# Exponential Stabilization of a Vectored-Thrust Vehicle Using Synergistic Potential Functions

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***Abstract*— In this paper, we design a hybrid controller based on the concept of centrally synergistic potential functions on the $n$-dimensional sphere to achieve global tracking of an attitude reference as well as exponential stabilization of the position and velocity of a vectored-thrust vehicle. The proposed hybrid attitude controller renders a partial attitude reference globally exponentially stable, and the full attitude reference globally attractive and locally exponentially stable. This controller is then combined with a position controller for the quadrotor vehicle for tracking of a given reference. Simulation results are provided so as to demonstrate the performance of the proposed controller.**

## I. Introduction

Mechanical systems that have rotational degrees of freedom are often described by elements of the $n$-dimensional sphere. These include, but are not limited to rigid-body dynamics, robotic manipulators, and three-dimensional pendulums. The book [1] has detailed descriptions of many of these systems as well as continuous controller synthesis techniques. However, it has been shown in [2] that given a continuous vector field over a compact manifold, there exists more than one equilibrium point, thus precluding global asymptotic stabilization of a given setpoint for systems with rotational degrees of freedom, via continuous feedback.

These topological obstructions have not deterred researchers from applying continuous control laws to the stabilization of complex systems involving rotational degrees of freedom. A research subject that has drawn considerable attention over the last few years is that of control of Unmanned Air Vehicles (UAVs). In particular, continuous control strategies have been used in the stabilization of vectored thrust aircraft, as described in [3], [4] [5], for example. In these works, the undesired equilibrium point is avoided by means of a control law which grows without bound near the undesired equilibrium point. In [6], a bounded controller is proposed but, as shown therein, there exists a submanifold of the rotation group which is invariant under the influence of the vector field and it is such that the state does not converge to the desired equilibrium point. Using discontinuous quaternion feedback it has been possible to overcome this limitation, as shown in [7]. The reader is referred to [8] for an exhaustive summary on the advantages and disadvantages of different control strategies for rigid-body dynamics.

It has been recently shown in [9] that neither continuous nor discontinuous controllers are able to provide robust global asymptotic stabilization of a setpoint for system evolving on compact manifolds. The inherent limitations of both continuous and discontinuous controllers have nurtured the development of hybrid control techniques, since it has been shown that hybrid systems satisfying the hybrid basic conditions are endowed with robustness to small measurement noise [10]. In particular, strategies for rigid-body stabilization by means of hybrid feedback have been provided in [11] and [12]. These controllers are instrumental in the recent techniques for global trajectory tracking of a vectored thrust vehicle, as illustrated in [13]. More recently, the controller design in [14] employed synergistic potential functions on $\mathbb{S}^2$ to achieve reduced attitude stabilization by means of a potential-induced gradient-based feedback and appropriate switching law that mitigated undesirable equilibria, achieving global asymptotic stabilization. In [15], we developed the ideas of synergistic potential functions on $\mathbb{S}^n$ to derive a controller for the reduced attitude stabilization of a rigid body such that the trajectories of the closed-loop system follow paths of least distance upon switching. Using the aforementioned strategy, we were able to prove global exponential stability of a given reference for the closed-loop system. In this paper, we extend that controller to attitude tracking for a rigid body. To accomplish this goal, we define the attitude error and represent it by means of two orthogonal unitary vectors and, using this representation of attitude, we show that it is possible to apply the hybrid controller induced by synergistic potential functions of $\mathbb{S}^2$ to track a given attitude reference. We show that the partial attitude reference is globally exponentially stable, the full attitude reference is globally attractive and locally exponentially stable. We also demonstrate how this strategy may be applied to the tracking of a reference position trajectory for a quadrotor vehicle. The proofs of the results in this paper are to appear elsewhere.

The paper is organized as follows. In Section II, we present some notation that is used throughout the paper. In sections III and IV, demonstrate the application of synergistic

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potential functions to the attitude tracking of rigid body and to position tracking for a vectored thrust vehicle, respectively. Moreover, we review the properties of the synergistic potential function that was introduced in [15]. In Section V, we provide a set of simulations, validating our results.

## II. Preliminaries & Notation

The set $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, equipped with the inner product $\langle u, v\rangle = u^\top v$, defined for each $u, v \in \mathbb{R}^n$, and the norm $|x| = \sqrt{\langle x, x\rangle}$. The set $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$ denotes the $n$-dimensional sphere and $x_0 + M\mathbb{B} := \{x \in \mathbb{R}^n : |x - x_0| \leq M\}$ denotes a ball with radius $M > 0$, centered at $x_0 \in \mathbb{R}^n$. $C^n(\mathscr{M}, \mathscr{N})$ denotes a function from $\mathscr{M}$ to $\mathscr{N}$ that is continuously differentiable up to order $n$. For $F \in C^n(\mathscr{M}, \mathscr{N})$ with $n \geq 1$, $\mathscr{M} \subset \mathbb{R}^m$, $\mathscr{N} \subset \mathbb{R}^k$ and $F(x) := (F_1(x_1, \ldots, x_m), \ldots, F_k(x_1, \ldots, x_m))$ for each $x \in \mathscr{M}$, we define

$$\frac{\partial F}{\partial x^\top} := \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1} & \frac{\partial F_k}{\partial x_2} & \cdots & \frac{\partial F_k}{\partial x_m} \end{bmatrix},$$

and $\frac{\partial F}{\partial x} := \frac{\partial F}{\partial x^\top}^\top$. If $F$ is a twice differentiable scalar function, then we define $\frac{\partial^2 F}{\partial x^\top \partial x} := \frac{\partial}{\partial x^\top}\left(\frac{\partial F}{\partial x}\right)$ and $\nabla F := \frac{\partial F}{\partial x}$.

