Hybrid Systems: stability and control

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Abstract: Modeling issues for hybrid dynamical systems are discussed and fundamental stability analysis tools are summarized. These tools are useful for the development of hybrid control algorithms.

Key Words: Hybrid systems, stability, control

1 Introduction

Hybrid dynamical systems are those that exhibit both continuous and discontinuous state evolution. These systems have been studied from a theoretical point of view for multiple decades and cover a wide range of physical processes and engineering systems. The framework of hybrid systems can be used to model mechanical systems with impacts, like a ball bouncing on the ground, and networks of biological oscillators where oscillator states make jumps when other oscillator states pass certain thresholds. Hybrid systems also address a wide range of control systems, including sample-and-hold control systems, reset control systems like those that employ the so-called "Clegg integrator", control systems that involve hysteresis, etc.

2 Mathematical modeling

Several models and solution concepts for hybrid systems have been proposed during the last decades, see, for example, the work of Tavernini [19], Michel and Hu [15], Lygeros et al. [14], Aubin et al. [1], and van der Schaft and Schumacher [20]. In this paper, we work in the framework outlined in [6] (related to concurrent approach in [4]) and established in [8]. A hybrid system is specified by its state space, which we take to be an open set in a Euclidean space, mappings that specify the continuous and discontinuous evolution, and sets in the state space where the continuous and discontinuous evolution are possible. We typically use $O \subset \mathbb{R}^n$ to denote the state space, $C \subset O$ to denote the *flow set*, that is, the set where continuous evolution is possible, $D \subset O$ to denote the *jump set*, that is, the set where discontinuous evolution is possible, $f : C \to \mathbb{R}^n$ to denote the map that determines continuous evolution, which we refer to as the *flow map*, according to the differential equation $\dot{x} = f(x)$, and $g : D \to \mathbb{R}^n$ to denote the map that determines discontinuous evolution, which we refer to as the *jump map*, according to the "difference" equation $x^+ = g(x)$. A hybrid system is summarized by the data $\mathcal{H} := (O, f, C, g, D)$ and/or the equations [7]

$$\mathcal{H}: \qquad x \in O \qquad \left\{ \begin{array}{rcll} \dot{x} & = & f(x) & x \in C \\ x^+ & = & g(x) & x \in D\,. \end{array} \right.$$

Below, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{N} = \{0, 1, 2, ...\}$, $|\cdot|$ denotes the Euclidean vector norm, and given a nonempty set $\mathcal{A}$, $|\cdot|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$.

Example 1 (bouncing ball)
Consider a ball bouncing on the ground with vertical position x_1 and vertical velocity x_2 as shown in Figure 1. Let the state space be $O = \mathbb{R}^2$. The equations of motion for the ball are given by

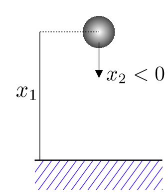

Figure 1: Bouncing ball.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\gamma,$$

when the ball is above the floor, where $\gamma > 0$ is the gravity constant. Then, the continuous dynamics of the ball are governed by the function f on the set C given by

$$f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix}, C := \{x \in O \mid x_1 > 0\}\ .$$

The ball bounces on the ground when its height is zero and its velocity indicates that it moves towards the floor. The sign convention in Figure 1 implies that the velocity at impacts has to be negative. The impacts are modeled by a

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jump map g that keeps the ball's height constant and decreases the magnitude of the velocity and reverses its direction. The discrete dynamics are given by

$$g(x) := \begin{bmatrix} x_1 \\ -\lambda x_2 \end{bmatrix}, \quad D := \{x \in O \mid x_1 = 0, x_2 < 0\},$$

where $\lambda \in [0, 1)$ is the restitution coefficient. The functions f and g and the sets C, D, and O defined above describe the bouncing ball system as a hybrid system $\mathcal{H}$. ◻

Example 2 (biological oscillator)
Consider the network of biological oscillators described by n fireflies where an oscillator is associated with each firefly's flashing mechanism [16]. Let $x \in \mathbb{R}^n$ be the state of the group of oscillators. The i-th component of the state x corresponds to the state of the i-th firefly which is reset to zero when it reaches the value one. The interaction between each firefly is modeled as coupling between the oscillators: when a firefly flashes, all the other firefly oscillators are shifted by a fixed amount of time $\varepsilon > 0$. During flows, the state of the ith biological oscillator, x_i, evolves in the interval $[0, 1]$ increasing its value according to the differential equation

$$\dot{x}_i = f_i(x_i)$$

where $f_i : [0, 1] \to \mathbb{R}_{>0}$ is continuous.
At jumps, oscillators with state equal to one reset to zero while all of the other state components change according to the rule

$$x_i^+ = \min\{1, x_i + \varepsilon\} \ .$$

Thus, the flow map is given by

$$f(x) := \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{bmatrix}$$

and the jump map can be constructed by first defining the function $g : \mathbb{R}^n \times \{1, \ldots, n\} \to \mathbb{R}^n$ as

$$g(x, i) := \begin{bmatrix} \min\{1, x_1 + \varepsilon\} \\ \vdots \\ \min\{1, x_{i-1} + \varepsilon\} \\ 0 \\ \min\{1, x_{i+1} + \varepsilon\} \\ \vdots \\ \min\{1, x_n + \varepsilon\} \end{bmatrix}$$

and then defining the set-valued jump map G as

$$G(x) := \bigcup_{\{i \in \{1,\ldots,n\} : x_i = 1\}} g(x, i) \ .$$

For state values x satisfying $x_i < 1$ for all $i \in \{1, \ldots, n\}$, this set-valued mapping is empty.
Since flows are allowed only when all state components are in the interval $[0, 1]$, then

$$C := \{x \in \mathbb{R}^n : x_i \in [0, 1] \quad \forall i \in \{1, \ldots, n\}\} =: [0, 1]^n \ .$$

