$. Let $\mathcal{B}_2(A_2) \subset \mathbb{R}^{n_2}$ be the basin of pre-attraction of A_2 for (C_2, F_2, G_2, D_2), i.e., the set of all $x \in \mathbb{R}^{n_2}$ such that every $\psi \in \mathcal{S}(x)$ is bounded and if it is complete, it converges to A_2. Directly from the definitions, one can gather that $\mathcal{B}_1(A_2) \subset \mathcal{B}_1(A_1)$ and $\Psi(\mathcal{L}_1(\mathcal{B}_1(A_2))) \subset \mathcal{B}_2(A_2)$. Here, $\mathcal{L}_1$ is the limit mapping (2) for (C_1, F_1, G_1, D_1).

Proposition 5.6: *If Assumptions 5.1 and 5.4 hold, then:*

(a) *The basin of attraction* $\mathcal{B}_1(A_2)$ *is open.*

(b) *Convergence from* $\mathcal{B}_1(A_2)$ *to* A_2 *is uniform, in the sense that for every compact set* $K_1 \subset \mathcal{B}_1(A_2)$ *and every neighborhood* U_2 *of* A_2 *there exists* $T > 0$ *such that, for every* $(\phi, \psi) \in \mathcal{S}(K_1)$*, every* $(t, k) \in \operatorname{dom} \psi$ *with* $t + k > T$, $\psi(t, k) \in U_2$.

VI. Scenario 2: partial state goes past Zeno

The second scenario builds upon the first scenario, by considering two hybrid systems evolving pre-Zeno time. One system is uniformly not Zeno, the other is like in the first scenario, and these two systems are decoupled. Then, the continuation of solutions past Zeno time depends on solutions to both pre-Zeno systems. The following assumption is in place throughout the section.

Assumption 6.1:

- (C_0, F_0, G_0, D_0) *satisfies hybrid basic assumptions and defines a forward complete hybrid system in* $\mathbb{R}^{n_0}$*;*
- (C_0, F_0, G_0, D_0) *is locally uniformly non-Zeno;*
- (C_1, F_1, G_1, D_1) *satisfies hybrid basic assumptions and defines a forward complete hybrid system in* $\mathbb{R}^{n_1}$*;*
- A_1 *is a closed pointwise SOT asymptotically stable set for* (C_1, F_1, G_1, D_1)*;*
- (C_2, F_2, G_2, D_2) *satisfies hybrid basic assumptions and defines a pre-forward complete hybrid system* $\mathbb{R}^{n_2}$*;*
- $\Psi : \mathbb{R}^{n_0+n_1} \rightrightarrows \mathbb{R}^{n_2}$ *is an outer semicontinuous and locally bounded set-valued mapping.*

The assumption that (C_0, F_0, G_0, D_0) be locally uniformly non-Zeno means: for every compact $K_0 \subset \mathbb{R}^{n_0}$ there exist $T, J > 0$ such that for every solution ϕ to (C_0, F_0, G_0, D_0) with $\phi(0, 0) \in K$, for every $(t, j), (t', j') \in \operatorname{dom} \phi$, if $|t - t'| \leq T$ then $|j - j'| \leq J$. This property is implied by, for example, dwell-time and average dwell-time conditions.

Definition 6.2: *A solution to the system above is a triple* (ϕ_0, ϕ_1, ψ) *such that*

- ϕ_0 *is a complete solution to* (C_0, F_0, G_0, D_0)*;*
- ϕ_1 *is a complete solution to* (C_1, F_1, G_1, D_1)*;*
- $\phi_1(0, 0) \in \mathcal{B}_1(A_1)$

- $\psi(T,0) \in \Psi\left(\phi_0(T,j(T)), \lim_{t+j\to\infty}\phi_1(t,j)\right)$, *where* $T = \text{length}_t \operatorname{dom}\phi_1$ *and* $j(T)$ *is any natural number such that* $(T, j(T)) \in \operatorname{dom}\phi_0$*;*
- ψ *is a maximal solution to* (C_2, F_2, G_2, D_2) *defined on a hybrid time domain from* $(T,0)$.

The following is a generalization of Theorem 5.3.

Theorem 6.3: *Suppose Assumption 6.1 holds. Let* $(\phi_{0,i}, \phi_{1,i}, \psi_i) \in \mathcal{S}(x_{0,i}, x_{1,i})$ *with* $x_{0,i} \to x_0$ *and* $x_{1,i} \to x_1 \in \mathcal{B}_1(A_1)$*. Then, there exists a graphically convergent subsequence of* $(\phi_{0,i}, \phi_{1,i}, \psi_i)$*, the limit* (ϕ_0, ϕ_1, ψ) *of which is a solution in* $\mathcal{S}(x_0, x_1)$*. Furthermore, without relabeling, for every* $\varepsilon > 0$*,* $\tau > 0$ *and all large enough* i*,* ψ_i *and* ψ *are* (τ, ε)-close*, in the sense of Definition 2.5 (a).*