Given a continuously differentiable function $V : \mathbb{S}^n \to \mathbb{R}_{\geq 0}$, its set of critical points is

$$\text{crit}(V) := \{x \in \mathbb{S}^n : \Pi(x)\nabla V(x) = 0\},$$

where $\Pi(x) = I_{n+1} - xx^\top$. $SO(3)$ denotes the set of orthogonal matrices with determinant equal to 1, i.e. if $R \in SO(3)$ then $R^\top R = I_3$ and $\det(R) = 1$.

A hybrid system $\mathcal{H} = (C, F, D, G)$ defined in $\mathbb{R}^n$ is defined as follows:

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x) \end{cases},$$

where $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ denotes the jump set, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map, where $\rightrightarrows$ denotes a map from sets in $\mathbb{R}^n$ to sets in $\mathbb{R}^n$. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} \left([t_j, t_j + 1] \times \{j\}\right),$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ is a compact hybrid domain.

Every solution $(t, j) \mapsto x(t, j)$ to a hybrid system is defined on a hybrid time domain $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0$, where $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers and $\mathbb{N}_0$ denotes the set of non-negative integers. A solution to a hybrid system is said to be *maximal* if it cannot be extended by flowing nor jumping, *complete* if its domain is unbounded, and *precompact* if it is complete and bounded (the reader is referred to [10, Chapter 2] for more information on solutions to hybrid systems).

The so-called *hybrid basic conditions* [10, Assumption 6.5] are reproduced next for the sake of completeness.

**Definition 1.** *The hybrid system $\mathcal{H}$ satisfies the hybrid basic conditions if:*

*(A1) $C$ and $D$ are closed subsets of $\mathbb{R}^n$;*
*(A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to $C$, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for every $x \in C$;*
*(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to $D$, and $D \subset \operatorname{dom} G$.*

Assuming that each maximal solution to $\mathcal{H}$ is complete, a compact set $\mathcal{A}$ is said to be: globally stable for $\mathcal{H}$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution $\phi$ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t, j)|_{\mathcal{A}} \leq \epsilon$ for each $(t, j) \in \operatorname{dom} \phi$; globally attractive for $\mathcal{H}$ if every maximal solution $x$ to $\mathcal{H}$ is complete and $\lim_{t+j\to\infty} |x(t, j)|_{\mathcal{A}} = 0$, where $|x|_{\mathcal{A}} := \min_{y\in\mathcal{A}} |x - y|$. A set $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}$ if it is both globally attractive and stable for $\mathcal{H}$. We say that a compact set $\mathcal{A}$ is globally exponentially stable in the $t$-direction if there exists $k, \lambda > 0$ such that, for each maximal solution $\phi$ to the hybrid system, the following holds:

$$|\phi(t, j)|_{\mathcal{A}} \leq k \exp(-\lambda t) \, |\phi(0, 0)|_{\mathcal{A}},$$

for each $(t, j) \in \operatorname{dom} \phi$ and $\sup_t \operatorname{dom} \phi := \sup\{t : (t, j) \in \operatorname{dom} \phi\} = \infty$. When this property holds for all maximal solutions starting from a neighborhood of $\mathcal{A}$, we say that $\mathcal{A}$ is locally exponentially stable.

## III. Tracking on $SO(3)$

The attitude of a rigid-body can be represented by a rotation matrix $R \in SO(3)$ with kinematics

$$\dot{R} = RS(\omega), \tag{1}$$

where $\omega \in \mathbb{R}^3$ denotes the angular velocity and

$$S(\omega) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

(c.f. [16]). Given attitude and angular velocity references $(R_d, \omega_d)$ satisfying

$$\dot{R}_d = R_d S(\omega_d), \quad \omega_d \in M\mathbb{B} \tag{2}$$

for some $M > 0$, we may define the attitude error as

$$\widetilde{R} = R^\top R_d. \tag{3}$$

Using (1) and (2), the derivative of (3) is given by

$$\dot{\widetilde{R}} = -S(\omega - \widetilde{R}\omega_d)\widetilde{R}. \tag{4}$$

where we have used the facts $S(\omega)^\top = -S(\omega)$ for each $\omega \in \mathbb{R}^3$ and $AS(v)A^{-1} = S(Av)$ for each non-singular $A \in \mathbb{R}^{3\times 3}$ and for each $v \in \mathbb{R}^3$. Defining $\widetilde{\omega} := \omega - \widetilde{R}\omega_d$, it follows from (4) that, for a constant vector $r \in \mathbb{S}^2$, the time derivative of $x := \widetilde{R}r$ is given by

