Results on Finite Time Stability for A Class of Hybrid Systems

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Abstract— **This paper introduces notions and tools for finite time stability of closed sets for a class of hybrid systems. The proposed finite time stability notion consists of both Lyapunov stability and finite time convergence. Several sufficient conditions for a closed set to have such a property for a hybrid system are established. Robustness of finite time stability to small perturbations is studied by the regularity of the system data. Relationships between finite time stability and asymptotic stability of sets are also investigated. Examples illustrating the results are discussed throughout the paper.**

I. Introduction

In contrast to asymptotic stability, which pertains to asymptotic convergence to a point (or a set), finite time stability is a concept that requires convergence of solutions in finite time. More precisely, a closed set $\mathcal{A}$ is finite time stable if the distance between any maximal solution and the set has stable behavior and converges to zero in finite time. For a continuous-time system $\dot{x} = f(x)$, the uniform version of such a property for a closed set $\mathcal{A}$ can be captured by the following $\mathcal{GKL}$ estimate: every solution $t \mapsto \phi(t)$ satisfies

$$|\phi(t)|_{\mathcal{A}} \leq \beta(|\phi(0)|_{\mathcal{A}}, t) \tag{1}$$

for each t in the domain of definition of ϕ, where β is a class-$\mathcal{GKL}$ function[1]; see, e.g., [1], [2]. The bound (1) implies that the Euclidean distance between the solution ϕ and the set $\mathcal{A}$ is upper bounded by a function of their initial distance and also decreases to zero in finite time. Over the past few decades, much work has been dedicated to this concept. In [2], necessary and sufficient conditions for finite time stability in continuous-time systems are explored when solutions are unique in forward time. In [3], finite-time converging controllers were developed for dynamical systems given in terms of *continuous finite time differential equations*. The authors in [4] established several necessary and sufficient conditions for continuous-time nonautonomous systems using Lyapunov functions. Finite time stability-like properties have been used in the design of observers [5], [6], consensus algorithms for multi-agent systems [7], and finite-time converging feedback controllers [8].

Unfortunately, the aforementioned finite time stability results cannot be applied directly to systems with variables that can change continuously and, at times, jump. These systems, known as *hybrid systems*, are capable of modeling a wide range of complex dynamical systems, including robotic, automotive, and power systems as well as natural processes. While a set stability theory in terms of Lyapunov functions is available, see [9], [10], finite time stability for such systems has not been thoroughly studied. The effort in [11] provides sufficient conditions for finite time stability of the origin for impulsive dynamical systems using scalar and vector Lyapunov functions, but its focus is on sufficient conditions that pertain to the continuous dynamics of the impulsive systems; in particular, finite time stability through jumps is not addressed. It should be noted that converse Lyapunov theorems and robustness results for finite time stability of general hybrid systems are also lacking.

Motivated by the applications of finite time stability and recent advancements in hybrid systems theory, in this paper, we introduce and study notions of finite time stability for a class of hybrid systems. In particular, this paper makes the following contributions to this problem:

1) For a class of hybrid systems in the framework of [9], we introduce uniform and nonuniform finite time stability notions for closed sets.
2) For the proposed finite time stability notions, sufficient Lyapunov-type conditions are presented. In particular, we provide conditions guaranteeing that maximal solutions to a hybrid system may converge to a finite time stable closed set via flows, jumps, or both.
3) Conditions assuring robustness to small perturbations of finite time stability of closed sets are presented.

These results are illustrated in examples throughout the paper.

The remainder of this paper is organized as follows. In Section II, some preliminaries on the hybrid systems framework and nonsmooth Lyapunov functions are briefly discussed. In Section III, the proposed notion and sufficient conditions for a set being finite time stable are presented and illustrated by examples. The results on robustness of finite time stability are in Section IV. Proofs will be published elsewhere due to space constraints.

II. Preliminaries

A. Notation

Given a set $S \subset \mathbb{R}^n$, the closure of S is the intersection of all closed sets containing S, denoted by $\overline{S}$; $\overline{\mathrm{con}}S$ is the closure of the convex hull of the set S. Given vectors $\nu \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $|\nu|$ defines the Euclidean vector norm $|\nu| = \sqrt{\nu^\top \nu}$, and $[\nu^\top \; w^\top]^\top$ is equivalent to (ν, w). Given a

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[1] A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized $\mathcal{KL}$-function ($\mathcal{GKL}$-function) if, for each fixed $t \geq 0$, the function $s \to \beta(s, t)$ is strictly increasing and continuous with $\beta(0, t) = 0$; for each fixed $s \geq 0$, the function $t \to \beta(s, t)$ is continuous and decreases to zero as $t \to T$ for some $T < \infty$.