Jumps are allowed when at least one state component is equal to one, then

$$D := \{x \in [0, 1]^n : \exists i \in \{1, \ldots, n\} : x_i = 1\} \ .$$

◻

Example 3 (sampled-data systems) Given a nonlinear control system

$$\dot{\xi} = \widetilde{f}(\xi, u), \quad \xi \in \mathbb{R}^p\,, \quad u \in \mathbb{R}^m\,, \tag{1}$$

let the control law to be implemented, by sample-and-hold with period T, be the state-feedback law $\kappa : \mathbb{R}^p \to \mathbb{R}^m$. A timer variable τ is used to keep track of when the sampling period has elapsed. The held value of the control is stored in the variable u. This variable becomes part of the state of the closed-loop system, which is given by

$$x := \begin{bmatrix} \xi \\ u \\ \tau \end{bmatrix}$$

with state space $\mathbb{R}^{p+m+1}$. Since the variable u is held constant during flows, the timer variable τ keeps track of elapsed time, and the state x evolves with the dynamics in (1), the flow map is given by

$$f(x) := \begin{bmatrix} \widetilde{f}(\xi, u) \\ 0 \\ 1 \end{bmatrix} .$$

Since, at the end of a sampling period, the variable u is updated by the feedback law κ (function of ξ), the timer should restart its count, and the state of the plant does not change, the jump map is given by

$$g(x) := \begin{bmatrix} \xi \\ \kappa(\xi) \\ 0 \end{bmatrix} .$$

The continuous evolution is allowed when the timer variable τ belongs to the interval $[0, T]$. In other words,

$$C := \left\{ \begin{bmatrix} \xi \\ u \\ \tau \end{bmatrix} \in \mathbb{R}^{p+m+1} : \tau \in [0, T] \right\} .$$

The jump evolution is allowed when the timer variable τ equals T, i.e.,

$$D := \left\{ \begin{bmatrix} \xi \\ u \\ \tau \end{bmatrix} \in \mathbb{R}^{p+m+1} : \tau = T \right\} .$$

◻

3 Solutions

The typical continuous and discontinuous behavior in a hybrid system produces motions (or solutions) that *flow* when the behavior is continuous and *jump* when the evolution is discontinuous. As in the case of classical continuous-time systems, we parametrize flows with the ordinary time variable $t \in \mathbb{R}_{\geq 0}$ while, as in the case of classical discrete-time systems, we parametrize jumps with the discrete time variable $j \in \mathbb{N}$. With this parametrization, a solution to a hybrid system $\mathcal{H}$ is given by a function, which we call a *hybrid arc*, defined on an extended time domain, which we call a *hybrid time domain*.

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if for every $(T, J) \in E$,

$$E \cap ([0, T] \times \{0, 1, ...J\}) = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. That is, for every element (T, J) of E, its truncation up to (T, J) can be written as the union of (well-defined) intervals in t indexed by the j variable. Then, a function $x : E \to \mathbb{R}^n$ is a *hybrid arc* if E is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is locally absolutely continuous.

Hybrid time domains are similar to the concept of hybrid time trajectories in [13],[14], and [1], and to the notion of time evolution in [20]. However, hybrid time domains give a more prominent role to the number of jumps j (cf. the definition of hybrid time set by Collins in [4]). On a hybrid time domain there is a natural ordering of points: we write $(t, j) \preceq (t', j')$ for $(t, j), (t', j') \in E$ if $t \leq t'$ and $j \leq j'$.

A hybrid arc x can be classified based on its hybrid time domain. We say it is *nontrivial* if $\operatorname{dom} x$ contains at least two points; *complete* if $\operatorname{dom} x$ is unbounded, i.e., if the projection of E onto $\mathbb{R}_{\geq 0}$ and/or $\mathbb{N}$ is unbounded; *Zeno* if it is complete and its domain is bounded in the t component.

A hybrid arc is a solution to the hybrid system only when the dynamics defined by the data (O, f, C, g, D) are satisfied during flows and jumps. More precisely, a hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in \overline{C} \cup D$, $x(t, j) \in O$ for all $(t, j) \in \operatorname{dom} x$, and

(S1) for all $j \in \mathbb{N}$ such that I^j has nonempty interior, where $I^j \times \{j\} = \operatorname{dom} x \cap ([0, \infty) \times \{j\})$,

$$x(t, j) \in C \text{ for all } t \in \operatorname{int} I^j, \quad \dot{x}(t, j) = f(x(t, j)) \text{ for almost all } t \in I^j;$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) = g(x(t, j)).$$

A solution to the bouncing ball system in Example 1 with its corresponding hybrid time domain is given in Figure 2. Note that solutions to hybrid systems $\mathcal{H}$ can be nonunique from points in the state space O that are both in C and D, and from which either behavior is possible, that is, both (S1) and (S2) hold.