$$\dot{x} = S(x)\widetilde{\omega}.$$

This is important because each element of $SO(3)$ can be represented by two orthogonal unitary vectors, thus we are able to cast the tracking problem on $SO(3)$ as the stabilization of the system

$$\dot{x}_1 = S(x_1)\widetilde{\omega}, \quad \dot{x}_2 = S(x_2)\widetilde{\omega}, \tag{5}$$

where $(x_1, x_2) := (\widetilde{R}r_1, \widetilde{R}r_2)$ and $r_1, r_2 \in \mathbb{S}^2$ are mutually orthogonal, that is, $r_1^\top r_2 = 0$. The controller design problem that we address in this paper is provided below.

**Problem 2.** *Given two mutually orthogonal vectors $r_1, r_2 \in \mathbb{S}^2$, design a hybrid controller $\mathcal{H}_c := (C_c, F_c, D_c, G_c)$ with state $x_c \in \mathcal{X}_c$ and input $(x_1, x_2) := (\widetilde{R}r_1, \widetilde{R}r_2)$ such that the set*

$$\mathcal{A}_{r_1} := \{z \in Z : x_1 = r_1\} \tag{6}$$

*with $z := (x_1, x_2, x_c)$ and $Z := \mathbb{S}^2 \times \mathbb{S}^2 \times \mathcal{X}_c$, is globally exponentially stable in the $t$-direction for the interconnection between $\mathcal{H}_c$ and the system* (5)*, and the set*

$$\mathcal{A}_R := \{z \in Z : x_1 = r_1, \ x_2 = r_2\} \tag{7}$$

*is globally attractive and locally exponentially stable.* □

To tackle this problem, we make use of the concept of centrally synergistic potential functions on $\mathbb{S}^n$ that was introduced in [14] and which we describe below.

**Definition 3.** *Given $r \in \mathbb{S}^n$ and a compact set $Q_r \subset \mathbb{R}^m$, for some $m > 0$, we say that $V_r \in C^1(\mathbb{S}^n \times Q_r, \mathbb{R})$ is a centrally synergistic potential function relative to $Q_r$, if it is positive definite relative to*

$$\mathcal{A}_{V_r} := \{r\} \times Q_r \tag{8}$$

*and if there exists $\delta > 0$ such that*

$$\mu_{V_r}(x, y) := V_r(x, y) - \min_{w \in Q_r} V_r(x, w) > \delta, \tag{9}$$

*for each $(x, y) \in \mathcal{E}(V_r)$, where $\mathcal{E}(V_r) := \{(x, y) \in \mathbb{S}^n \times Q_r : r \in \operatorname{crit}(V_r^y)\backslash\{r\}\}$ with $V_r^y(x) := V_r(x, y)$. We also say that $V_r$ has* synergy gap exceeding *$\delta$.* □

Centrally synergistic potential functions of $\mathbb{S}^n$ satisfy the following very important properties which are introduced in [15].

**Proposition 4.** *Let $r \in \mathbb{S}^n$, let $Q_r \subset \mathbb{R}^m$ be a compact set, $V_r \in C^1(\mathbb{S}^n \times Q_r, \mathbb{R})$, and $V_r^y \in C^1(\mathbb{S}^n, \mathbb{R})$ be defined as $V_r^y(x) = V(x, y)$ for each $(x, y) \in \mathbb{S}^n \times Q_r$. Then, the following holds: 1) If $V_r$ is positive definite relative to* (8)*, then $r \in \operatorname{crit}(V_r^y)$ for each $y \in Q_r$; 2) The function* (9) *is continuous and the map*

$$\varrho_{V_r}(x) := \operatorname{argmin}\{V_r(x, y) : y \in Q_r\}, \tag{10}$$

*defined for each $x \in \mathbb{S}^2$, is outer semicontinuous.* □

As shown in [14] and [15], this class of functions induces a natural hybrid controller for the global asymptotic stabilization of a reference $r \in \mathbb{S}^n$ that uses a gradient-based control law during flows and appropriate switching near undesired equilibrium points of the closed-loop system. In this paper, we propose an adaptation of the aforementioned hybrid controller, given by

$$\underbrace{\begin{array}{ll} \dot{y}_1 & = 0 \\ \dot{y}_2 & = 0 \end{array}}_{z \in C_R} \quad \underbrace{\begin{array}{ll} y_1^+ & \in \varrho_{V_{r_1}}(x_1) \\ y_2^+ & \in \varrho_{V_{r_2}}(x_2) \end{array}}_{z \in D_R} \tag{11}$$

with output

$$\widetilde{\omega}(z) := k_1 S(x_1) \nabla V_{r_1}^{y_1}(x_1) + k_2 x_1 x_1^\top S(x_2) \nabla V_{r_2}^{y_2}(x_2), \tag{12}$$