function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by dom f, i.e., dom $f := \{x \in \mathbb{R}^m : \ f(x) \text{ is defined}\}$. The range of f is denoted by rge f, i.e., rge $f := \{f(x) : x \in \operatorname{dom} f\}$. The right limit of the function f is defined as $f^+(x) := \lim_{\nu \to 0^+} f(x + \nu)$ if it exists. The function f is said to belong to $\mathcal{C}^2$ if its derivative is continuously differentiable. Given a point $x \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_\mathcal{A} := \inf_{y\in\mathcal{A}} |x-y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, also written $\alpha \in \mathcal{K}$, if α is zero at zero, continuous, strictly increasing; it is said to belong to class-$\mathcal{K}_\infty$, also written $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded; α is positive definite, also written $\alpha \in \mathcal{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\varphi \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \varphi(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \varphi(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a matrix $A \in \mathbb{R}^{n\times n}$, eig$(A)$ is the set of eigenvalues of A. Given a real number $x \in \mathbb{R}$, ceil(x) denotes the next larger integer of x. The set of positive semidefinite matrices with dimension $p \times p$ is denoted by $\mathcal{SP}^{p\times p}$.

B. Preliminaries on Hybrid Systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, F, D, G) and is defined by

$$\begin{aligned} \dot{z} &\in F(z) \quad z \in C, \\ z^+ &\in G(z) \quad z \in D, \end{aligned} \tag{2}$$

where $z \in \mathbb{R}^n$ is the state, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the flow map capturing the continuous dynamics and C defines the flow set on which F is effective. The map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time.[2] A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ. A hybrid system $\mathcal{H}$ is said to satisfy the hybrid basic conditions if it satisfies [9, Assumption 6.5]. The definition of uniform global pre-asymptotic stability (UGpAS) for a set is given in [9, Definition 3.6].

We refer the reader to [9] for more details on these notions and the hybrid systems framework.

[2]A solution to $\mathcal{H}$ is defined in [9, Definition 2.6]. The domain dom $\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom}\phi$, the set dom $\phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J}(I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps.

C. Preliminaries on Nonsmooth Lyapunov Functions

For a hybrid system $\mathcal{H}$, let $V : O \to \mathbb{R}$ be continuous on O and locally Lipschitz on a neighborhood of C. The generalized gradient (in the sense of Clarke) of V at $z \in C$, denoted by $\partial V(z)$, is a closed, convex, and nonempty set equal to the convex hull of all limits of the sequence $\nabla V(z_i)$, where z_i is any sequence converging to z while avoiding an arbitrary set of measure zero containing all the points at which V is not differentiable (as V is locally Lipschitz, ∇V exists almost everywhere). The (Clarke) generalized directional derivative of V at z in the direction of v can be expressed as

$$V^\circ(z, v) = \max_{\zeta\in\partial V(z)} \langle \zeta, v\rangle. \tag{3}$$

Then, for any solution $t \mapsto z(t)$ to $\dot{z} \in F(z)$,

$$\frac{d}{dt}V(z(t)) \leq V^\circ(z(t), \dot{z}(t)) \tag{4}$$

for almost all t in the domain of definition of z, where $\frac{d}{dt}V(z(t))$ is understood in the standard sense since V is locally Lipschitz. For more details on generalized gradient, see, e.g., [12].

To bound the increase of the function V along solutions to the hybrid system $\mathcal{H}$, following [13], we define the function $u_C : O \to [-\infty, +\infty)$ as

$$u_C(z) := \begin{cases} \max\limits_{v\in F(z)} \max\limits_{\zeta\in\partial V(z)} \langle \zeta, v\rangle & z \in C \\ -\infty & \text{otherwise} \end{cases} \tag{5}$$

In particular, for any solution ϕ to $\mathcal{H}$, and any t where $\frac{d}{dt}V(\phi(t, j))$ exists, we have

$$\frac{d}{dt}V(\phi(t, j)) \leq u_C(\phi(t, j)). \tag{6}$$

Moreover, in order to bound the change in V after jumps, we define the following quantity:

$$u_D(z) := \begin{cases} \max\limits_{\zeta\in G(z)} V(\zeta) - V(z) & z \in D \\ -\infty & \text{otherwise} \end{cases} \tag{7}$$

Then, for any solution ϕ to $\mathcal{H}$ and for any $(t_{j+1}, j), (t_{j+1}, j+1) \in \operatorname{dom}\phi$, it follows that

$$V(\phi(t_{j+1}, j + 1)) - V(\phi(t_{j+1}, j)) \leq u_D(\phi(t_{j+1}, j)). \tag{8}$$

Note that when F is a single-valued map, $u_C(z) = V^\circ(z, F(z))$ for each $z \in C$. When G is a single-valued map, $u_D(z) = V(G(z)) - V(z)$ for each $z \in D$.

III. Finite Time Stability

A. Finite Time Stability Notions

Inspired by the notion in [2], we introduce the following finite time stability notion for hybrid systems $\mathcal{H}$.