4 Stability

Control engineers focused on inducing stability and the appropriate long-term trends in hybrid control systems will be interested in stability properties for hybrid systems and in tools for establishing asymptotic stability. Because hybrid

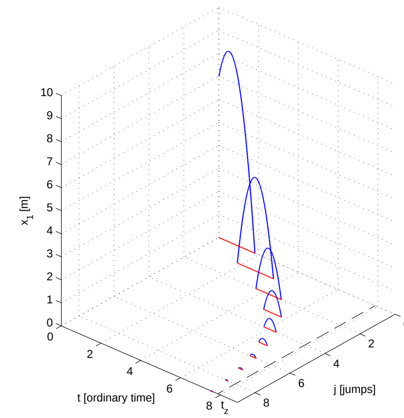

Figure 2: Height of the ball as a function of (t, j) on its hybrid time domain.

systems often contain logic variables, timers, etc. that don't necessarily converge to a point, it is natural in hybrid systems to consider asymptotic stability of compact sets rather than of simple equilibrium points. The notion of uniform global asymptotic stability is relevant here. It entails the following:

Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. The set $\mathcal{A}$ is said to be

- *uniformly globally stable* for $\mathcal{H}$ if there exists $\gamma \in \mathcal{K}_\infty$ such that any solution x to $\mathcal{H}$ satisfies $|x(t, j)|_\mathcal{A} \leq \gamma(|x(0, 0)|_\mathcal{A})$ for all $(t, j) \in \operatorname{dom} x$;
- *uniformly globally pre-attractive* for $\mathcal{H}$ if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution x to $\mathcal{H}$ with $|x(0, 0)|_\mathcal{A} \leq r$, $(t, j) \in \operatorname{dom} x$ and $t + j \geq T$ imply $|x(t, j)|_\mathcal{A} \leq \varepsilon$;
- *uniformly globally pre-asymptotically stable* for $\mathcal{H}$ if it is both uniformly globally stable and uniformly globally pre-attractive.

It turns out that uniform global pre-asymptotic stability is equivalent to the existence of $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and such that any solution x to $\mathcal{H}$ satisfies

$$|x(t, j)|_\mathcal{A} \leq \alpha_1 \left(\frac{\alpha_2(|x(0, 0)|_\mathcal{A})}{\exp{(t + j)}} \right) \qquad \forall (t, j) \in \operatorname{dom} x$$

where $|x|_\mathcal{A}$ denotes the distance of x to the set $\mathcal{A}$.

A sufficient condition for uniform global pre-asymptotic stability is given in the following Lyapunov conditions. Indeed, the set $\mathcal{A}$ is uniformly globally pre-asymptotically stable if $G(D) \subset \overline{C} \cup \overline{D}$ and there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$\begin{aligned} \alpha_1(|x|_\mathcal{A}) \leq V(x) &\leq \alpha_2(|x|_\mathcal{A}) && \forall x \in C \cup D \\ \sup_{f \in F(x)} \langle \nabla V(x), f \rangle &\leq -\rho\left(|x|_\mathcal{A}\right) && \forall x \in C \\ \sup_{g \in G(x)} V(g) - V(x) &\leq -\rho\left(|x|_\mathcal{A}\right) && \forall x \in D\,. \end{aligned}$$

Example 4 (bouncing ball, revisited) The reader is encouraged to verify for the bouncing ball that the function

$$V(x) := (1 + \theta \arctan(x_2)) \left(\frac{1}{2}x_2^2 + \gamma x_1\right)$$

is a Lyapunov function for the set $\mathcal{A}$ taken to be the origin when $\theta > 0$ is chosen appropriately. ◘

Example 5 (linear sampled-data systems) Consider sampled-data systems as discussed earlier, but with linear functions $\widetilde{f}(\xi, u) = A\xi + Bu$ and $\kappa(\xi) = K\xi$. This corresponds to a hybrid system with state x where $x_1 = \begin{bmatrix} \xi \\ u \end{bmatrix}$, $x_2 = \tau$,

$$f(x) = \begin{bmatrix} A_f x_1 \\ 1 \end{bmatrix}, \quad C = \{x \mid x_2 \in [0, T]\}$$
$$g(x) = \begin{bmatrix} A_g x_1 \\ 0 \end{bmatrix}, \quad D = \{x \mid x_2 = T\}$$

where

$$A_f := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \; A_g := \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}.$$

