A Decentralized Consensus Algorithm for Distributed State Observers with Robustness Guarantees

Yuchun Li and Ricardo G. Sanfelice

Abstract— **Motivated by the design of distributed observers with good performance and robustness to measurement and communication noise, the problem of obtaining a global estimate, over a graph, of a common process is considered. We propose a global consensus algorithm that satisfies a pre-specified rate of convergence and has optimized robustness to both communication and measurement noise. The convergence rate and the robustness to communication and measurement noise of the proposed consensus algorithm are characterized in terms of (nonlinear and linear) optimization problems. The stability and robustness properties of the proposed algorithm are shown analytically and validated numerically.**

I. Introduction

The problem of estimating the state $x \in \mathbb{R}^n$ of the linear time-invariant system

$$\dot{x} = Ax \tag{1}$$

in a distributed fashion through output measurements of the form $y = Cx$ have lead to enlightening results in distributed estimation and consensus. Distributed Kalman filtering is employed to achieve spatially-distributed estimation tasks in [1] and for sensor networks in [2], [3], [4], [5]. In [6], switching topologies are used to design distributed observers for a leader-follower problem in multi-agent systems. In [7], a synthesis method for distributed estimation that guarantees H_∞-performance of the estimates with respect to model and measurement disturbances is presented.

Along with distributed estimation, consensus for distributed systems has also been extensively studied in the literature. Static consensus, which corresponds to the agreement on the initial conditions of the agents, modeled via a directed graph, are reported in [8], [9], [10]. For dynamic average consensus, [11] proposes a decentralized algorithm that guarantees asymptotic agreement of a signal over strongly connected and weight-balanced graphs. In [12], a dynamic consensus algorithm that is robust to communication time delay is proposed. In [13], an observer-type of consensus algorithm is designed for linear systems without noise. In [14], the stability and convergence properties of integral and propotional-integral consensus is thoroughly studied. On the other hand, the literature on consensus design with robustness to measurement and communication noise is not as rich. In [15], the effect of additive noise with zero mean on the variables of a consensus algorithm is discussed, while in [16], a performance region-based approach is used for the design of distributed H_∞-based consensus. In [17], Kalman filter based consensus is shown to be robust to state and input noise. In [18], the author studies consensus under random topologies with noise and link failures.

In this paper, we propose a decentralized consensus algorithm for distributed state observers with robustness guarantees. In [19], interconnected Luenberger observers for linear time-invariant systems are proposed to guarantee performance specifications and minimizing the effect of measurement noise. However, the setting in [19] does not include noise in the signals communicated between the agents. Moreover, it does not provide a way to design a consensus algorithm for the generation of a global state estimate, even for the nominal case. Unfortunately, in distributed state estimation, communication and measurement noise are two factors that cannot be ignored for many practical applications. In fact, algorithms for distributed estimation should be capable of generating local and global estimates that are robust and have good performance. Motivated by the lack of such algorithms, we propose a consensus algorithm to generate a global estimate from the (time-varying) local estimates generated by the distributed state observer in [19], under communication and measurement noise. The performance and robustness of the proposed algorithms can be specified at the design stage. To accomplish that, we propose tools to synthesize the distributed observer and the consensus algorithm using H_∞ theory and optimization techniques. The designs using these tools guarantee that the agents agree on the average of all local (time-varying) estimates, asymptotically and robustly with respect to communication and measurement noise.

The remainder of this paper is organized as follows. After a motivational example in Section II, the problem of interested is formulated in Section III-B. Section III introduces the proposed consensus algorithm. Design methods are presented in Section IV-A and Section IV-B. Examples illustrate the results established throughout the paper. Proofs will be published elsewhere due to space constraints.

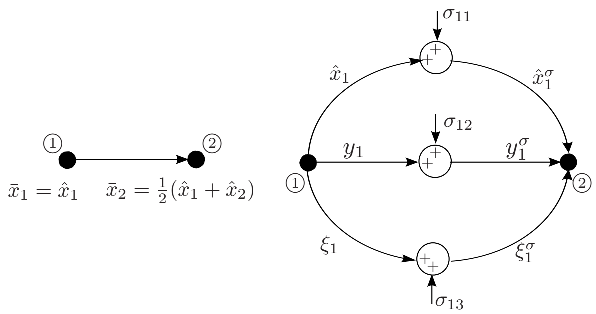
II. Motivational Example

Consider the scalar plant

$$\dot{x} = ax, \quad y = x, \tag{2}$$

where $a < 0$. It is shown in [19] that a distributed state observer can be designed to satisfy a pre-specified rate of convergence constraint and have an optimized H_∞ gain from measurement noise to estimation error. Suppose there are two connected agents such that agent 1 can transmit information to agent 2, but agent 2 cannot send data to agent 1, as shown in Figure 1(a). Following [19], a distributed state observer

Y. Li and R. G. Sanfelice are with the Department of Computer Engineering, University of California, Santa Cruz, CA 95064. Email: yuchunli,ricardo@ucsc.edu Research partially supported by NSF Grants no. ECS-1150306 and CNS-1544396, and by AFOSR Grant FA9550-16-1-0015.



(a) Graph connectivity, self link is not shown.

(b) Information over the network under the effect of communication noise.