Definition 3.1: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$, a closed set $\mathcal{A} \subset \mathbb{R}^n$, an open neighborhood $\mathcal{N}$ of $\mathcal{A}$, and a function $T : \mathcal{N} \to [0, \infty)$, called the settling-time function. The closed set $\mathcal{A}$ is said to be

1) *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N})$ with $|\phi(0, 0)|_\mathcal{A} \leq \delta$, we have $|\phi(t, j)|_\mathcal{A} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}\phi$;

2) *uniformly stable* for $\mathcal{H}$ if there exists a function $\alpha \in \mathcal{K}_\infty$ such that any solution $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N})$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \alpha(|\phi(0,0)|_\mathcal{A})$ for all $(t,j) \in \operatorname{dom}\phi$;
3) *finite time attractive* (FTA) for $\mathcal{H}$ if for every solution $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N})$, $\sup_{(t,j)\in\operatorname{dom}\phi} t \geq T(\phi(0,0))$ and
$$\lim_{(t,j)\in\operatorname{dom}\phi : t+j \nearrow T(\phi(0,0))} |\phi(t,j)|_\mathcal{A} = 0; \tag{9}$$
4) *finite time stable* (FTS) for $\mathcal{H}$ if it is stable and FTA for $\mathcal{H}$;
5) *uniformly finite time stable* (UFTS) if it is uniformly stable and FTA for $\mathcal{H}$.

The global version of the notions defined in Definition 3.1 can be obtained when the set $\mathcal{N}$ is chosen as $\mathcal{N} = \mathbb{R}^n$.

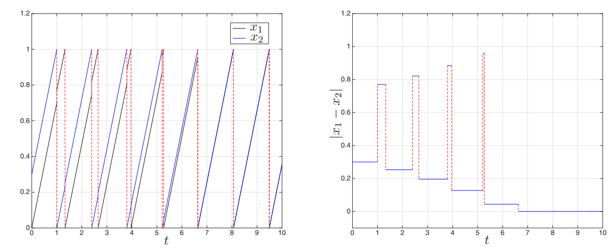
Remark 3.2: Note that for a given $\phi \in \mathcal{S}(\mathcal{N})$ with $\phi(0,0) = \xi$, $T(\xi)$ can be decomposed as $T(\xi) = T^\star(\xi) + J^\star(\xi)$ for some functions $T^\star : \mathcal{N} \to \mathbb{R}_{\geq 0}$ and $J^\star : \mathcal{N} \to \mathbb{N}$, with
$$\lim_{(t,j)\in\operatorname{dom}\phi : t+j \to T^\star(\xi)+J^\star(\xi)} |\phi(t,j)|_\mathcal{A} = 0.$$
Moreover, $(T^\star(\xi), J^\star(\xi)) \in \operatorname{dom}\phi$ if $\operatorname{dom}\phi$ is a compact hybrid time domain.

The following examples illustrate the notions in Definition 3.1.

Example 3.3: Inspired from [14, Example 14], consider the hybrid system $\mathcal{H} = (C, F, D, G)$ with state $z = (x, \tau) \in \mathbb{R} \times [0,1]$ and data given by[3]
$$\begin{aligned} F(z) &= \begin{bmatrix} -k|x|^\alpha \operatorname{sgn}(x) \\ 1 \end{bmatrix} & z \in C = \mathbb{R} \times [0,1], \\ G(z) &= \begin{bmatrix} -x \\ 0 \end{bmatrix} & z \in D = \mathbb{R} \times \{1\}, \end{aligned} \tag{10}$$
where $\alpha \in (0,1)$ and $k > 0$. Each maximal solution $\phi = (\phi^x, \phi^\tau)$ to $\mathcal{H}$ from $\phi(0,0) = (x_0, \tau_0)$ satisfies
$$\phi(t,0) = \left(|x_0|^{1-\alpha} - k(1-\alpha)t\right)^{\frac{1}{1-\alpha}} \operatorname{sgn}(x_0) \tag{11}$$
for all $0 \leq t \leq \min\left\{1-\tau_0, \frac{|x_0|^{1-\alpha}}{k(1-\alpha)}\right\}$. Let $\bar{N}$ be such that
$$\frac{|x_0|^{1-\alpha}}{k(1-\alpha)} + \tau_0 - 2 \leq \bar{N} \leq \frac{|x_0|^{1-\alpha}}{k(1-\alpha)} + \tau_0 - 1,$$
where $\bar{N}$ is an integer. If $\bar{N} \leq -1$, we obtain
$$\frac{|x_0|^{1-\alpha}}{k(1-\alpha)} \leq 1 - \tau_0.$$
From (11), we have that $\phi(t^\star, 0) = 0$ where $t^\star = \frac{|x_0|^{1-\alpha}}{k(1-\alpha)}$. Furthermore, $\phi(t,j) = 0$ for all $(t,j) \in \operatorname{dom}\phi$ such that $t \geq t^\star$ according to (10). When $\bar{N} \geq 0$, we have
$$\phi(1-\tau_0, 1) = -\left(|x_0|^{1-\alpha} - k(1-\alpha)(1-\tau_0)\right)^{\frac{1}{1-\alpha}} \operatorname{sgn}(x_0). \tag{12}$$
Moreover, after $\bar{N}+1$ jumps,
$$\begin{aligned} &|\phi(1-\tau_0+\bar{N}, \bar{N}+1)| \\ &\quad = \left|\left(|x_0|^{1-\alpha} - k(1-\alpha)(1-\tau_0-\bar{N})\right)^{\frac{1}{1-\alpha}}\right|. \end{aligned} \tag{13}$$

[3]The function $\operatorname{sgn} : \mathbb{R} \to \{-1, 1\}$ is defined as $\operatorname{sgn}(x) = 1$ if $x \geq 0$, and $\operatorname{sgn}(x) = -1$ otherwise.