where $k_1, k_2 > 0$ are controller parameters, $z := (x_1, x_2, y_1, y_2)$ belongs to $Z := \{(x_1, x_2, y_1, y_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \times Q_{r_1} \times Q_{r_2} : \langle x_1, x_2 \rangle = 0\}$, $V_{r_i}^{y_i}(x_i) := V_{r_i}(x_i, y_i)$ for each $i \in \{1, 2\}$ and for each $z \in Z$, $V_{r_1}$ and $V_{r_2}$ are centrally synergistic potential functions with respect to $Q_{r_1}$ and $Q_{r_2}$, respectively, for orthogonal unitary vectors $r_1, r_2 \in \mathbb{S}^2$. The flow and jump sets are given by $C_R = C_{R_1} \cap C_{R_2}$ and $D_R = D_{R_1} \cup D_{R_2}$, respectively, where, for each $i \in \{1, 2\}$,

$$\begin{aligned} C_{R_i} &:= \{z \in Z : \mu_{V_{r_i}}(x_i, y_i) \le \delta\}, \\ D_{R_i} &:= \left\{z \in Z : \mu_{V_{r_i}}(x_i, y_i) \ge \delta\right\}, \end{aligned} \tag{13}$$

for some $\delta > 0$.

The interconnection between the controller (11) and (5) yields the closed-loop hybrid system $\mathcal{H}_R = (C_R, F_R, D_R, G_R)$, given by

$$\begin{aligned} F_R(z) &:= (S(x_1)\widetilde{\omega}, S(x_2)\widetilde{\omega}, 0, 0) & \forall z \in C_R, \\ G_R(z) &:= (x_1, x_2, \varrho_{V_{r_1}}(x_1), \varrho_{V_{r_2}}(x_2)) & \forall z \in D_R. \end{aligned} \tag{14}$$

Note that the control law (12) has two components: the first is normal to $x_1$ and the second is collinear to it. As a consequence, global exponential stability in the $t$-direction is guaranteed for (6) but not (7), as shown next.

**Theorem 5.** *Given $r_1, r_2 \in \mathbb{S}^2$ satisfying $r_1^\top r_2 = 0$ and synergistic potential functions relative to $Q_{r_1}$ and $Q_{r_2}$, denoted by $V_{r_1}$ and $V_{r_2}$, respectively, if, for each $(x_2, y_2) \in \mathbb{S}^2 \times Q_{r_2}$, $r_1^\top S(x_2) \nabla V_{r_2}^{y_2}(x_2) = 0$ and $x_2^\top r_1 = 0$ implies that $x_2 = r_2$ or $\mu_{V_{r_2}}(x_2, y_2) > \delta$, then the set* (6) *is globally asymptotically stable for the hybrid system* (14) *and the set* (7) *globally attractive.*

Next we show that, if the centrally synergistic potential function $V_{r_1}$ and its derivative satisfy some quadratic bounds, then the set (6) is globally exponentially stable in the $t$-direction for the hybrid system (13).

**Corollary 6.** *Let $V_{r_1}$ denote a centrally synergistic potential functions relative to $r_1$. If there exist $\underline{\alpha}, \overline{\alpha}, \eta > 0$ such that*

$$\underline{\alpha}\, |x_1 - r_1|^2 \le V_{r_1}(x_1, y_1) \le \overline{\alpha}\, |x_1 - r_1|^2 \ \ \forall z \in Z \tag{15a}$$

$$\left|\Pi(x_1) \nabla V_{r_1}^{y_1}(x_1)\right|^2 \ge \eta V_{r_1}(x_1, y_1) \ \forall z \in C_R, \tag{15b}$$

*then the set* (6) *is globally exponentially stable in the $t$-direction for the hybrid system* (14)*.*

Next, we provide particular examples of centrally synergistic potential functions $V_{r_1}$ and $V_{r_2}$, to be used in stabilization on $SO(3)$. The full description of these functions can be found in [15]. Let $\mathcal{V} := \mathbb{S}^2 \times (\mathbb{S}^2 \backslash \{r\})$ and let us consider the following function:

$$V_r(x, y) := \frac{1 - r^\top x}{1 - r^\top x + k\,(1 - y^\top x)}, \tag{16}$$

with $k > 0$ and defined for each $(x, y) \in \mathcal{V}$, whose gradient is given by

$$\nabla V_r^y(x) = \frac{k V_r(x, y) y - (1 - V_r(x, y)) r}{1 - r^\top x + k\,(1 - y^\top x)},$$

for each $x \in \mathbb{S}^2$ and satisfies

$$|\Pi(x)\nabla V_r(x,y)|^2 = \frac{2kV_r(x,y)\left(1-V_r(x,y)\right)\left(1-r^\top y\right)}{\left(1-x^\top r+k\left(1-y^\top x\right)\right)^2},$$

for each $x \in \mathbb{S}^2$. Given $r \in \mathbb{S}^2$, $\gamma \in \mathbb{R}$ satisfying $-1 \leq \gamma < 1$, we define the set $Q_r \subset \mathbb{S}^2$ as

$$Q_r = \left\{y \in \mathbb{S}^2 : r^\top y \leq \gamma\right\}. \tag{17}$$

The boundary of $Q_r$, denoted $\partial Q_r$, is

$$\partial Q_r = \left\{y \in \mathbb{S}^2 : r^\top y = \gamma\right\}.$$

The following result was introduced in [15].