Fig. 1. Two agents connected via a direct graph.

takes the form[1]

$$
\begin{aligned}
&\dot{\hat{x}}_1 = a\hat{x}_1 - K_{11}(\hat{y}_1 - y_1),\\
&\dot{\hat{x}}_2 = a\hat{x}_2 - K_{22}(\hat{y}_2 - y_2) - K_{21}(\hat{x}_1^\sigma - y_1^\sigma),\\
&\hat{y}_1 = \hat{x}_1, \quad \hat{y}_2 = \hat{x}_2, \quad y_1 = x + m_1, \quad y_2 = x + m_2,\\
&\hat{x}_1^\sigma = \hat{x}_1 + \sigma_{11}, \quad y_1^\sigma = y_1 + \sigma_{12},\\
&\bar{x}_1 = \hat{x}_1, \quad \bar{x}_2 = \frac{1}{2}(\hat{x}_1 + \hat{x}_2),
\end{aligned} \tag{3}
$$

where, for each $i \in \{1, 2\}$, $\hat{x}_i$ is a state associated to agent i, $\hat{y}_i$ is the output of agent i, $\hat{x}_1^\sigma$ is the corrupted version of $\hat{x}_1$ due to communication noise that is received at agent 2, m_i is the measurement noise when agent i takes measurements of the plant's state x, and σ_{11}, σ_{12} are the communication noises that affect $\hat{x}_1$ and y_1, respectively, as shown in Figure 1(b). Variables $\bar{x}_1, \bar{x}_2$ provide local estimates at agent 1 and 2, respectively. The gains K_{11}, K_{21}, K_{22} are to be determined. Note that m_i's are assumed to be independent. As agent 1 sends information of $\hat{x}_1, y_1$ to agent 2, the information is also influenced by communication noise, namely, agent 2 receives the corrupted version of $\hat{x}_1$ and y_1, given by $\hat{x}_1^\sigma = \hat{x}_1 + \sigma_{11}$ and $y_1^\sigma = y_1 + \sigma_{12}$, respectively.

By defining $e_i = \hat{x}_i - x$, $\tilde{e}_i = \bar{x}_i - x$, for each $i \in \{1, 2\}$, $m = (m_1, m_2)$, $\tilde{e} = (\tilde{e}_1, \tilde{e}_2)$ and $e = (e_1, e_2)$, we can rewrite system (3) in the form

$$
\dot{e} = A_e e + B_1 m + B_2 \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix}, \quad \tilde{e} = C_e e, \tag{4}
$$

where

$$
A_e = \begin{bmatrix} a - K_{11} & 0 \\ -K_{21} & a - K_{22} \end{bmatrix}, \quad C_e = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix},
$$
$$
B_1 = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ -K_{21} & K_{21} \end{bmatrix}.
$$

The quantity $\frac{1}{2}(\hat{x}_1 + \hat{x}_2)$ provides a global estimate of x, namely, the average of the estimate at each agent. However, not every agent in the network can compute this quantity. In the particular example considered, agent 1 is not capable of computing it since it does not receive $\hat{x}_2$.

In this paper, we propose a decentralized consensus algorithm to guarantee a pre-specified rate of convergence constraint as well as robustness to measurement and communication noise. For the particular distributed state observer leading to (4), our consensus algorithm is given by

$$\dot{\xi}_1 = \dot{\hat{x}}_1 - \alpha_1(\xi_1 - \hat{x}_1), \tag{5a}$$
$$\dot{\xi}_2 = \dot{\hat{x}}_2 - \alpha_2(\xi_2 - \hat{x}_2) - \omega_{21}(\xi_2 - \xi_1^\sigma) - \theta_{21}(\hat{x}_2 - \hat{x}_1^\sigma), \tag{5b}$$
$$\xi_1^\sigma = \xi_1 + \sigma_{13}, \tag{5c}$$

where ξ_1 and ξ_2 are consensus variables attached to agent 1 and agent 2, respectively, and $\alpha_1, \alpha_2, \omega_{21}, \theta_{21}$ are parameters to be designed. The transmission of information ξ_1 is also corrupted by communication noise which is denoted by σ_{13}, i.e., the signal ξ_1^σ shown in Figure 1(b). By defining $\delta_i = \xi_i - \frac{1}{2}(\hat{x}_1 + \hat{x}_2)$ for each $i \in \{1, 2\}$, we can rewrite (5) as

$$
\begin{aligned}
\dot{\delta}_1 &= -\alpha_1\delta_1 + \frac{1}{2}\alpha_1(e_1 - e_2) + \frac{1}{2}(\dot{e}_1 - \dot{e}_2),\\
\dot{\delta}_2 &= -(\alpha_2 + \omega_{21})\delta_2 + \omega_{21}\delta_1 + \left(\frac{1}{2}\alpha_2 - \theta_{21}\right)(e_2 - e_1)\\
&\quad + \frac{1}{2}(\dot{e}_2 - \dot{e}_1) + \omega_{21}\sigma_{13} + \theta_{21}\sigma_{11}.
\end{aligned} \tag{6}
$$

Let $\delta = (\delta_1, \delta_2)$ and $\sigma = (\sigma_{11}, \sigma_{12}, \sigma_{13})$. To see the effect of noises m and σ, assume m and σ are constant, i.e., $m_1 \equiv m_2 \equiv m^\star$ and $\sigma_{11} = \sigma_{12} = \sigma_{13} \equiv \sigma^\star$. Then, the steady state of (4) and (6) is given by

$$
\delta^\star = \begin{bmatrix} 0 \\ \dfrac{\omega_{21} + \theta_{21}}{\alpha_2 + \omega_{21}} \end{bmatrix} \sigma^\star + \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \end{bmatrix} m^\star, \tag{7}
$$

$$
e^\star = -\begin{bmatrix} \dfrac{K_{11}}{a - K_{11}} \\ \dfrac{K_{21}a + K_{22}(a - K_{11})}{(a - K_{11})(a - K_{22})} \end{bmatrix} m^\star, \tag{8}
$$

where

$$
\bar{c}_1 = \frac{a(K_{21} + K_{22} - K_{11})}{2(K_{11} - a)(K_{22} - a)}, \tag{9}
$$

$$
\bar{c}_2 = \frac{a(\alpha_2 - \omega_{21} - 2\theta_{21})(K_{11} - K_{21} - K_{22})}{2(\alpha_2 + \omega_{21})(K_{11} - a)(K_{22} - a)}. \tag{10}
$$