Therefore, using the property that
$$1 - \tau_0 + \bar{N} \leq \frac{|x_0|^{1-\alpha}}{k(1-\alpha)} \leq 1 - \tau_0 + \bar{N} + 1,$$
it implies that ϕ converges to 0 between the $(\bar{N}+1)$-th jump and the $(\bar{N}+2)$-th jump. In fact,
$$\phi\left(\frac{|x_0|^{1-\alpha}}{k(1-\alpha)}, \bar{N}+1\right) = 0$$
and $\phi(t,j) = 0$ for all $(t,j) \in \operatorname{dom}\phi$ such that $t \geq \frac{|x_0|^{1-\alpha}}{k(1-\alpha)}$. Therefore, the set $\{0\} \times [0,1]$ is FTA. $\triangle$



(a) The projection of the states x_1 and x_2 on the t direction.

(b) The projection of the Euclidean distance between x_1 and x_2 on the t direction

Fig. 1. The trajectories of the states x_1, x_2 for two fireflies in (14) and the Euclidean distance between them. Parameters used are $\gamma = 0.7$, $x_1(0,0) = 0$, $x_2(0,0) = 0.3$ and $\tilde{\varepsilon} = 0.1$.

Example 3.4: Consider the model of interacting fireflies in [10, Example 25]. The time of flashes of a firefly is determined by the firefly's internal clock. In between flashes, the internal clock gradually increases at a common rate $\gamma > 0$. When it reaches a certain threshold, a flash occurs and the clock is instantly reset to 0. In a group of fireflies, the flash of one firefly affects the internal clock of all other fireflies. That is, when a firefly witnesses a flash from another firefly, its internal clock instantly increases to a value closer to the threshold. To model the internal clock of the i-th firefly and to simplify the analysis, we consider a normalized clock, namely, the clock, denoted by x_i, takes values in the interval $[0,1]$ and flashes occur when x_i reaches the threshold 1. In between flashes, the clock state flows toward the threshold according to $\dot{x}_i = \gamma$. The resulting hybrid system $\mathcal{H}$ for two fireflies has state $x = (x_1, x_2) \in \mathbb{R}^2$ and data
$$\begin{aligned} F(x) &:= \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} & \forall x \in C, \\ G(x) &:= \begin{bmatrix} g((1+\tilde{\varepsilon})x_1) \\ g((1+\tilde{\varepsilon})x_2) \end{bmatrix} & \forall x \in D, \end{aligned} \tag{14}$$
where $C := [0,1] \times [0,1]$ and $D := \{x \in C : \max\{x_1, x_2\} = 1\}$. The parameter $\varepsilon > 0$ represents the effect on the timer of a firefly when another firefly's timer expires, i.e., the timer increases $(1+\tilde{\varepsilon})$ times its current value. The set-valued map g is defined as $g(s) = s$ when $s < 1$, $g(s) = 0$ when $s > 1$ and $g(s) = \{0, 1\}$ when $s = 1$. Then, the set of interest is $\mathcal{A} = \{x \in C : x_1 = x_2\}$, which defines the situation when both fireflies flash at the same time, namely, synchronized flashing. It can be shown that the compact set $\mathcal{A}$ is finite time stable for the system $\mathcal{H}$ from any open subsets $\widetilde{\mathcal{N}} \subset$

$\{x \in C : |x_1 - x_2| \neq (1+\tilde{\varepsilon})/(2+\tilde{\varepsilon})\}$[4] such that $\mathcal{A} \subset \widetilde{\mathcal{N}}$. A rigorous analysis will be carried out in Example 3.11. A simulation is shown in Figure 1. [5] △

B. Sufficient Conditions for Finite Time Stability

In this section, we propose sufficient conditions that guarantee finite time stability of a closed set for a hybrid system $\mathcal{H}$. The following result characterizes the scenario where, during flows, the distance of each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N})$ to a closed set $\mathcal{A}$ strictly decreases, where the set $\mathcal{N}$ is an open neighborhood of $\mathcal{A}$.