Corollary 6.4: *Suppose Assumption 6.1 holds. For every compact set* $K \subset \mathbb{R}^{n_0} \times \mathcal{B}_1(A_1)$ *and every* $\tau > 0$*,* $\varepsilon > 0$*, there exist* $\delta > 0$ *with the following property: for every* $(\phi_0, \phi_1, \psi) \in \mathcal{S}(K + \delta\mathbb{B})$ *there exists* $(\phi_0', \phi_1', \psi') \in \mathcal{S}(K)$ *such that* ψ *and* ψ' *are* (τ, ε)*-close.*

VII. Examples

A. Consensus in Zeno time and beyond

Consider two agents with dynamics given by

$$\dot{z}_i = c_i u_i \qquad z_i, u_i \in \mathbb{R}^{n_z}, i \in \{1,2\} \tag{3}$$

where c_1 and c_2 are positive constants. An algorithm for consensus that requires transmission of information between the agents at discrete events, reaches consensus with Zeno, and after that allows z_1 and z_2 to be controlled by a well-posed hybrid control algorithm with state of dimension n_c inducing globally asymptotic stability of a compact set $A_2 \subset \mathbb{R}^{n_z+n_z+n_c}$ is given next – note that since A_2 is asymptotically stable, every maximal solution to the closed loop with this controller is complete and approaches A_2.

A timer τ triggers the transmission of information between agents when it reaches zero. In between events, the timer decreases, $\dot{\tau} = -1$. A memory state a stores the average of z_1 and z_2 at transmission events. When τ reaches 0, τ updates to $\min\{t_1, t_2\}$, where

$$t_i := \frac{\sqrt{|a - z_i|}}{c_i}, \quad i = 1,2$$

and a is mapped to the average $\dfrac{1}{2}(z_1 + z_2)$. The inputs are assigned to $u_i = \dfrac{a - z_i}{\sqrt{|a - z_i|}}$, $i \in \{1,2\}$. This way, the timer resets when one of the agents reduces its distance from a by a factor of 4. This algorithm can be modeled as a well-posed hybrid controller, leading to the closed-loop system:

- (C_1, F_1, D_1, G_1), $n_1 = 3n_z + 1$, state $\eta = (z_1, z_2, a, \tau)$,

$$C_1 = \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times [0, \infty),$$

$$F_1(\eta) = \begin{bmatrix} c_1 \frac{a - z_1}{\sqrt{|a - z_1|}} \\ c_2 \frac{a - z_2}{\sqrt{|a - z_2|}} \\ 0 \\ -1 \end{bmatrix} \quad \forall \eta \in C_1,$$

$$D_1 = \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \{0\},$$

$$G_1(\eta) = \begin{bmatrix} z_1 \\ z_2 \\ \frac{1}{2}(z_1 + z_2) \\ \min\{t_1, t_2\} \end{bmatrix} \quad \forall \eta \in D_1.$$

This system is nominally well-posed and forward complete. Moreover, the set $A_1 = \{\eta \in C_1 \mid z_1 = z_2 = a, \tau = 0\}$ is closed and pointwise SOT asymptotically stable with basin of attraction $\mathcal{B}_1(A_1) = \mathbb{R}^{n_1}$. In fact,

$$W(\eta) = \text{co}\{z_1, z_2, a\} \times [0, \max\{\tau, \min\{t_1, t_2\}\}]$$

is a weak set-valued Lyapunov function, as defined in [13, Definition 4.4], and pointwise asymptotic stability can be concluded by invariance arguments, as in [13, Theorem 4.5]. Indeed,

- $W(\eta) = \{\eta\}$ for each $\eta \in A_1$;
- W is locally bounded and continuous;
- During flows, z_i flow to a along the directions $\frac{a - z_i}{|a - z_i|}$ and a remains constant, and thus $\text{co}\{z_1, z_2, a\}$ is not increasing. Also, τ decreases, t_i decrease, and thus $\max\{\tau, \min\{t_1, t_2\}\}$ decreases. So, every solution $t \mapsto \phi(t)$ to $\dot{\eta} = F_1(\eta), \eta \in C_1$ satisfies
$$W(\phi(t)) \subset W(\phi(0)) \qquad \forall t \in \operatorname{dom}\phi$$
- At jumps, z_i don't change, a is updated to $\text{ave}(z_1, z_2) \in \text{co}\{z_1, z_2, a\}$, and thus $\text{co}\{z_1, z_2, a\}$ does not increase. Also, when $\tau = 0$, $[0, \max\{\tau, \min\{t_1, t_2\}\}] = [0, \min\{t_1, t_2\}]$, and an update of τ to $\min\{t_1, t_2\}$ does not increase the set $[0, \max\{\tau, \min\{t_1, t_2\}\}]$. Hence, for each $\eta \in D_1$,
$$W(G_1(\eta)) \subset W(\eta).$$