Suppose K_{11} and K_{22} are fixed in order to satisfy a rate of convergence constraint[2], i.e., $K_{11} = K_{22}$. Then, by choosing $K_{21} = \frac{(K_{11} - a)K_{22}}{a}$, we have $e_2^\star = 0$. To make the steady state error $(\delta^\star, e^\star)$ corresponding to measurement noise $m^\star$ zero, we pick

$$
\alpha_2 - \omega_{21} - 2\theta_{21} = 0. \tag{11}
$$

A simulation of this scenario is shown in Figure 2(a), where the effect of measurement noise is shown. On the other hand, to make the steady-state error corresponding to communication noise $\sigma^\star$ zero, we pick

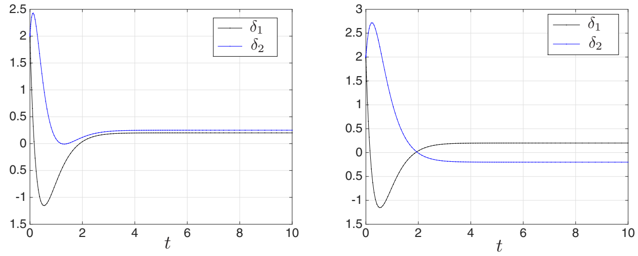
$$
\omega_{21} + \theta_{21} = 0. \tag{12}
$$

A simulation of this scenario is shown in Figure 2(b), where the effect of measurement noise is depicted. However, note that if equalities (11) and (12) are simultaneously satisfied, then $\alpha_2 = -\omega_{21}$. In this case, the system in (6) has an eigenvalue at zero, which does not lead to an exponentially convergent (to zero) state δ. In fact, this dilemma is generic and similar to the typical tradeoff between convergence rate and robustness to measurement noise for an observer, see, e.g., [20], [21], [22].

The idea behind the proposed consensus algorithm illus-

[1]In (3), the terms $a\hat{x}_i - K_{ii}(\hat{y}_i - y_i)$ follow from the definitions of injection terms for Luenberger observers. The term $-K_{21}(\hat{x}_1^\sigma - y_1^\sigma)$ is an innovative injection term that utilizes information of agent 1 and dynamically couples the first and second equation in (3).

[2]Note that since A_e is a lower triangular matrix, the eigenvalues of A_e are given by $a - K_{11}$ and $a - K_{22}$.



(a) Parameters are such that $\alpha_2 - \omega_{21} - 2\theta_{21} = 0$, in particular, $\alpha_1 = \alpha_2 = 2.5$, $\omega_{21} = 1.5$, $\theta_{21} = 0.5$.

(b) Parameters are such that $\omega_{21} + \theta_{21} = 0$, in particular, $\alpha_1 = \alpha_2 = 2.5$, $\omega_{21} = 1.5$, $\theta_{21} = -1.5$.

Fig. 2. Comparison of steady state error according to different design strategies. Parameters are $a = -0.5$, $K_{11} = K_{22} = 2$, $K_{21} = \frac{(K_{11}-a)K_{22}}{a}$, $m_1 = m_2 = \sigma_{11} = \sigma_{12} = \sigma_{13} = 0.5$.

trated in the example above generalizes to the case where N agents can measure the plant's output over a graph. The purpose of the next two sections is making this precise.

III. A Dynamic Consensus Algorithm For Distributed State Observers

A. Notation

Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and the notation $[u^\top \; v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t\geq 0} |m(t)|$. The set of complex numbers is denoted by $\mathcal{C}_0$. The set $\mathbb{N}$ denotes the set of natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \dots\}$. For a transfer function $\mathcal{C}_0 \ni s \mapsto T(s) \in \mathcal{C}_0^{n\times m}$, the H_∞ norm is defined as $||T||_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(T(j\omega))$, where $\bar{\sigma}(T(j\omega)) =: \max\{|\lambda|^{\frac{1}{2}} : \; \lambda \in \text{eig}(T(j\omega)^H T(j\omega))\}$ with $T(j\omega)^H$ being the conjugate transpose of $T(j\omega)$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; and $A * B$ defines the Khatri-Rao product. Given $N \in \mathbb{N}$, $I_N \in \mathbb{R}^{N\times N}$ defines the identity matrix, 1_N is the vector of N ones, and $\Pi_N := I_N - \frac{1}{N} 1_N 1_N^\top$. Denote $A = \{a_{ij}\}_{M\times N} \in \mathbb{R}^{M\times N}$ with the (i, j)-th entry being a_{ij}. A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, G)$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \dots, N\}$, and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to j, denoted by (i, j), implies that agent i can send information to agent j. The adjacency matrix of the digraph Γ is denoted by $G = (g_{ij}) \in \mathbb{R}^{N\times N}$, where $g_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise. The in-degree of agent i are defined by $d^{in}(i) = \sum_{j=1}^N g_{ji}$. The in-degree matrix D is the diagonal matrix with entries $D_{ii} = d^{in}(i)$, for all $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L}$, is defined as $\mathcal{L} = D - G^\top$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{I}(i) := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$.