Theorem 3.5: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathcal{N} \subset \mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N}) \subset \mathcal{N}$. The set $\mathcal{A}$ is UFTS for $\mathcal{H}$ if there exist a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c_1 > 0, c_2 \in [0,1)$ such that

1) *for every $\xi \in \mathcal{N} \cap (\overline{C} \cup D) \setminus \mathcal{A}$, each $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ satisfies*
$$\frac{V^{1-c_2}(\xi)}{c_1(1-c_2)} \leq \sup_{(t,j) \in \operatorname{dom} \phi} t;$$
2) *there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}})$ for all $z \in (C \cup D \cup G(D)) \cap \mathcal{N}$ and*
$$u_C(z) + c_1 V^{c_2}(z) \leq 0 \quad \forall z \in C \cap \mathcal{N}, \tag{15}$$
$$u_D(z) \leq 0 \quad \forall z \in D \cap \mathcal{N}, \tag{16}$$

where the functions u_C and u_D are defined in (5) *and* (7)*, respectively. Furthermore, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0,0) = \xi$:*

a) *the settling-time function $T : \mathcal{N} \cap (\overline{C} \cup D) \to [0, \infty)$ satisfies*
$$T(\xi) \leq T^\star(\xi) + J^\star(\xi),$$
where $T^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$, and $J^\star(\xi)$ is such that $(T^\star(\xi), J^\star(\xi)) \in \operatorname{dom} \phi$;
b) *$|\phi(t,j)|_{\mathcal{A}} = 0$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t \geq T^\star(\xi)$.*

Remark 3.6: Assumption 1) in Theorem 3.5 is satisfied if the domain of each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N})$ is unbounded in the t direction. A result in a similar spirit, but for small ordinary time asymptotic stability can be found in [15, Proposition 3.2]. Moreover, when the jump set is empty and the flow set is such that $\mathcal{N} \subset C$, $\mathcal{H}$ reduces to a continuous-time system on $\mathcal{N}$, and the result in Theorem 3.5 reduces to a result for continuous-time systems; see, e.g., [2].

Remark 3.7: From Definition 3.1, if a closed set $\mathcal{A}$ is (uniformly) FTS for $\mathcal{H}$ with settling-time function $T : \mathcal{N} \cap (\overline{C} \cup D) \to \mathbb{R}_{\geq 0}$, where $\mathcal{N}$ is an open neighborhood of $\mathcal{A}$, then the set $\mathcal{A}$ is also (respectively, uniformly) pre-asymptotically stable (see [9, Definition 3.6]) for $\mathcal{H}$ with basin of attraction $\mathcal{N}$. However, the reverse implication is not true.

The following example illustrates Theorem 3.5.

[4]Solutions from the set $\{x \in C : |x_1 - x_2| = (1+\tilde{\varepsilon})/(2+\tilde{\varepsilon})\}$ do not converge to $\mathcal{A}$.

[5]Code at `https://github.com/HybridSystemsLab/FTSFireflies`

Example 3.8: Consider the system in Example 3.3, the function $V : \mathbb{R} \times [0,1] \to \mathbb{R}_{\geq 0}$ given by $V(z) = \frac{1}{2}x^2$ for each $z \in C$, and the compact set $\mathcal{A} = \{0\} \times [0,1]$. We have that, for each $z \in C$,
$$\langle \nabla V(z), F(z) \rangle = -k|x|^{1+\alpha} = -2^{\frac{1+\alpha}{2}} k V(z)^{\frac{1+\alpha}{2}}. \tag{17}$$
Then, condition (15) is satisfied with $\mathcal{N} = \mathbb{R} \times \mathbb{R}$, $c_1 = 2^{\frac{1+\alpha}{2}} k > 0$ and $c_2 = \frac{1+\alpha}{2} \in (0,1)$. Moreover, for all $z \in D$,
$$V(G(z)) - V(z) = 0, \tag{18}$$
which verifies the condition in (16). Note that the condition in item 1) follows since every maximal solution to $\mathcal{H}$ in (10) is complete (with its domain of definition unbounded in the t direction); e.g., by applying [9, Proposition 6.10]. Therefore, by Theorem 3.5, the set $\{0\} \times [0,1]$ is UFTS. △

Inspired by Example 3.4, the following result characterizes the scenario where the distance of a solution $\phi \in \mathcal{S}_{\mathcal{H}}$ to a closed set $\mathcal{A}$ strictly decreases after jumps.

Theorem 3.9: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathcal{N} \subset \mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N}) \subset \mathcal{N}$. The set $\mathcal{A}$ is UFTS for $\mathcal{H}$ if there exist a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c > 0$ such that

1) *for every $\xi \in \mathcal{N} \cap (\overline{C} \cup D) \setminus \mathcal{A}$, each $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ satisfies*
$$\operatorname{ceil}\left(\frac{V(\xi)}{c}\right) \leq \sup_{(t,j) \in \operatorname{dom} \phi} j;$$
2) *there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with $\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}})$ for each $z \in (C \cup D \cup G(D)) \cap \mathcal{N}$ such that*
$$\begin{aligned} u_C(z) &\leq 0 & \forall z \in C \cap \mathcal{N}, \\ u_D(z) &\leq -\min\{c, V(z)\} & \forall z \in D \cap \mathcal{N}. \end{aligned} \tag{19}$$

Furthermore, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0,0) = \xi$:

a) *the settling-time function $T : \mathcal{N} \cap (\overline{C} \cup D) \to [0, \infty)$ satisfies*
$$T(\xi) \leq T^\star(\xi) + J^\star(\xi),$$
where $J^\star(\xi) = \operatorname{ceil}\left(\frac{V(\xi)}{c}\right)$ and $T^\star(\xi)$ is such that $(T^\star(\xi), J^\star(\xi)) \in \operatorname{dom} \phi$ and $(T^\star(\xi), J^\star(\xi) - 1) \in \operatorname{dom} \phi$;
b) *$|\phi(t,j)|_{\mathcal{A}} = 0$ for all $(t,j) \in \operatorname{dom} \phi$ such that $j \geq J^\star(\xi)$.*

A similar result is established when the set $\mathcal{A}$ is asymptotically stable.