Define $H := \exp(A_f T)A_g$ and note that the matrix H indicates the evolution of the variable x_1 at sampling times just before jumps. In particular

$$x_1(t_{j+2}, j+1) = Hx_1(t_{j+1}, j) \quad \forall j \in \mathbb{N}.$$

Suppose there exists a positive definite symmetric matrix P such that $H^T PH - P$ is negative definite; equivalently, the eigenvalues of H all have magnitude less than one. Under this condition, the compact set $\mathcal{A} := \{x : x_1 = 0\ ,\ x_2 \in [0, T]\}$ is uniformly globally asymptotically stable. The reader is encouraged to verify that the function $V_2(x)$ is a Lyapunov function for the hybrid system with respect to this set $\mathcal{A}$ where $W(z) := z^T Pz$, $V_1(x) := W(\exp(A_f(T - x_2))x_1)$ and

$$V_2(x) := \exp(-\sigma x_2)V_1(x)$$

as long as $\sigma > 0$ is taken to be sufficiently small. ◘

5 Generalized Solutions

The presence of state perturbations in hybrid systems can dramatically change its behavior if the data (O, f, C, g, D) of $\mathcal{H}$ fail to satisfy certain properties. This can occur even if the magnitude of the perturbation is arbitrarily small and the functions f and g have nice regularity properties, e.g. they are smooth. This section addresses this issue and motivates the concept of *generalized solutions to hybrid systems*.

5.1 Hybrid systems with state perturbations

A hybrid system $\mathcal{H} = (O, f, C, g, D)$ with a state perturbation e is denoted by $\mathcal{H}_e$ and can be written in the suggestive form:

$$\mathcal{H}_e :\ x + e \in O \left\{ \begin{array}{rcll} \dot{x} & = & f(x+e) & x+e \in C \\ x^+ & = & g(x+e) & x+e \in D\,. \end{array} \right.$$

Solutions to $\mathcal{H}_e$ are defined for only a class of state perturbations. The hybrid arc e is an *admissible state perturbation* if $\operatorname{dom} e$ is a hybrid time domain and the function $t \to e(t, j)$ is measurable on $\operatorname{dom} e \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}_e$ with admissible state perturbation e if $\operatorname{dom} x = \operatorname{dom} e$, $x(0,0) + e(0,0) \in \overline{C} \cup D$, $x(t, j) + e(t, j) \in O$ for all $(t, j) \in \operatorname{dom} x$, and

(S1$_e$) for all $j \in \mathbb{N}$ such that I^j has nonempty interior, where $I^j \times \{j\} := \operatorname{dom} x \cap ([0, +\infty) \times \{j\})$,

$$x(t, j) + e(t, j) \in C \text{ for all } t \in \operatorname{int} I^j,$$
$$\dot{x}(t, j) = f(x(t, j) + e(t, j)) \text{ for almost all } t \in I^j;$$

(S2$_e$) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) + e(t, j) \in D, \;\; x(t, j+1) = g(x(t, j) + e(t, j)).$$

The following example illustrates one possible effect of state perturbations in hybrid systems.

Example 6 (rotate and converge) Consider the hybrid system $\mathcal{H}$ with data given by

$$O = \mathbb{R}^2, \qquad f(x) := \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \qquad C := \mathbb{R}^2 \setminus D$$
$$g(x) = 0, \qquad D := \left\{ x \in \mathbb{R}^2 \mid x_1 > 0, x_2 = 0 \right\}.$$

Solutions to $\mathcal{H}$ converge to the origin in at most one jump. In fact, solutions starting from $\xi \in C \cup D$, $\xi \neq 0$, flow until they reach the set D, from where a jump to the origin follows. After that jump, solutions flow for all time at the origin. The solution starting from $\xi = 0 \in C \cup D$ flows for all time.

Solutions to $\mathcal{H}$ with state perturbation, that is, solutions to $\mathcal{H}_e$ with an admissible state perturbation e have a total different behavior. Solutions to $\mathcal{H}_e$ from points $\xi \in C \cup D$, $\xi \neq 0$, can miss the jump at the set D and flow for all time. Let $x : [0, \infty) \times \{0\} \to \mathbb{R}^2$ be such that $\dot{x}(t, 0) = f(x(t, 0))$, and let $t_i \geq 0$, $i \in \mathbb{N}$, be such that $x(t_i, 0) \in D$ for each $i \in \mathbb{N}$. Define the admissible state perturbation $e : [0, \infty) \times \{0\} \to \mathbb{R}^2$ to be $e(t, 0) = 0$ for all $t \neq t_i$, $i \in \mathbb{N}$, and $e(t_i, 0) = [0\ \varepsilon_2]^T$, for some $\varepsilon_2 > 0$. It follows that for all $(t, j) \in \operatorname{dom} x = [0, \infty) \times \{0\}$

$$x(t, j) + e(t, j) \in C \text{ for all } t \in \mathbb{R},$$
$$\dot{x}(t, j) = f(x(t, j) + e(t, j)) \text{ for almost all } t \in \mathbb{R}\,.$$

Then, x is a solution to $\mathcal{H}_e$ with admissible state perturbation e for any $\varepsilon_2 > 0$.