B. Problem Statement

Our goal is to design a consensus algorithm such that each agent achieves consensus on the average of all local estimates provided by a distributed state observer under communication and measurement noise. For this purpose, we adapt the distributed state observer from [19], for the plant in (1) and over a network of N agents connected via a digraph Γ, in which each agent has a local state observer that uses measurements of the plant and of its neighbors. After adding communication noise, which is particular to the problem studied in this paper, for each $i \in \mathcal{V}$, the agent i runs a local state observer given by[3]

$$\dot{\hat{x}}_i = A\hat{x}_i - K_{ii}(\hat{y}_i - y_i) - \sum_{j\neq i, j\in\mathcal{V}} g_{ji} K_{ij} (C_j \hat{x}_j^\sigma - y_j^\sigma), \quad (13a)$$

$$\hat{x}_i^\sigma = \hat{x}_i + \sigma_{i1}, \quad y_i^\sigma = y_i + \sigma_{i2}, \quad (13b)$$

$$\bar{x}_i = \frac{1}{d^{in}(i)} \sum_{j=1}^N g_{ji} \hat{x}_j, \quad \hat{y}_i = C_i \hat{x}_i \quad (13c)$$

where $\hat{x}_i$ denotes the state variable of the observer, $\bar{x}_i$ denotes the local estimate of x at agent i, K_{ij}'s are estimation gains to be designed, y_i denotes the measurement of y in (1) taken by the i-th agent with measurement noise m_i, that is,

$$y_i = C_i x + m_i \in \mathbb{R}^{p_i},$$

where p_i is the dimension of the measurement taken at the i-th agent. The information that the i-th agent obtains from its neighbors is given by $\hat{x}_j^\sigma$ and y_j^σ for each $j \in \mathcal{I}(i)$, which correspond to the values of $\hat{x}_j$ and y_j corrupted by communication noise σ_{i1} and σ_{i2}, respectively. The collection of local state observers in (13) connected via the digraph Γ defines a *distributed state observer* [19].

While convergence of the local estimates was characterized in [19], the algorithm therein does not provide a mechanism to obtain a globally agreed quantity of the estimate of x (neither for the nominal case nor when noise in the measurements are present). Unless the graph is all-to-all connected, not every agent can compute $\frac{1}{N}\sum_{j=1}^N \hat{x}_j(t)$. To make the quantity $\frac{1}{N}\sum_{j=1}^N \hat{x}_j(t)$ accessible to each agent, a possible solution is to use a dynamic consensus algorithm: obtain information from neighbors and make an internal state of each agent approach the quantity $\frac{1}{N}\sum_{j=1}^N \hat{x}_j(t)$ asymptotically. Note that this quantity converges to zero when A is Hurwitz, the observers are convergent, and noise signals are zero.

C. Distributed consensus algorithm

In this paper, we propose a *distributed consensus algorithm for the local estimates* $\hat{x}_i$. Following the idea in [11], the algorithm is given by

$$\dot{\xi}_i = \dot{\hat{x}}_i - \alpha_i(\xi_i - \hat{x}_i) - \sum_{j\neq i, j\in\mathcal{V}} g_{ji}\omega_{ij}(\xi_i - \xi_j^\sigma) - \sum_{j\neq i, j\in\mathcal{V}} g_{ji}\theta_{ij}(\hat{x}_i - \hat{x}_j^\sigma), \quad (14a)$$

$$\xi_i^\sigma = \xi_i + \sigma_{i3}, \quad (14b)$$

where, for each $i \in \mathcal{V}$, $\xi_i \in \mathbb{R}^n$ is the consensus variable attached to agent i; $\alpha_i, \omega_{ij}, \theta_{ij}$ are constant gains; and ξ_i^σ

[3]Within this work, it is assumed that Γ has $g_{ii} = 1$ for all $i \in \mathcal{V}$ due to that each agent has access to the measurement y_i of the plant's state x.

is the communication noise (σ_{i3}) corrupted value of ξ_i. Note that the algorithm uses information of $\dot{\hat{x}}_i$, ξ_i as well as information received from its neighbors, i.e., ξ_j^σ, $\hat{x}_j^\sigma$ for each $j \in \mathcal{I}(i)$. In particular, ξ_j^σ is defined in (13), and the transmissions of ξ_i are corrupted by communication noise σ_{i3}, where $\hat{x}_i$'s are the estimates generated by agent i using the local observer in (13a). For each $i, j \in \mathcal{V}$, the gains $\alpha_i, \omega_{ij}, \theta_{ij} \in \mathbb{R}$ are design parameters.

To analyze the interconnection between (13) and (14), for each $i \in \mathcal{V}$, define $e_i := \hat{x}_i - x$ and the associated vector $e := (e_1, \ldots, e_N)$, and $\tilde{e}_i := \bar{x}_i - x$ and the associated vector $\tilde{e} := (\tilde{e}_1, \ldots, \tilde{e}_N)$. Furthermore, define $\delta_i = \xi_i - \frac{1}{N}\sum_{j=1}^{N} \hat{x}_j$ and $\delta = (\delta_1, \cdots, \delta_N)$, the measurement noise vector $m := (m_1, \ldots, m_N)$, and the communication noise vector $\sigma = (\sigma_{11}, \ldots, \sigma_{N1}, \sigma_{12}, \ldots, \sigma_{N2}, \sigma_{13}, \ldots, \sigma_{N3})$. Then, it follows that

$$\dot{e}_i = Ae_i - \sum_{j\in\mathcal{V}} g_{ji}K_{ij}C_j e_j + \sum_{j\in\mathcal{V}} g_{ji}K_{ij}m_j - \sum_{j\neq i, j\in\mathcal{V}} g_{ji}K_{ij}C_j\sigma_{j1} + \sum_{j\neq i, j\in\mathcal{V}} g_{ji}K_{ij}\sigma_{j2}, \tag{15a}$$