Theorem 3.10: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathcal{N} \subset \mathbb{R}^n$ with $\mathcal{N}$ open such that $G(\mathcal{N}) \subset \mathcal{N}$. The set $\mathcal{A}$ is UFTS to $\mathcal{H}$ if

1) *the set $\mathcal{A}$ is uniformly asymptotically stable with basin of attraction including $\mathcal{N}$,*
2) *there exists a neighborhood $U \subset \mathcal{N}$ of $\mathcal{A}$ such that:*
   2.1) *for every $\phi \in \mathcal{S}_{\mathcal{H}}(U \cap (\overline{C} \cup D))$, $(t, 1) \in \operatorname{dom} \phi$ for some $t \in \mathbb{R}_{\geq 0}$;*
   2.2) *$G((D \cap U) \setminus \mathcal{A}) \subset \mathcal{A}$.*

The following example illustrates Theorem 3.10.

Example 3.11: Consider the system in Example 3.4. To show the FTS property of the set $\mathcal{A}$, let $k = \frac{\tilde{\varepsilon}}{2+\tilde{\varepsilon}}$ and consider the function

$$V(x) := \min\{|x_1 - x_2|, 1+k-|x_1-x_2|\} \quad \forall x \in \mathcal{X}, \tag{20}$$

where

$$\mathcal{X} := \left\{x \in \mathbb{R}^2 : V(x) < \tfrac{1+k}{2}\right\} = \left\{x \in \mathbb{R}^2 : |x_1 - x_2| \neq \tfrac{1+k}{2}\right\}.$$

This function V is continuously differentiable on the open set $\mathcal{X} \setminus \mathcal{A}$ and it is Lipschitz on $\mathcal{X}$. Following [10, Example 25], let $m^\star = \frac{1+k}{2}$ and $m \in (0, m^\star)$, $K_m = \{x \in C \cup D : V(x) \leq m\}$, and define $C_m = C \cap K_m$ and $D_m = D \cap K_m$. By definition of V, it follows that

$$\langle \nabla V(x), F(x) \rangle = 0 \quad \forall x \in C_m \setminus \mathcal{A}. \tag{21}$$

Now consider $x \in D_m$. Since V is symmetric on the variables x_1 and x_2, without loss of generality, consider the case $x = (1, x_2)$, where $x_2 \in [0,1] \setminus \{1/(2+\tilde{\varepsilon})\}$. Then,

$$V(x) = \min\{1 - x_2, k + x_2\}, \tag{22}$$

$$V(G(x)) = \min\{g((1+\tilde{\varepsilon})x_2), 1+k-g((1+\tilde{\varepsilon})x_2)\}. \tag{23}$$

When $g((1+\tilde{\varepsilon})x_2) = (1+\tilde{\varepsilon})x_2$, there are two cases

- if $x_2 < 1/(2+\tilde{\varepsilon})$, $V(x) = k + x_2 > (1+\tilde{\varepsilon})x_2 \geq V(G(x))$;
- if $x_2 > 1/(2+\tilde{\varepsilon})$, $V(x) = 1 - x_2 \geq V(G(x))$.

Therefore, the set $\mathcal{A}$ is globally asymptotically stable for the system $\mathcal{H}_m = (C_m, F, D_m, G)$ and using [9, Proposition 6.10], every maximal solution to $\mathcal{H}_m$ is complete. Furthermore, given $\tilde{\varepsilon} > 0$, for $\varepsilon = \tilde{\varepsilon}/(1+\tilde{\varepsilon})$ and pick m such that $(\mathcal{A}+\varepsilon\mathbb{B}) \cap C \subset C_m$, we have that for all $x \in D_m \cap (\mathcal{A}+\varepsilon\mathbb{B})$,

$$G(x) = 0 \in \mathcal{A}. \tag{24}$$

Then, it follows from Theorem 3.10 that $\mathcal{A}$ is finite time stable for the system $\mathcal{H}_m = (C_m, F, D_m, G)$ with $\mathcal{N} = \{x \in C \cup D : V(x) < m\}$. △

Next, we establish a result similar to Theorem 3.9 when maximal solutions converge to a closed set $\mathcal{A}$ through jumps.

Proposition 3.12: Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$ and a closed nonempty set $\mathcal{A} \subset \mathbb{R}^n$. If a nonempty set $\widetilde{\mathcal{A}}$ is globally finite time attractive for $\mathcal{H}$, and there exists $\delta > 0$ such that $\widetilde{\mathcal{A}} + \delta\mathbb{B} \subset G^{-1}(\mathcal{A})$ and no flows from the set $(\widetilde{\mathcal{A}} + \delta\mathbb{B}) \setminus \mathcal{A}$ are possible, then the set $\mathcal{A}$ is globally FTA, where $G^{-1}(\mathcal{A}) := \{z \in D : G(z) \subset \mathcal{A}\}$.