Solutions to $\mathcal{H}_e$ from points $\xi \in C \cup D$, $\xi = 0$ can jump for ever, for a particular admissible noise. Let $x : \{0\} \times \mathbb{N} \to \mathbb{R}^2$ be such that $x(0, j+1) = g(x(0, j)) = 0$ for all $j \in \mathbb{N}$. Define the admissible state perturbation $e : \{0\} \times \mathbb{N} \to \mathbb{R}^2$ to be $e(0, j) = [\varepsilon_1\ 0]^T$ for all $j \in \mathbb{N}$ for some $\varepsilon_1 > 0$. It follows that for all $(t, j) \in \operatorname{dom} x = \{0\} \times \mathbb{N}$

$$x(t, j) + e(t, j) \in D, \;\; x(t, j+1) = g(x(t, j) + e(t, j)).$$

Then, x is a solution to $\mathcal{H}_e$ with admissible state perturbation e for any $\varepsilon_1 > 0$. Comparing this solution with the solution without noise, even though their value is zero for all (t,j) in their domain, their hybrid time domains are totally different. ◻

Example 7 (discontinuous jump map) Consider the hybrid system $\mathcal{H}$ with data given by

$$O = \mathbb{R}^2, \qquad f(x) \equiv \emptyset, \qquad C := \emptyset \tag{2}$$

$$g(x) = \begin{cases} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} & x_1 \neq 0 \\ 0 & x_1 = 0 \end{cases}, \qquad D := \mathbb{R}^2 \,. \tag{3}$$

Solutions to $\mathcal{H}$ always jump for every point in $\mathbb{R}^2$. From $\xi = [\xi_1\ \xi_2]^T \in \mathbb{R}^2$ with $\xi_1 \neq 0$, they first jump to $\xi_1 = 0$ from where they jump to the origin and stay there jumping for ever. From every $\xi = [\xi_1\ \xi_2]^T \in \mathbb{R}^2$ with $\xi_1 = 0$, solutions reach the origin in one jump and stay there jumping for ever.

Solutions with state perturbation can fail to converge to the origin. Let $x : \{0\} \times \mathbb{N} \to \mathbb{R}^2$ be such that $x(0,0) = [\xi_1\ \xi_2]^T \in \mathbb{R}^2$, $\xi_1, \xi_2 \neq 0$, $x(0,j) = [0\ \xi_2]^T$ for all $j > 0$, $j \in \mathbb{N}$. Define the admissible state perturbation $e : \{0\} \times \mathbb{N} \to \mathbb{R}^2$ to be $e(0,0) = 0$, $e(0,j) = [\varepsilon_1\ 0]^T$ for all $j > 0$, $j \in \mathbb{N}$ for some $\varepsilon_1 > 0$. (Similar construction is possible for points ξ with $\xi_1 = 0$.) It follows that for all $(t,j) \in \operatorname{dom} x = \{0\} \times \mathbb{N}$

$$x(t,j) + e(t,j) \in D, \;\; x(t,j+1) = g(x(t,j) + e(t,j)).$$

Then, x is a solution to $\mathcal{H}_e$ with admissible state perturbation e for any $\varepsilon_1 > 0$. The solution x does not converge to the origin. ◻

Note that the solutions in the examples above exhibit the same behavior for arbitrarily small state perturbation, even in the limit when its magnitude goes to zero.

5.2 Hermes and Krasovskii solutions

The examples in Section 5.1 suggest that solutions obtained by taking the limit of sequences of solutions with vanishing state perturbations in hybrid systems can lead to solutions that differ from the nominal solutions. These solutions are called *Hermes solutions*.

Before defining Hermes solutions, a metric that measures the distance between hybrid arcs is introduced. Such metric is needed since hybrid arcs do not necessarily share the same hybrid time domain. Given $\tau \geq 0$ and $\varepsilon > 0$, we say that two hybrid arcs x_1 and x_2 are *(τ,ε)-close* if

(a) for all $(t,j) \in \operatorname{dom} x_1$ with $t + j \leq \tau$ there exists s such that $(s,j) \in \operatorname{dom} x_2$, $|t - s| < \varepsilon$, and

$$|x_1(t,j) - x_2(s,j)| < \varepsilon,$$

(b) for all $(t,j) \in \operatorname{dom} x_2$ with $t + j \leq \tau$ there exists s such that $(s,j) \in \operatorname{dom} x_1$, $|t - s| < \varepsilon$, and

$$|x_2(t,j) - x_1(s,j)| < \epsilon.$$

A hybrid arc x is a *Hermes solution* to $\mathcal{H}$ if the restriction of x, denoted x', to each compact subset of $\operatorname{dom} x$ that is a hybrid time domain is such that there exist a sequence $\{x'_i\}_{i=1}^{\infty}$ of hybrid arcs and $\{e_i\}_{i=1}^{\infty}$ a sequence of admissible state perturbations such that

- x'_i is a solution to $\mathcal{H}_e$ with state perturbation e_i for each $i \in \mathbb{N}$;
- for each $\varepsilon > 0$ there exists i_0 such that for all $i > i_0$, x'_i and x' are (τ,ε)-close, where $\tau = T + J$ and (T,J) is the supremum of $\operatorname{dom} x'$;
- the sequence of $\sup_{(t,j)\in \operatorname{dom} e_i} |e_i(t,j)|$ converges to 0.