$$\dot{\delta}_i = -\alpha_i\delta_i - \sum_{j\in\mathcal{V}} g_{ji}w_{ij}(\delta_i - \delta_j) + \frac{1}{N}\sum_{j\in\mathcal{V}}(\dot{e}_i - \dot{e}_j + \alpha_i(e_i - e_j)) - \sum_{j\in\mathcal{V}} g_{ji}\theta_{ij}(e_i - e_j) + \sum_{j\neq i, j\in\mathcal{V}} g_{ji}(\omega_{ij}\sigma_{j3} + \theta_{ij}\sigma_{j1}), \tag{15b}$$

which can be rewritten in the compact form

$$\dot{e} = A_e e + B_1 m + B_2\sigma, \tag{16a}$$

$$\dot{\delta} = A_\delta\delta + A_c e + B_3 m + B_4\sigma, \tag{16b}$$

$$\tilde{e} = C_e e, \tag{16c}$$

where

$$\begin{aligned}
&A_e = I_N \otimes A - (\mathcal{K} * G^\top)C_g, \quad C_e = (D^{-1} * G^\top) \otimes I_n,\\
&B_1 = \mathcal{K} * G^\top, \quad \bar{G} = G - I_N, \quad \bar{\Pi}_N = \Pi_N \otimes I_n,\\
&B_2 = \begin{bmatrix} -(\mathcal{K} * \bar{G}^\top)C_g & \mathcal{K} * \bar{G}^\top & 0_{nN\times nN}\end{bmatrix},\\
&A_\delta = -(\mathrm{diag}(\alpha) + \mathcal{L}_\omega) \otimes I_n,\\
&A_c = \bar{\Pi}_N(A_e + \mathrm{diag}(\alpha) \otimes I_n) - \mathcal{L}_\theta \otimes I_n,\\
&B_3 = \bar{\Pi}_N B_1, \quad \bar{G}_\theta = \bar{G}^\top * \Theta, \quad \bar{G}_\omega = \bar{G}^\top * \Omega,\\
&B_4 = \bar{\Pi}_N B_2 + \begin{bmatrix}\bar{G}_\theta \otimes I_n & 0_{nN\times nN} & \bar{G}_\omega \otimes I_n\end{bmatrix},\\
&\mathcal{L}_\omega = \mathrm{diag}(\bar{G}_\omega 1_N) - \bar{G}_\omega, \quad \mathcal{L}_\theta = \mathrm{diag}(\bar{G}_\theta 1_N) - \bar{G}_\theta,\\
&C_g = \mathrm{diag}(C_1, C_2, \cdots, C_N),
\end{aligned}$$

and gains

$$\mathcal{K} = \{K_{ij}\}_{N\times N}, \quad \Omega = \{\omega_{ij}\}_{N\times N}, \quad \Theta = \{\theta_{ij}\}_{N\times N},$$

where G is the adjacency matrix, D is the in-degree matrix, the Khatri-Rao product $\mathcal{K} * G^\top$ is such that $\mathcal{K}$ is treated as $N \times N$ block matrices with K_{ij}'s as blocks. By defining

$$\mathcal{A} := \begin{bmatrix} A_e & 0 \\ A_c & A_\delta \end{bmatrix}, \quad \mathcal{B}_1 := \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, \quad \mathcal{B}_2 := \begin{bmatrix} B_2 \\ B_4 \end{bmatrix}, \tag{17}$$

and the total error $\bar{e} = (e, \delta)$, the interconnection in (16) can be written as

$$\dot{\bar{e}} = \mathcal{A}\bar{e} + \mathcal{B}_1 m + \mathcal{B}_2\sigma. \tag{18}$$

Remark 3.1: $I_N \otimes A$ defines a block diagonal matrix with matrix A in each of the N blocks (of dimension $n \times n$). The matrix $\mathcal{K} * G^\top$ defines the gain matrix for the graph, while $C_e = (D^{-1} * G^\top) \otimes I_n$ generates the estimation matrix for each agent by averaging the local estimates obtained from its neighbors. When the estimation algorithm in (15) is properly designed, A_e is Hurwitz and e converges to zero asymptotically. It further implies that, due to the property of A_δ, the steady state of δ in (16b) depends on the supremum of "inputs" e, σ and m.

IV. Design Using Optimization Formulation

A. Decoupled Design

The design of the closed-loop system in (18) can be decomposed into two steps. The distributed state observer in (13) can be designed first, while the distributed consensus algorithm in (14) is designed after. In this section, we discuss several particular optimization problems for such decoupled design procedure.

For the distributed consensus in (14), the design specifications of interest are the rate of convergence and the H_∞ gain from communication noise σ and from measurement noise m to the consensus error δ. In particular, to guarantee a particular rate of convergence of system (16), the eigenvalues of the error system (16) will be assigned to the left of the vertical line at $-h$ in the s-plane, where $h > 0$ is the convergence rate specification. Following [23], the eigenvalues of the matrix A_δ are located in the region $\mathcal{R} := \{s \in \mathcal{C}_0 : Re(s) < -h\}$ if and only if there exists a matrix $P_S = P_S^\top > 0$ such that

$$A_\delta^\top P_S + P_S A_\delta + 2hP_S < 0. \tag{19}$$

Recall from the motivational example in Section II that there is a tradeoff when choosing parameters to reduce the effect of measurement noise and communication noise simultaneously. To determine the performance of our distributed consensus algorithm (14), we introduce the criterion of global H_∞ gain from noise vector (m, σ) to consensus error vector δ. Recall that finding the H_∞ gain of a transfer function $T(s) = \mathcal{C}(sI - \mathcal{A})\mathcal{B} + \mathcal{D}$ is equivalent to the feasibility of a certain matrix inequality. Then, we can establish the following optimization problem for the design of the distributed consensus algorithm according to a pre-specified global H_∞ gain from (m, σ) to δ.