Remark 3.13: A sufficient condition guaranteeing that no flows from the set $(\widetilde{\mathcal{A}} + \delta\mathbb{B}) \setminus \mathcal{A}$ are possible is when the flow set C is closed and $F(z) \cap T_C(z) = \emptyset$ for all $z \in C \cap ((\widetilde{\mathcal{A}} + \delta\mathbb{B}) \setminus \mathcal{A})$.

The following corollary considers the situation when the hybrid system $\mathcal{H}$ has linear flow and jump dynamics with a dwell-time behavior.

Corollary 3.14: Consider a hybrid system $\mathcal{H}$ with state $z = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and the closed set $\mathcal{A} = \{0\} \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The set $\mathcal{A}$ is globally FTA for $\mathcal{H}$ if the following holds:

1) *the flow map and jump map are single valued and their x_1 components are linear, i.e., $F(z) = (Ax_1, f(x_2))$ for all $z \in C$ and $G(z) = (Bx_1, g(x_2))$ for all $z \in D$ with $A \in \mathbb{R}^{n_1 \times n_1}$, $B \in \mathbb{R}^{n_1 \times n_1}$, $f : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, and $g : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$;*
2) *for each $\phi \in \mathcal{S}_\mathcal{H}$, $\sup_{(t,j) \in \operatorname{dom}\phi} j \geq n_1 + 1$, where n_1 is the dimension of x_1 component, and the flow time between every two consecutive jumps after the first jump are identical, i.e., there exists $\gamma > 0$ such that $t_{j+1} - t_j = \gamma$ for all $j \in \mathbb{N} \setminus \{0\}$ and $j \leq n_1 + 1$;*
3) *the matrix $B\exp(A\gamma)$ is nilpotent, where A, B come from item 1) and γ from item 2).*

Furthermore, for each $\phi \in \mathcal{S}_\mathcal{H}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\phi(0,0) = \xi$:

a) *there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ satisfying $T(\xi) \leq T^\star(\xi) + J^\star(\xi)$, where $J^\star(\xi) = n_1 + 1$ and $T^\star(\xi)$ is such that $(T^\star(\xi), J^\star(\xi)) \in \operatorname{dom}\phi$ and $(T^\star(\xi), J^\star(\xi) - 1) \in \operatorname{dom}\phi$;*
b) *$|\phi(t,j)|_\mathcal{A} = 0$ for all $(t,j) \in \operatorname{dom}\phi$ such that $j \geq J^\star(\xi)$.*

Remark 3.15: The second component x_2 of the state in the system in Corollary 3.14 can be arbitrary, but it would typically be involved in a mechanism that guarantees that the property in item 2) holds. Due to this, x_2 may include variables that behave like a timer. If the hybrid system $\mathcal{H}$ with linear flow and jump dynamics in Corollary 3.14 is such that $C = \emptyset$, then, the result in Corollary 3.14 is similar to the results about deadbeat convergence for discrete-time systems; see, e.g., [16].

Example 3.16: Consider a hybrid system with state $z = (x_1, x_2)$, $x_1 = (x_{11}, x_{12})$ and $z \in \mathcal{X} := \mathbb{R}^2 \times [0,1]$, its data $\mathcal{H} = (C, f, D, g)$ is given by

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} && z \in C \\ z^+ &= \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} && z \in D \end{aligned} \tag{25}$$

where $C = \{z \in \mathcal{X} : x_2 \in [0,1]\}$, $D = \{z \in \mathcal{X} : x_2 = 0\}$, and

$$G = \frac{1}{5}\begin{bmatrix} 2\cos(1) - \sin(1) & -\cos(1) - 2\sin(1) \\ 4\cos(1) - 2\sin(1) & -2\cos(1) - 4\sin(1) \end{bmatrix}. \tag{26}$$

Consider the set $\mathcal{A} = \{0\} \times \{0\} \times [0,1]$ and a solution $\phi = (\phi^{x_1}, \phi^{x_2}) \in \mathcal{S}_\mathcal{H}$. Then, $\phi^{x_1}(t,0) = \exp(At)\phi^{x_1}(0,0)$ for all $t \in [0, 1 - \phi^{x_2}(0,0)]$, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{27}$$

Furthermore, after three jumps,

$$\begin{aligned} &\phi^{x_1}(3 - \phi^{x_2}(0,0), 3) \\ &\quad = (G\exp(A))^2 G\exp(A(1 - \phi^{x_2}(0,0)))\phi^{x_1}(0,0). \end{aligned}$$