Hermes solutions to $\mathcal{H} = (O,f,C,g,D)$ are captured by the solutions to the hybrid system resulting from *regularizing* $\mathcal{H}$ via the following closure operation, referred to as *Krasovskii regularization* , given by [17]

$$\widehat{C} := \overline{C} \cap O\,, \;\; \widehat{D} := \overline{D} \cap O$$

$$\forall x \in \widehat{C} \;\;\; \widehat{f}(x) := \bigcap_{\delta > 0} \overline{\operatorname{con}} f((x + \delta\mathbb{B}) \cap C),$$

$$\forall x \in \widehat{D} \;\;\; \widehat{g}(x) := \bigcap_{\delta > 0} \overline{g((x + \delta\mathbb{B}) \cap D)}.$$

Note that at points in C, respectively D, where f, respectively g, is continuous, $\widehat{f}(x) = f(x)$, respectively $\widehat{g}(x) = g(x)$. Moreover, when C, respectively D, is closed relative to O, the regularization does not change C, respectively D. (A set S, subset of an open set $O \in \mathbb{R}^n$, is closed relative to O if $S = \overline{S} \cap O$.)

A hybrid arc x is a *Krasovskii solution* to $\mathcal{H} = (O,f,C,g,D)$ if (S1) and (S2) hold with $\dot{x} = f(x)$ and $x^+ = g(x)$ replaced by $\dot{x} \in \widehat{f}(x)$ and $x^+ \in \widehat{g}(x)$, respectively, and C and D replaced by $\widehat{C}$ and $\widehat{D}$, respectively.

Under locally boundedness assumptions on f and g, it follows that

A hybrid arc x is a Hermes solution to $\mathcal{H}$ if and only if is a Krasovskii solution to $\mathcal{H}$.

This implies that every limiting solution obtained from any sequence of solutions to $\mathcal{H}_e$ with admissible state perturbation converging to zero is a Krasovskii solution to $\mathcal{H}$, and that every Krasovskii solution to $\mathcal{H}$ can be reproduced with arbitrary precision by solutions to $\mathcal{H}_e$ with admissible state perturbation.

Example 8 (bouncing ball, revisited) The regularization of the bouncing ball system in Example 1 is given by

$$\begin{aligned} &\widehat{C} = \overline{C} = \{x \in O \mid x_1 \geq 0\} \\ &\widehat{D} = \overline{D} = \{x \in O \mid x_1 = 0, x_2 \leq 0\}\,, \\ &\qquad \forall x \in \widehat{C} \;\; \widehat{f}(x) = f(x), \;\; \forall x \in \widehat{D} \;\; \widehat{g}(x) = g(x) \end{aligned}$$

The only solution added by the regularization is a solution starting from $x = 0$. (Note that from the origin there is no solution to the bouncing ball system in Example 1.) This solution has a hybrid time domain given by $\{0\} \times \mathbb{N}$. This corresponds to a special case of Zeno solution called *discrete*. ◘

Example 9 (rotate and converge, revisited) The regularization of the hybrid system in Example 6 is given by

$$\begin{aligned}
&\widehat{C} = \overline{C} = \mathbb{R}^2 \\
&\widehat{D} = \overline{D} = \{x \in O \mid x_1 \geq 0, x_2 = 0\}, \\
&\qquad \forall x \in \widehat{C} \;\; \widehat{f}(x) = f(x), \;\; \forall x \in \widehat{D} \;\; \widehat{g}(x) = g(x)
\end{aligned}$$

The solution to the system in Example 6 that always flows is a Hermes solution as it can be obtained as the limiting operation of sequences of solutions with vanishing state perturbation. Moreover, since the regularized hybrid system has flow set $C = \mathbb{R}^2$, it is also a Krasovskii solution. Similarly for the solution to the system in Example 6 that always jumps – it is both a Hermes and Krasovskii solution. ◘

Example 10 (discontinuous jump map, revisited) The regularization of the hybrid system in Example 7 is given by

$$\begin{aligned}
&\widehat{C} = \overline{C} = \emptyset, \; \widehat{D} = \overline{D} = \mathbb{R}^2 \\
&\forall x \in \widehat{C} \quad \widehat{f}(x) = \emptyset, \\
&\forall x \in \widehat{D} \quad \widehat{g}(x) = \begin{cases} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} & x_1 \neq 0 \\ \left\{0, \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right\} & x_1 = 0 \end{cases}
\end{aligned}$$

The solution to the system in Example 7 that always jumps and stays away from the origin is a Hermes solution as it can be obtained as the limiting operation of sequences of solutions with vanishing state perturbation ε_1. Moreover, since the regularized hybrid system has jump map including $[0\; x_2]^T$ when $x_1 = 0$, then it is also a Krasovskii solution – the one corresponding to always choosing $[0\; x_2]^T$ at jumps. ◘

The results above generalize, to the hybrid setting, a result for differential equations initially reported by Hermes in [10] and expanded upon by Hàjek in [9]. Following the lines of [5], a similar result but for the state feedback case also holds for hybrid systems, see [17].