Proposition 4.1: Given a plant as in (1) *and a digraph* Γ, *the global* H_∞ *gain of the transfer function from* (m, σ) *to* δ *in* (18) *is less than or equal to* γ *if and only if the following inequality is feasible for some* $P_H = P_H^\top > 0$*:*

$$\begin{bmatrix} \mathrm{He}(\mathcal{A}, P_H) & P_H\mathcal{B} & \mathcal{C}^\top \\ \mathcal{B}^\top P_H & -\gamma I & 0 \\ \mathcal{C} & 0 & -\gamma I \end{bmatrix} < 0, \tag{20}$$

where $\mathcal{B} := [\mathcal{B}_1\ \mathcal{B}_2]$, *and* $\mathcal{C} := [0\ \ 1] \otimes I_{nN}$.

Remark 4.2: The global H_∞ gain from (m, σ) to δ determines the overall effect of the total noise (m, σ) on the distributed consensus of local estimates provided by (14). To determine the effect of the noise (m, σ) on the local estimate δ_i, the H_∞ gain from (m, σ) to δ_i can also be characterized in (20) by replacing $\mathcal{C}$ with $\mathcal{C}_i$, where $\mathcal{C}_i$ is the sub-matrix of $\mathcal{C}$ from the $(in - n + N + 1)$-th row to the $(in + N)$-th row.

Then, by combining the rate of convergence constraint in (19) and the H_∞ constraint in (20), we can perform the synthesis of the distributed consensus algorithm of global estimates in (14) using the following result.

Theorem 4.3: For the plant in (1) *with distributed state observer* (13) *and distributed consensus algorithm* (14)*, given a digraph* Γ*, and a gain* $\mathcal{K}$ *such that* $\mathcal{A}$ *is Hurwitz, the rate of convergence of* (16) *is larger than or equal to* $h > 0$ *and the* H_∞ *gain from* (m, σ) *to* δ *in* (16) *is minimized if and only if there exist matrices* α, Θ, Ω, P_S*, and* P_H *such that the following optimization problem is feasible:*

$$\min \gamma \tag{21a}$$

$$\text{s.t. } \operatorname{He}(A_\delta, P_S) + 2hP_S < 0, \tag{21b}$$

$$\begin{bmatrix} \operatorname{He}(\mathcal{A}, P_H) & P_H\mathcal{B} & \mathcal{C}^\top \\ \mathcal{B}^\top P_H & -\gamma I & 0 \\ \mathcal{C} & 0 & -\gamma I \end{bmatrix} < 0, \tag{21c}$$

$$P_S = P_S^\top > 0, \; P_H = P_H^\top > 0. \tag{21d}$$

Remark 4.4: The problem in (21) can be solved offline by using, e.g., [24], and the resulting observers along with the consensus algorithm are decentralized. Note that the two constraints in (21b) and (21c) are nonlinear due to the fact that Ω in A_δ is multiplied by P_S and P_H in the operation of $\operatorname{He}(A_\delta, P_S)$ and $\operatorname{He}(\mathcal{A}, P_H)$, respectively. Moreover, if the local H_∞ gain from noise (m, σ) to δ_i at agent i is of interest, then, the matrix $\mathcal{C}$ in (21c) can be replaced by $\mathcal{C}_i$. When the effect of communication noise (measurement noise, respectively) is of major concern, the matrix $\mathcal{B}$ in (21) can be replaced by $\mathcal{B}_2$ ($\mathcal{B}_1$, respectively).

Next, we provide an example to illustrate the results above.

Example 4.5: Consider the plant in (2) with $a = -0.5$, and the motivational example with two agents connected as in Figure 1(a). Suppose that the distributed state observers in (4) is designed to satisfy a rate of convergence equal to 2.5 with parameters $K_{11} = K_{22} = 2$, $K_{12} = 0$ and $K_{21} = -4.7$. Then, we can design the distributed consensus algorithm of local estimates by solving (21). The resulting gains are

$$\alpha = (7.14, 14.24), \; \Omega = \begin{bmatrix} 0 & 0 \\ 0.52 & 0 \end{bmatrix}, \; \Theta = \begin{bmatrix} 0 & 0 \\ -3.6 & 0 \end{bmatrix}.$$

The resulting H_∞ gain from (m, σ) to δ is approximately 3.17. If instead, these two agents are all-to-all connected, then, by solving (21), the resulting gains are

$$\alpha = (3.5, 2.5), \; \Omega = \begin{bmatrix} 0 & -0.88 \\ 0.28 & 0 \end{bmatrix}, \; \Theta = \begin{bmatrix} 0 & 0.79 \\ 0.06 & 0 \end{bmatrix},$$

and the resulting H_∞ gain from (m, σ) to δ is approximately 1.65.

B. Joint Observer-Consensus Design

As shown in Example 4.5, when K_{11}, K_{21}, K_{22} for the distributed state observer are fixed, the effect of measurement and communication noise on the estimation error e are fixed. In such a case, we can only tune the performance of the distributed consensus algorithm. On the other hand, we can simultaneously design the consensus algorithm in (14) and the estimation algorithm in (13). In particular, for a fixed graph Γ, we can design $\mathcal{K}$, α, Θ, Ω simultaneously. For such a design, note that $\mathcal{A}$ is a lower block triangular matrix, so its eigenvalues consist of the eigenvalues of A_e and of A_δ. Imposing a rate of convergence on $\mathcal{A}$ is equivalent to imposing it on both A_δ and A_e. In other words, the eigenvalues of a matrix $\mathcal{A}$ are located in the region $\mathcal{D} := \{s \in \mathcal{C}_0 : Re(s) < -h\}$ if and only if there exist matrices $P_1 = P_1^\top > 0$ and $P_2 = P_2^\top > 0$ such that

$$\operatorname{He}(A_\delta, P_1) + 2hP_1 < 0, \tag{22a}$$

$$\operatorname{He}(A_e, P_2) + 2hP_2 < 0. \tag{22b}$$

Then, we can establish the following result for a joint observer-consensus design.