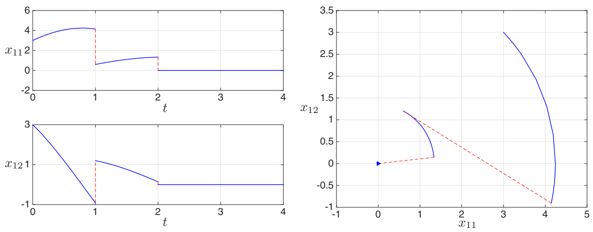
Note that $G\exp(A)$ is a nilpotent matrix, i.e., all eigenvalues are located at zero. In fact, since

$$\exp(A) = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix},$$

which is an invertible matrix, we have that $G = G_0(\exp(A))^{-1}$ for any given nilpotent matrix G_0. By letting

$$G_0 = \begin{bmatrix} 2/5 & -1/5 \\ 4/5 & -2/5 \end{bmatrix},$$

we obtain (26). Then, $(G\exp(A))^2 = 0$. Therefore, the solution ϕ converges to $\mathcal{A}$ within 3 jumps. Furthermore, since the time between two consecutive jumps is equal to one, $\mathcal{A}$ is uniformly finite time stable to $\mathcal{H}$ with $T = 3$. A simulation



(a) The projections of trajectories of x_{11} and x_{12} on the t direction. (b) Phase plot of x_{11} and x_{12}, the origin is denoted by the blue triangle.

Fig. 2. The trajectories of components x_{11}, x_{12} of solutions to (25). Initial condition is $z(0, 0) = (3, 3, 1)$.

is shown in Figure 2, where finite time convergence is approached within 2 jumps.[6] Note that the trajectory shown in Figure 2 reaches $\mathcal{A}$ after the second jump due to the fact that $G\exp(A)G$ is zero in this example. △

IV. Robustness of Finite Time Stability

In this section, we explore the robustness of finite time stability to perturbations. In particular, we consider the following perturbed system

$$\begin{array}{ll} \dot{z} \in F(z + d_1) + d_2 & (z + d_1) \in C \\ z^+ \in G(z + d_1) + d_2 & (z + d_1) \in D \end{array} \quad (28)$$

where d_1 denotes the perturbation on the state z and d_2 captures the unmodeled dynamics. By defining $v = (d_1, d_2)$, the system in (28) can be written as a hybrid system $\mathcal{H}_v$ given by

$$\begin{array}{ll} \dot{z} \in F_v(z, v) & (z, v) \in C_v \\ z^+ \in G_v(z, v) & (z, v) \in D_v \end{array} \quad (29)$$

where $C_v = \{(z, v) : z + d_1 \in C\}$, $D_v = \{(z, v) : z + d_1 \in D\}$, $F_v(z, v) = F(z + d_1) + d_2$ and $G_v(z, v) = G(z + d_1) + d_2$. Then, a function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is used to define the ρ-perturbation [9, Definition 6.27] of $\mathcal{H}$ in (2), denoted $\mathcal{H}_\rho$, and given by

$$\left\{ \begin{array}{lll} \dot{z} & \in F_\rho(z) & z \in C_\rho \\ z^+ & \in G_\rho(z) & z \in D_\rho \end{array} \right. \quad (30)$$

where

$$C_\rho = \{z \in \mathbb{R}^n : (z + \rho(z)\mathbb{B}) \cap C \neq \emptyset\},$$

$$F_\rho(z) = \overline{\text{con}} F((z + \rho(z)\mathbb{B}) \cap C) + \rho(z)\mathbb{B}$$

for all $z \in \mathbb{R}^n$,

$$D_\rho = \{z \in \mathbb{R}^n : (z + \rho(z)\mathbb{B}) \cap D \neq \emptyset\},$$

$$G_\rho(z) = \{\nu \in \mathbb{R}^n : \nu \in g + \rho(g)\mathbb{B},\ g \in G((z + \rho(z)\mathbb{B}) \cap D)\}$$

for all $z \in \mathbb{R}^n$. Using results from [9, Lemma 7.20], we have the following robustness result for perturbations (d_1, d_2) of size ρ.

[6] Code at `https://github.com/HybridSystemsLab/FTSNilpotency`

Theorem 4.1: Suppose $\mathcal{H}$ satisfies the hybrid basic conditions and a compact set $\mathcal{A}$ is FTS for $\mathcal{H}$. Then, the compact set $\mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable for $\mathcal{H}$, i.e., for each $\varepsilon > 0$, every continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive on $\mathcal{N} \setminus \mathcal{A}$, and each compact set $K \subset \mathcal{N}$, there exist $\beta \in \mathcal{KL}$ and $\delta \in (0, 1)$ such that every $\phi \in \mathcal{S}_{\mathcal{H}_{\delta\rho}}(K)$ satisfies

$$|\phi(t, j)|_{\mathcal{A}} \leq \beta(|\phi(0, 0)|_{\mathcal{A}}, t + j) + \varepsilon \quad (31)$$

for all $(t, j) \in \operatorname{dom} \phi$.

V. Conclusion

A notion of finite time stability consisting of both stability and finite time attractivity was proposed for hybrid systems. Sufficient conditions guaranteeing the new notion were proposed. The conditions conveniently isolate the properties needed when finite time convergence occurs via flows or via jumps. Conditions for robustness of the new notion to perturbations, though generic, rely on the regularity of the data of the hybrid system so as to preserve the finite time stability property semiglobally and practically.

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