**Theorem 7.** *Given $r \in \mathbb{S}^2$ and $\gamma \in [-1,1)$, let $Q_r(r,\gamma)$ be given by* (17)*. Then, considering the definitions* (10)*, the following holds for the function $V_r$ given in* (16)

$$\varrho_{V_r}(x) = \begin{cases} Q_r(r,\gamma) & \text{if } x = r \\ -x & \text{if } 1 > r^\top x \geq -\gamma \\ \gamma r - \frac{\sqrt{1-\gamma^2}\Pi(r)x}{|\Pi(r)x|} & \text{if } -1 < r^\top x < -\gamma \\ \partial Q_r(r,\gamma) & \text{if } r^\top x = -1. \end{cases} \tag{18a}$$

$$\nu_{V_r}(x) = \begin{cases} \dfrac{1-r^\top x}{1-r^\top x+2k} & \text{if } r^\top x \geq -\gamma \\ \dfrac{1-r^\top x}{1-r^\top x + k\left(1-\gamma+\sqrt{1-\gamma^2}\,|\Pi(r)x|\right)} & \text{if } r^\top x < -\gamma \end{cases} \tag{18b}$$

*for each $x \in \mathbb{S}^2$, where $\nu_{V_r}(x) := \min_{y\in Q_r} V(x,y)$.* □

For any given $\gamma \in [-1,1)$, the function (16) is a centrally synergistic potential function relative to $Q_r$ with synergy gap exceeding $\delta$, for any

$$\delta \in \left(0, \frac{1+\gamma}{2/k+1+\gamma}\right). \tag{19}$$

**Theorem 8.** *The function $V_r \in C^1(\mathbb{S}^n \times Q_r$ given in* (16) *satisfies* (15) *with*

$$\underline{\alpha} = \frac{1}{2(1+k+\sqrt{1+2k\gamma+k^2})}, \tag{20a}$$

$$\overline{\alpha} = \frac{1}{2(1+k-\sqrt{1+2k\gamma+k^2})}, \tag{20b}$$

$$\eta = \frac{2k(1-V^\star)(1-\gamma)}{\left(1+k+\sqrt{1+2k\gamma+k^2}\right)^2}, \tag{20c}$$

*where $V^\star := \max\{V_r(x,y) : \mu_{V_r}(x,y) \leq \delta, (x,y) \in \mathbb{S}^2 \times Q_r\}$.* □

The next result shows that, using the right controller parameters, it is possible to meet the conditions of Theorem 5.

**Proposition 9.** *Given two mutually orthogonal unitary vectors $r_1, r_2 \in \mathbb{S}^2$ and $V_{r_2}$ given by* (16)*, for each $k > 0$ and $\gamma \in (-1,0)$, if*

$$\delta \in \left(0, -\frac{4k\gamma}{4+4k+k(1-\gamma^2)}\right) \tag{21}$$

*then $r_1^\top S(x_2)\nabla V_{r_2}^{y_2}(x_2) = 0$ and $x_2^\top r_1 = 0$ implies that $x_2 = r_2$ or $\mu_{V_{r_2}}(x_2,y_2) > \delta$.*

For the particular constructions of $V_{r_1}$ and $V_{r_2}$ given in (16), we are able to assert a local stability property for the system (5), in addition to the global exponential stability property that was proved in Corollary 6.

**Proposition 10.** *Given $k > 0$, $\gamma \in (-1,1)$ and centrally synergistic potential functions with respect to $Q_{r_1}$ and $Q_{r_2}$, denoted by $V_{r_1} : \mathbb{S}^2 \times Q_{r_1} \to \mathbb{R}_{\geq 0}$ and $V_{r_2} : \mathbb{S}^2 \times Q_{r_2} \to \mathbb{R}_{\geq 0}$, respectively, and given by* (16)*, the set*

$$\mathcal{C} := \{(x_1,x_2) \in SO(3) : x_1 = r_1,\ x_2 = r_2\}$$

*is locally exponentially stable for the system* (5) *with $\omega$ given by* (12).

The proposed controller is particularly tailored for the stabilization of the quadrotor vehicle presented in the following section. It was shown in [15] that the proposed controller tracks a reference vector in $\mathbb{S}^2$ through geodesics, thus it is more appropriate for the application at hand than other stabilization strategies such as the one in [17].

## IV. Exponential Stabilization of a Vectored Thrust Vehicle

In this section, we design a controller for a vectored thrust vehicle that tracks a given reference, using the controller presented in the previous section. The dynamics of this kind of vehicles are described by the following equations:

$$\dot{p} = v \tag{22a}$$

$$\dot{v} = Rr_1u + g \tag{22b}$$

$$\dot{R} = RS(\omega), \tag{22c}$$

where $g \in \mathbb{R}^3$ is the gravity vector, $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the position and velocity, respectively, of the body attached frame relative to the inertial reference frame, expressed in inertial coordinates, $r_1 \in \mathbb{S}^2$ is the thrust direction which is fixed with respect to the body of the vehicle, $u \in \mathbb{R}$ denotes the thrust force, $R \in SO(3)$ is the rotation matrix that maps vectors in body coordinates to inertial coordinates, and $\omega \in \mathbb{R}^3$ represents the the angular velocity, which is considered as an input. The model for vectored thrust vehicles (22) can be found in [3] and [13], for example.