Theorem 4.6: For the plant in (1) *with the distributed state observer* (13) *and the distributed consensus algorithm* (14)*, given a digraph* Γ*, the rate of convergence of* (18) *is larger than or equal to* h *and the* H_∞ *gain from* σ *to consensus error* δ *in* (16) *is minimized if and only if there exist matrices* $\mathcal{K}$, α, Θ, Ω, P_1, P_2 *and* P_H *such that the following optimization problem is feasible:*

$$\min \gamma$$

$$\text{s.t. } \operatorname{He}(A_\delta, P_1) + 2hP_1 < 0, \tag{23a}$$

$$\operatorname{He}(A_e, P_2) + 2hP_2 < 0, \tag{23b}$$

$$\begin{bmatrix} \operatorname{He}(\mathcal{A}, P_H) & P_H\mathcal{B}_2 & ([0\ 1] \otimes I_{nN})^\top \\ \mathcal{B}_2^\top P_H & -\gamma I & 0 \\ [0\ 1] \otimes I_{nN} & 0 & -\gamma I \end{bmatrix} < 0, \tag{23c}$$

$$P_1 = P_1^\top > 0, \; P_2 = P_2^\top > 0, \; P_H = P_H^\top > 0. \tag{23d}$$

Example 4.7: Consider Example 4.5 with $a = -0.5$ and a rate of convergence constraint equal to 2.5. Now, we let $\mathcal{K}$ be the gain to be chosen. By solving (23), we obtain

$$\alpha = (2.50, 4.22), \quad \Omega = \begin{bmatrix} 0 & 0 \\ 1.35 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} 0 & 0 \\ 0.35 & 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} 2 & 0 \\ 0.55 & 2 \end{bmatrix}.$$

The resulting H_∞ gain from (m, σ) to δ is approximately 0.82, which is a significant improvement over a gain equal to 3.17 obtained in Example 4.5. On the other hand, consider the same plant with two agents that are all-to-all connected, then, by solving (23), the resulting gains are

$$\alpha = (10.98, 10.98), \quad \Omega = \begin{bmatrix} 0 & 1.32 \\ 1.32 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} 0 & 3.31 \\ 3.31 & 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} 2.95 & 0.95 \\ 0.95 & 2.95 \end{bmatrix},$$

the resulting H_∞ gain from (m, σ) to δ is approximately 0.32, and the resulting H_∞ gain from m to $\tilde{e}$ is approximately 0.88. △

Recall from the motivational example that when measurement noise is assumed to be zero, the optimal strategy to reduce the effect of communication noise is to not communicate at all. However, such a strategy also eliminates the possibility of reducing the effect of measurement noise, which is a key capability of distributed state observers. The following result provides one way to formulate a design with a pre-specified H_∞ gain from m to $\tilde{e}$.

Theorem 4.8: For the plant in (1) *with estimation system in* (13) *and consensus algorithm in* (14)*, given a digraph* Γ*, the rate of convergence of* (18) *is larger than or equal to* h*, the* H_∞ *gain from* m *to estimation error* $\tilde{e}$ *in* (16) *is no larger than* $\gamma^\star$ *and the* H_∞ *gain from* σ *to consensus error* δ *in* (16) *is minimized if and only if there exist matrices* $\mathcal{K}$*,* α*,* Θ*,* Ω*,* P_1*,* P_2*,* P_H*, and* P_e *such that the following optimization problem is feasible:*

$$\min \gamma$$

$$\text{s.t. } \mathrm{He}(A_\delta, P_1) + 2hP_1 < 0, \quad (24a)$$

$$\mathrm{He}(A_e, P_2) + 2hP_2 < 0, \quad (24b)$$

$$\begin{bmatrix} \mathrm{He}(\mathcal{A}, P_H) & P_H\mathcal{B}_2 & ([0\ 1]\otimes I_{nN})^\top \\ \mathcal{B}_2^\top P_H & -\gamma I & 0 \\ [0\ 1]\otimes I_{nN} & 0 & -\gamma I \end{bmatrix} < 0, \quad (24c)$$

$$\begin{bmatrix} \mathrm{He}(A_e, P_e) & P_e B_1 & C_e^\top \\ B_1^\top P_e & -\gamma^\star I & 0 \\ C_e & 0 & -\gamma^\star I \end{bmatrix} < 0, \quad (24d)$$

$$P_1 = P_1^\top > 0,\ P_2 = P_2^\top > 0, \quad (24e)$$

$$P_H = P_H^\top > 0,\ P_e = P_e^\top > 0. \quad (24f)$$

Remark 4.9: Note that the constraints defined in (24a) and (24c) are nonlinear due to the fact that Ω in A_δ is multiplied with P_1, P_H in the operation of $\mathrm{He}(A_\delta, P_1)$ and $\mathrm{He}(\mathcal{A}, P_H)$, respectively. The constraints defined in (24b) and (24d) are also nonlinear and not jointly convex due to the products $P_2\mathcal{K}$ and $P_e\mathcal{K}$. With the structure of the distributed state observer in (13), the H_∞ gain $\gamma^\star$ from m to estimation error $\tilde{e}$ can be chosen much smaller than that of a Luenberger observer as in [19].