6 Basic Conditions and their consequences

Note that $\widehat{\mathcal{H}}$ has set-valued dynamics due to the regularization procedure. Hybrid systems with set-valued dynamics appear also when perturbations are explicitly modeled and, for the particular case of the jump map, when multiple possibilities at jumps are possible, like in the case of hybrid control with logic variables. From now on, we consider the general case of a hybrid system $\mathcal{H}$ with data (O, F, C, G, D), where F and G are set-valued mappings, and can be written in the compact form

$$\mathcal{H}: \qquad x \in O \qquad \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D\,. \end{cases}$$

Below, a set-valued mapping $\phi : S \rightrightarrows \mathbb{R}^n$, where $S \subset O$, is outer semicontinuous relative to S if for any $x \in S$ and any sequence $\{x_i\}_{i=1}^{\infty}$ with $x_i \in S$, $\lim_{i\to\infty} x_i = x$ and any sequence $\{y_i\}_{i=1}^{\infty}$ with $y_i \in \phi(x_i)$ and $\lim_{i\to\infty} y_i = y$ we have $y \in \phi(x)$.
We say that $\mathcal{H}$ satisfies the *hybrid basic conditions* if

(A0) $O \subset \mathbb{R}^n$ is an open set;

(A1) C and D are sets closed relative to O;

(A2) $F : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$;

(A3) $G : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $G(x)$ is nonempty and $G(x) \subset O$ for all $x \in D$.

The hybrid basic conditions are automatically satisfied by the Krasovskii regularization of a hybrid systems in Section 5.2.
A hybrid arc x is a solution to $\mathcal{H} = (O, F, C, G, D)$ if it satisfies (S1) and (S2) with $\dot{x} = f(x)$ and $x^+ = g(x)$ replaced by $\dot{x} \in F(x)$ and $x^+ \in G(x)$, respectively. (When $\mathcal{H}$ is regular, that is, satisfies the hybrid basic conditions, solutions to $\mathcal{H}$ as just defined coincide with Krasovskii solutions.)
When $\mathcal{H}$ satisfies the hybrid basic conditions, then the following sequential compactness of solutions property holds [8]:

For every (locally eventually) bounded with respect to O sequence of solutions $x_i : \operatorname{dom} x_i \to \mathbb{R}^n$ to $\mathcal{H}$, there exists a subsequence graphically converging to a solution to $\mathcal{H}$.

By a locally eventually bounded with respect to O sequence of solutions x_i we mean that for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset O$ such that for all $i > i_0$, all $(t, j) \in \operatorname{dom} x_i$ with $t + j < m$, $x_i(t, j) \in K$.
When $\mathcal{H}$ satisfies the hybrid basic conditions and is such that all solutions from some compact set $K \subset O$ are complete, then the following outer-semicontinuous dependence of solutions holds [8]:

For any $\varepsilon > 0$ and $\tau \in \mathbb{R}_{\geq 0}$ there exists $\delta > 0$ such that: for any solution $x' \in \mathcal{S}_{\mathcal{H}}(K + \delta\mathbb{B})$ there exists a solution x to $\mathcal{H}$ with $x(0,0) \in K$ such that x' and x are (τ, ε)-close.

7 Stability revisited

Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. The set $\mathcal{A}$ is said to be

- *stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}\, x$;

- *globally pre-attractive* if each solution x to $\mathcal{H}$ is bounded or complete, and in the latter case satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

- *globally pre-asymptotically stable* if it is both stable and globally pre-attractive.

For a hybrid systems satisfying the hybrid basic conditions with state space $O = \mathbb{R}^n$ and a compact set $\mathcal{A} \subset \mathbb{R}^n$, the following statements are equivalent:

1. The set $\mathcal{A}$ is globally pre-asymptotically stable.
2. The set $\mathcal{A}$ is uniformly globally pre-asymptotically stable.

Under the same assumptions above, a Lyapunov characterization of global pre-asymptotic stability holds as in the case of uniform pre-asymptotic stability in Section 4. The following statements are equivalent:

1. The set $\mathcal{A}$ is globally pre-asymptotically stable.
2. (a) $G(\mathcal{A}) \subset \mathcal{A}$;
   (b) There exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\begin{array}{rclll} \alpha_1(|x|_{\mathcal{A}}) \leq & V(x) & \leq & \alpha_2(|x|_{\mathcal{A}}) & x \in C \cup D \\ \max\limits_{f\in F(x)} & \langle \nabla V(x), f \rangle & < & 0 & \forall x \in C \backslash \mathcal{A} \\ \max\limits_{g\in G(x)} & V(g) - V(x) & < & 0 & \forall x \in D \backslash \mathcal{A}\,; \end{array}$$

3. For each $\lambda > 0$ there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\begin{array}{rcll} \alpha_1(|x|_{\mathcal{A}}) \leq V(x) & \leq & \alpha_2(|x|_{\mathcal{A}}) & x \in \mathbb{R}^n \\ \max\limits_{f\in F(x)} \langle \nabla V(x), f \rangle & \leq & -\lambda V(x) & \forall x \in C \\ \max\limits_{g\in G(x)} V(g) & \leq & \exp(-\lambda) V(x) & \forall x \in D\,. \end{array}$$

Implication from item 1 to item 3 constitutes a "converse Lyapunov theorem" and can be used to establish robustness of asymptotic stability to various types of perturbations, including slowly-varying, weakly jumping parameters, temporal regularization, small average-dwell time jumping, etc.
General statements about the existence of smooth Lyapunov functions for hybrid systems and the ensuing robustness can be found in [3] and [2].