To design the controller, we split the problem in two: firstly, we consider $u$ and $R$ as inputs and stabilize the position subsystem; secondly, we use the position controller to define an attitude reference for the attitude tracking controller and use the angular velocity as the input.

Assuming that the reference trajectory is a smooth path $t \mapsto r_d(t) = (p_d, \dot{p}_d, \ddot{p}_d, \dddot{p}_d)(t)$ satisfying

$$\dot{r_d} \in M_p\mathbb{B}$$

for some $M_p > 0$, we define the tracking errors

$$\widetilde{p} := p - p_d \quad \widetilde{v} := v - \dot{p}_d$$

which, using (22), are characterized by the following equations of motion:

$$\dot{\widetilde{p}} = \widetilde{v}, \qquad \dot{\widetilde{v}} = R r_1 u + g - \ddot{p}_d. \tag{23}$$

The stabilization of the position subsystem described by (23) amounts to the stabilization of three double integrators in parallel, hence we may use a linear controller of the form

$$w(\widetilde{p}, \widetilde{v}) := K \begin{bmatrix} \widetilde{p} \\ \widetilde{v} \end{bmatrix}, \tag{24}$$

for each $(\widetilde{p}, \widetilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$. Then, if

$$u(\widetilde{p}, \widetilde{v}, R) := r_1^\top R^\top (w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g), \tag{25a}$$

$$R r_1 = \rho := \frac{w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g}{|w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g|}, \tag{25b}$$

the error dynamics (23) can be rewritten as follows:

$$\dot{\widetilde{p}} = \widetilde{v}, \quad \dot{\widetilde{v}} = w(\widetilde{p}, \widetilde{v}), \tag{26}$$

meaning that the acceleration of the vehicle is equal to the control law for the double integrator system. From the definition of $\rho$ in (25b) it is clear that

$$w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g \neq 0 \tag{27}$$

is a pivotal condition to guarantee that the controller is well defined. To address this issue, we provide an auxiliary result in Proposition 12 which guarantees that, for each compact set of initial position and velocity tracking errors, there exists a controller gain $K$ such that the condition (27) is satisfied.

Since the full attitude reference $R_d \in SO(3)$ to be tracked by the attitude controller must satisfy (2) in addition to $R_d r_1 = \rho$, then

$$S(r_1)\omega_d = S(r_1)^2 R_d^\top \frac{\dot{w} + p_d^{(3)}}{|w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g|}.$$

Let $\widetilde{R} := R^\top R_d$ denote the attitude error and let the thrust input be given by (25a). Then, it is possible to write the dynamics of the error variables using (22) as follows:

$$\dot{\widetilde{p}} = \widetilde{v} \tag{28a}$$

$$\dot{\widetilde{v}} = R r_1 r_1^\top R^\top (w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g) + g - \ddot{p}_d \tag{28b}$$

$$\dot{\widetilde{R}} = -S(\widetilde{\omega})\widetilde{R}, \tag{28c}$$

with $\widetilde{\omega} := \omega - \widetilde{R}\omega_d$. Suppose that we are given a unitary vector $r_2 \in \mathbb{S}^2$ that is orthogonal to $r_1 \in \mathbb{S}^2$, then it is possible to represent $\widetilde{R}$ by the two orthogonal vectors $x_1 := \widetilde{R} r_1$ and $x_2 := \widetilde{R} r_2$, as done in Section III. Using these definitions, it is possible to describe the full system as follows:

$$\dot{r}_d \in M_p \mathbb{B} \tag{29a}$$

$$\dot{R}_d = R_d S(\omega_d) \tag{29b}$$

$$\dot{\widetilde{p}} = \widetilde{v} \tag{29c}$$

$$\dot{\widetilde{v}} = R_d x_1 x_1^\top R_d^\top (w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g) + g - \ddot{p}_d \tag{29d}$$

$$\dot{x}_1 = S(x_1)\widetilde{\omega} \tag{29e}$$

$$\dot{x}_2 = S(x_2)\widetilde{\omega} \tag{29f}$$

where $\widetilde{\omega} \in \mathbb{R}^3$ is the input and

$$\omega_d := S(r_1) R_d^\top \frac{\dot{w} + p_d^{(3)}}{|w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g|}.$$