Furthermore, suppose there exists a weakly invariant set M_1 for (C_1, F_1, D_1, G_1) that is not contained in A_1 and on which W is constant. Then, by definition of weak invariance, there exists a complete solution $(t,j) \mapsto \phi(t,j)$ that stays in M_1 and, for some set W_0, $W(\phi(t,j)) = W_0$ for each $(t,j) \in \operatorname{dom}\phi$. Since M_1 is not contained in A_1, there exists $(t,j) \in \operatorname{dom}\phi$ such that $\phi(t,j) \notin A_1$. If (t,j) is not a jump time, then at least one of the z_i components of ϕ flows towards a, implying that $W(\phi(t,j))$ does not remain equal to W_0. If (t,j) is a jump time, since $\phi(t,j) \notin A_1$, the a^+ is not equal to the either z_1 or z_2 solution components, from where flows towards a would follow. This also contradicts W staying equal to W_0 along the solution. Then, [13, Theorem 4.5] implies that A_1 is PAS. The SOT property can be certified using $V(\eta) = \frac{1}{2}\sum_{i=1}^{2}|z_i - a|^2$ in Proposition 3.3.

Due to the continuity of the right-hand side of (3), the well-posed hybrid control algorithm to be used beyond Zeno leads to a well-posed hybrid inclusion (C_2, F_2, D_2, G_2) with state χ and a compact set A_2 that is pre-asymptotically stable. The state χ has components that correspond to the agent states z_1 and z_2. For these solution components to be properly continued after Zeno, it suffices to pick Ψ such that $\Psi(\eta) = (z_1, z_2)$ for each $\eta \in \mathbb{R}^{n_1}$.

The above arguments show that Assumption 5.1 holds. By Proposition 5.6, convergence from $\mathcal{B}_1(A_2) = \mathbb{R}^{n_1}$ is uniform. Namely, from compact sets in $\mathbb{R}^{n_1}$, solutions to the entire system are such that the χ components reach neighborhoods of A_2 uniformly in hybrid time.

B. Bouncing ball with horizontal velocity

Consider a point-mass model of a ball bouncing on the ground, at zero height, and moving horizontally without friction, so that after the impacts with the ground accumulate, the ball rolls horizontally. A model of this system in the proposed framework is given by

- (C_0, F_0, G_0, D_0) with $n_0 = 1$, state $\zeta \in \mathbb{R}$,

$$C_0 = \mathbb{R}, \qquad F_0(\zeta) = v_x \quad \forall \zeta \in C_0$$
$$D_0 = \emptyset, \qquad G_0 \text{ arbitrary}$$

  where $v_x \in \mathbb{R}$ denotes the horizontal velocity;
- (C_1, F_1, G_1, D_1) with $n_1 = 2$, state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$,

$$C_1 = [0, \infty) \times \mathbb{R}, \qquad F_1(\eta) = \begin{bmatrix} \eta_2 \\ -\gamma \end{bmatrix}$$
$$D_1 = \{0\} \times (-\infty, 0], \qquad G_1(\eta) = \begin{bmatrix} \eta_1 \\ -\lambda \eta_2 \end{bmatrix}$$

  where $\gamma > 0$ denotes the gravity constant and $\lambda \in [0, 1)$ the restitution coefficient;
- $A_1 = \{(0, 0)\} \in \mathbb{R}^2$;
- (C_2, F_2, G_2, D_2), $n_2 = 3$, state $\chi = (\chi_1, \chi_2, \chi_3) \in \mathbb{R}^3$

$$C_2 = \{0\} \times \{0\} \times \mathbb{R}, \qquad F_2(\chi) = \begin{bmatrix} 0 \\ 0 \\ v_x \end{bmatrix} \quad \forall \chi \in C_2$$
$$D_2 = \emptyset, \qquad G_2 \text{ arbitrary}$$

- $\Psi : \mathbb{R}^{n_0+n_1} \rightrightarrows \mathbb{R}^{n_2}$ defined as $\Psi(\zeta, \eta) = \begin{bmatrix} \zeta \\ 0 \\ 0 \end{bmatrix}$ for each $(\zeta, \eta) \in \mathbb{R} \times \mathbb{R}^2$.

Systems (C_0, F_0, G_0, D_0) and (C_1, F_1, G_1, D_1) capture the horizontal and vertically motion of the ball, respectively, before Zeno, while (C_2, F_2, G_2, D_2) models its evolution after Zeno. For simplicity, the horizontal motion is described by a first-order system. Assumption 6.1 holds. Moreover, (C_0, F_0, G_0, D_0) has only continuous solutions. Pointwise SOT asymptotic stability of A_1 for (C_1, F_1, G_1, D_1) follows from the fact that the bouncing ball system with only vertical motion with $\lambda \in [0, 1)$ is Zeno and that A_1 is an isolated globally asymptotically stable point; see, e.g., [14, Example 2.12 and Example 3.19]. A typical solution reaches Zeno at a point $(\zeta, \eta) = (\zeta^*, 0)$ and, after the solution's Zeno time, flows away from it. The map Ψ trivially satisfies the required conditions. Then, by Theorem 6.3 and Corollary 6.4, solutions to entire system depend upper semicontinuously with respect to initial conditions. In fact, as the solutions are unique, the dependence is continuous.

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