Example 4.10: Consider the plant in (2) with $a = -0.5$, and a distributed state observer with two agents connected via an all-to-all connectivity graph. The specifications of interest are convergence rate greater than or equal to 2.5 and H_∞ gain from m to $\tilde{e}$ no larger than 0.6 (which is smaller than 0.88 obtained in Example 4.7). By solving (24), we obtain

$$\alpha = (13.75, 93.80), \quad \Omega = \begin{bmatrix} 0 & 8.52 \\ 0.34 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} 0 & 1.55 \\ 7.96 & 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} 1.11 & -5.77 \\ 0.14 & 2.90 \end{bmatrix}.$$

The resulting H_∞ gain from σ to δ is approximately 1.52, and the resulting H_∞ gain from (m, σ) to δ is approximately 2.34. △

V. Conclusion

The proposed distributed consensus algorithm for distributed state observers has the capability of attaining fast rate of convergence without necessarily jeopardizing robustness to measurement and communication noise in the H_∞ sense. An optimization-based design method is proposed to separately and jointly determine the parameters of distributed state observer and the proposed distributed consensus algorithm. Furthermore, the resulting distributed consensus algorithm is exponentially stable and robust to measurement and communication noise.

References

[1] J. Cortés. Distributed Kriged Kalman filter for spatial estimation. *IEEE Transactions on Automatic Control*, 54(12):2816–2827, 2009.

[2] D. P. Spanos, R. Olfati-Saber, and R. M. Murray. Approximate distributed Kalman filtering in sensor networks with quantifiable performance. In *Proc. of 4th International Symposium on Information Processing in Sensor Networks*, Piscataway, NJ, USA, 2005.

[3] P. Alriksson and A. Rantzer. Distributed Kalman filtering using weighted averaging. In *Proc. of 17th International Symposium on Mathematical Theory of Networks and Systems*, 2006.

[4] R. Olfati-Saber. Distributed Kalman filtering for sensor networks. In *Proc. of 46th IEEE Conference on Decision and Control*, pages 5492–5498, 2007.

[5] R. Carli, A. Chiuso, L. Schenato, and S. Zampieri. Distributed Kalman filtering based on consensus strategies. *IEEE Journal on Selected Areas in Communications*, 26(4):622–633, 2008.

[6] Y. Hong, G. Chen, and L. Bushnell. Distributed observers design for leader-following control of multi-agent networks. *Automatica*, 44(3):846–850, 2008.

[7] J. Wu, V. Ugrinovskii, and F. Allgower. Cooperative $\mathcal{H}_\infty$-estimation for large-scale interconnected linear systems. In *Proc. of American Control Conference*, pages 2119–2124, July 2015.

[8] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, Jan 2007.

[9] W. Ren and E. Atkins. Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, 17(10-11):1002–1033, 2007.

[10] M. Mesbahi and M. Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton University Press, 2010.

[11] S. S. Kia, J. Cortés, and S. Martinez. Dynamic average consensus under limited control authority and privacy requirements. *International Journal of Robust and Nonlinear Control*, 25(13):1941–1966, 2015.

[12] D. P. Spanos, R. Olfati-Saber, and R. M. Murray. Dynamic consensus on mobile networks. In *IFAC world congress*, pages 1–6. Prague Czech Republic, 2005.

[13] Z. Li, Z. Duan, and G. Chen. Dynamic consensus of linear multi-agent systems. *Control Theory & Applications, IET*, 5(1):19–28, 2011.

[14] R. A. Freeman, P. Yang, and K. M. Lynch. Stability and convergence properties of dynamic average consensus estimators. In *Proc. of IEEE Conference on Decision and Control*, pages 398–403, 2006.

[15] L. Xiao, S. Boyd, and S. J. Kim. Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing*, 67(1):33–46, 2007.

[16] Y. Zhao, Z. Duan, G. Wen, and G. Chen. Distributed H_∞ consensus of multi-agent systems: a performance region-based approach. *International Journal of Control*, 85(3):332–341, 2012.

[17] R. Olfati-Saber. Kalman-consensus filter: Optimality, stability, and performance. In *Proc. of IEEE Conference on Decision and Control, held jointly with the Chinese Control Conference.*, pages 7036–7042, 2009.

[18] S. Kar and J. M. F. Moura. Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise. *IEEE Transactions on Signal Processing*, 57(1):355–369, 2009.

[19] Y. Li and R. G. Sanfelice. Interconnected observers for robust decentralized estimation with performance guarantees and optimized connectivity graph. *IEEE Transactions on Control of Network Systems*, PP(99):1–1, 2015.

[20] L. K. Vasiljevic and H. K. Khalil. Differentiation with high-gain observers the presence of measurement noise. In *Proc. of 45th IEEE Conference on Decision and Control*, pages 4717–4722, Dec. 2006.

[21] J. H. Ahrens and H. K. Khalil. High-gain observers in the presence of measurement noise: A switched-gain approach. *Automatica*, 45(4):936–943, 2009.

[22] R. G. Sanfelice and L. Praly. On the performance of high-gain observers with gain adaptation under measurement noise. *Automatica*, 47(10):2165–2176, 2011.

[23] M. Chilali and P. Gahinet. H_∞ design with pole placement constraints: an LMI approach. *IEEE Transactions on Automatic Control*, 41(3):358–367, March 1996.

[24] J. Fiala, M. Kočvara, and M. Stingl. Penlab: A matlab solver for nonlinear semidefinite optimization. *arXiv preprint arXiv:1311.5240*, 2013.