Let $\zeta := (r_d, R_d, \widetilde{p}, \widetilde{v}, z) \in \mathcal{Z}$ and $\mathcal{Z} := \mathbb{R}^{12} \times SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times Z$, where $z$ and $Z$ are given in Section III. Then the interconnection between (28) and the attitude controller (11) with the modified output

$$\widetilde{\omega}(\zeta) := k_p \frac{S(x_1)\nabla V_{r_1}^{y_1}(x_1)\, |\nabla_v V_p|\, |w(\widetilde{p}, \widetilde{v}) + \ddot{p}_d - g|}{|S(x_1)\nabla V_{r_1}^{y_1}(x_1)|\, \sqrt{V_p(\widetilde{p}, \widetilde{v})}} + k_1 S(x_1)\nabla V_{r_1}^{y_1}(x_1) + k_2 x_1 x_1^\top S(x_2)\nabla V_{r_2}^{y_1}(x_2)$$

where $k_p > 0$, $V_p$ given by

$$V_p(\widetilde{p}, \widetilde{v}) := \begin{bmatrix} \widetilde{p}^\top & \widetilde{v}^\top \end{bmatrix} P_\epsilon \begin{bmatrix} \widetilde{p} \\ \widetilde{v} \end{bmatrix},$$

for each $\widetilde{p}, \widetilde{v} \in \mathbb{R}^6$, with $P \in \mathbb{R}^{6\times 6}$ positive definite, yields the closed-loop hybrid system

$$\begin{aligned} \dot{\zeta} \in F(\zeta) &:= \begin{pmatrix} M_p\mathbb{B} \\ R_d S(\omega_d) \\ \widetilde{v} \\ R_d x_1 u + g - \ddot{p}_d \\ F_R(z) \end{pmatrix} & \zeta \in C \\ \zeta^+ \in G(\zeta) &= \begin{pmatrix} r_d \\ R_d \\ \widetilde{p} \\ \widetilde{v} \\ G_R(z) \end{pmatrix} & \zeta \in D \end{aligned} \tag{30}$$

where $C := \{\zeta \in \mathcal{Z} : z \in C_R\}$, $D := \{\zeta \in \mathcal{Z} : z \in D_R\}$, $u$ is given by (25a).

**Theorem 11.** *For each $M_p < g$, for each compact set $U \subset \mathbb{R}^6$, each $k > 0$, $\gamma \in (-1, 0)$, $\delta \in \mathbb{R}$ satisfying* (21)*, there exists $K \in \mathbb{R}^{3\times 6}$ satisfying* (31) *such that*

$$\mathcal{A} := \mathcal{A}_p \times \mathcal{A}_{r_1}$$

*is exponentially stable in the $t$-direction and the set*

$$\mathcal{B} := \mathcal{A}_p \times \mathcal{A}_R$$

*is attractive from $U \times Z$ for the hybrid system* (30)*.*

## V. Simulation Results

In this section, we provide simulation results for the closed-loop hybrid system (30). In the simulations, we have considered the normalized gravity vector $g = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top$, that is, the gravity is aligned with the $x$-axis of the inertial reference frame. To design the position controller, we have selected the controller gain $K$ for the double integrator system which minimizes

$$J = \int_0^\infty \begin{bmatrix} x(t) & \dot{x}(t) \end{bmatrix} \left( \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix} + K^\top K \right) \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} dt,$$

where $x$ and $\dot{x}$ denote the position and velocity along the $x$-axis direction, respectively. The attitude controller parameters are $k_p = 1$, $\gamma = -0.5$, $k_1 = 1$, $k_2 = 1$, $k = 1$ and $\delta = 0.1$, which satisfies (21) and (19). We have designed a circular trajectory $p_d$ with radius equal to 1 m and such that the vehicle performs 3 revolutions per minute with respect to the $z$-axis.

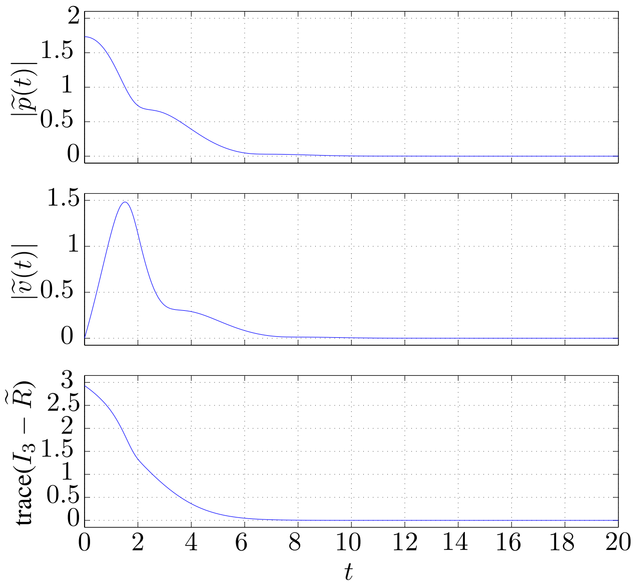


Fig. 1. Evolution of the tracking errors with time.

Figure 1 shows the evolution of position, velocity and attitude error in time, for an initial condition

$$p_0 = \begin{bmatrix}2\\1\\1\end{bmatrix},\ v_0 = \begin{bmatrix}0\\0.3\\0\end{bmatrix},\ y_{1_0} \approx \begin{bmatrix}0.59\\0.18\\0.79\end{bmatrix},\ y_{2_0} \approx \begin{bmatrix}0.09\\-0.07\\0.99\end{bmatrix},$$

$$R_0 = \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}.$$

It is possible to verify that the system behaves as desired since it converges to the reference trajectory. In the first two seconds, the vehicle exchanges its potential energy for kinetic energy, increasing its velocity as it falls towards the reference. Throughout the first few seconds, the attitude controller drives the orientation of the vehicle towards the desired orientation. As the thrust vector converges to the commanded thrust vector, the linear controller for the position subsystem is able to regulate the position tracking error as well as the velocity tracking error.[1]

## VI. Conclusions

In this paper, we developed the concept of centrally synergistic potential functions and showed that they can be applied to the stabilization of an attitude reference for a rigid-body vehicle. Moreover, we applied this controller to the stabilization of both position and orientation of a vectored thrust vehicle.

## References


[1] A. M. Bloch. *Nonholonomic Mechanics and Control*. Springer, April 2003.

[2] S. P. Bhat and D. S. Bernstein. A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, 39(1):63–70, January 2000.

[3] M.-D. Hua, T. Hamel, P. Morin, and C. Samson. Control of VTOL vehicles with thrust-tilting augmentation. *Automatica*, 52:1–7, February 2015.

[4] A. Roza and M. Maggiore. A Class of Position Controllers for Underactuated VTOL Vehicles. *IEEE Transactions on Automatic Control*, 59(9):2580–2585, September 2014.

[5] P. Casau, D. Cabecinhas, and C. Silvestre. Almost global stabilization of a vertical take-off and landing aircraft in hovered flight. In *American Control Conference (ACC)*, pages 3247–3252, June 2012.

[6] T. Lee. Robust Adaptive Attitude Tracking on With an Application to a Quadrotor UAV. *IEEE Transactions on Control Systems Technology*, 21(5):1924–1930, September 2013.

[7] O. E. Fjellstad and T.I. Fossen. Quaternion feedback regulation of underwater vehicles. In *Proceedings of the 3rd IEEE Conference on Control Applications*, pages 857–862 vol.2, August 1994.

[8] N.A. Chaturvedi, A.K. Sanyal, and N.H. McClamroch. Rigid-Body Attitude Control. *IEEE Control Systems*, 31(3):30–51, June 2011.

[9] C. G. Mayhew and A. R. Teel. On the topological structure of attraction basins for differential inclusions. *Systems & Control Letters*, 60(12):1045–1050, 2011.

[10] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[11] C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Quaternion-Based Hybrid Control for Robust Global Attitude Tracking. *IEEE Transactions on Automatic Control*, 56(11):2555–2566, November 2011.

[12] C. G. Mayhew and A. R. Teel. Synergistic Hybrid Feedback for Global Rigid-Body Attitude Tracking on SO(3). *IEEE Transactions on Automatic Control*, 58(11):2730–2742, November 2013.

[13] P. Casau, R. G. Sanfelice, R. Cunha, D. Cabecinhas, and C. Silvestre. Robust global trajectory tracking for a class of underactuated vehicles. *Automatica*, 58:90–98, August 2015.

[14] C. G. Mayhew and A. R. Teel. Global stabilization of spherical orientation by synergistic hybrid feedback with application to reduced-attitude tracking for rigid bodies. *Automatica*, 49(7):1945–1957, July 2013.

[15] P. Casau, C.G. Mayhew, R.G. Sanfelice, and C. Silvestre. Global exponential stabilization on the n-dimensional sphere. In *Proceedings of the 2015 American Control Conference (ACC)*, pages 3218–3223, July 2015.

[16] F. Bullo and A. D. Lewis. *Geometric Control of Mechanical Systems*, volume 49. Springer, 2004.

[17] Taeyoung Lee. Global Exponential Attitude Tracking Controls on SO(3). *IEEE Transactions on Automatic Control*, 60(10):2837–2842, October 2015.


## Appendix

**Proposition 12.** *For each $M_p < g$ and for each compact set $U \subset \mathbb{R}^6$, there exists $K \in \mathbb{R}^{3\times 6}$ satisfying*

$$(A + BK)^\top P + P(A + BK) + Q \preceq 0 \tag{31}$$

*for some positive definite symmetric matrices $P \in \mathbb{R}^{6\times 6}$ and $Q \in \mathbb{R}^{6\times 6}$, with*

$$A := \begin{bmatrix}0 & 0\\0 & I_3\end{bmatrix} \quad B := \begin{bmatrix}0\\I_3\end{bmatrix}$$

*such that*

$$\mathcal{A}_p := \{(\widetilde{p}, \widetilde{v}) \in \mathbb{R}^6 : \widetilde{p} = \widetilde{v} = 0\},$$

*is exponentially stable from $U$ for* (26) *and, for each solution $t \mapsto (\widetilde{p}, \widetilde{v})(t)$ to the closed-loop system, $|w(\widetilde{p}(t), \widetilde{v}(t))| \le g - M_p$ for all $t \ge 0$, where $w$ is given by* (24).

[1]The code for this simulation can be found at: `https://github.com/pcasau/ACC2016`.