8 Invariance principles

Invariance principles for hybrid systems that parallel those for continuous and discrete-time systems proposed by LaSalle [11, 12] are introduced in this section. The concept of invariance we use involves both forward and backward invariance. The prefix "weak" is used to indicate that the invariance notion involves only a particular solution to satisfy the invariance property.
Below, we denote the range of the solution x by $\operatorname{rge} x$, i.e. $\operatorname{rge} x = x(\operatorname{dom} x)$.
Given a hybrid system $\mathcal{H}$, a set $M \subset O$ is *weakly invariant* if it is both [18]:

- Weakly forward invariant: *if for each point $\xi \in M$ there exists at least one complete solution x to $\mathcal{H}$ starting from ξ and satisfying $\operatorname{rge} x \subset M$.*
- Weakly backward invariant: *if for each point $\xi' \in M$ and every positive number N there exists a point ξ from which there exists at least one solution x to $\mathcal{H}$ starting from ξ that is equal to ξ' for some $(t^*, j^*) \in \operatorname{dom} x$ with the property that $t^* + j^* \geq N$, and such that $x(t, j) \subset M$ for all $(t, j) \succeq (t^*, j^*)$.*

For a hybrid system $\mathcal{H}$ satisfying the hybrid basic conditions, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, and a nonempty set $U \subset \mathbb{R}^n$ such that for all $z \in U$

$$u_C(z) \leq 0 \quad \text{and } u_D(z) \leq 0 \tag{4}$$

where

$$u_C(x) \;:=\; \begin{cases} \max\limits_{f\in F(x)} \langle \nabla V(x), f \rangle & x \in C \\ -\infty & \text{otherwise} \end{cases}$$
$$u_D(x) \;:=\; \begin{cases} \max\limits_{g\in G(x)} \{V(g) - V(x)\} & x \in D \\ -\infty & \text{otherwise,} \end{cases}$$

every solution x to $\mathcal{H}$ that is complete, bounded, and $\operatorname{rge} x \subset U$ is such that converges to the largest weakly invariant set contained in

$$\left[u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G(u_D^{-1}(0))\right)\right] \cap V^{-1}(r) \cap U \tag{5}$$

for some constant $r \in V(U)$.

Example 11 (bouncing ball, revisited) For the bouncing ball system with regular data in Example 8, consider the continuously differentiable function $V(x) = \frac{1}{2}x_2^2 + \gamma x_1$. It follows that

$$\langle \nabla V(x), f(x) \rangle = 0 \quad \text{for all } x \in C\,. \tag{6}$$

and that

$$V(g(x)) - V(x) = -\frac{1}{2}(1 - \lambda^2)x_2^2 - \gamma x_1 \leq 0$$

for all $x \in D$. Then

$$u_C(x) \;:=\; \begin{cases} 0 & x \in C \\ -\infty & \text{otherwise} \end{cases}$$
$$u_D(x) \;:=\; \begin{cases} -\frac{1}{2}(1 - \lambda^2)x_2^2 - \gamma x_1 & x \in D \\ -\infty & \text{otherwise,} \end{cases}$$

Since u_C and u_D are never positive, for $U = \mathbb{R}^2$ we have

$$u_C(z) \leq 0 \quad \text{and } u_D(z) \leq 0\,. \tag{7}$$

Therefore, every precompact solution to the bouncing ball system converges to the largest weakly invariant set in (5) where

$$u_C^{-1}(0) = C, \;\; g(u_D^{-1}(0)) = u_D^{-1}(0) = \{x \in O \mid x = 0\}\,.$$

Then, it the set (5) is given by

$$\{x \in O \mid x_1 \geq 0\} \cap V^{-1}(r) \cap U\,.$$

Suppose $r > 0$ and consider solutions starting in this set. There exists a finite $(t, 0)$ at which $x_1(t, 0) = 0$ and then a jump occurs. Since $u_D(x)$ is strictly negative away from the origin, V needs to decrease after the jump. This shows that it is impossible for the a invariant set to exists in (5) with positive r. Then, the only invariant set in (5) is for $r = 0$. This set is the origin. Then precompact solutions converge to the origin. ◻

The following special cases to the invariance principle above hold when

(a) x is Zeno: then, for some $r \in V(U)$, it approaches the largest weakly invariant subset of

$$V^{-1}(r) \cap U \cap u_D^{-1}(0) \cap G(u_D^{-1}(0)); \tag{8}$$

(b) if x is s.t., for some $\gamma > 0$, $J \in \mathbb{N}$, and all $j \geq J$, $t_{j+1} - t_j \geq \gamma$ (i.e. the elapsed time between jumps is eventually bounded below by γ): then, for some $r \in V(U)$, x approaches the largest weakly invariant subset of

$$V^{-1}(r) \cap U \cap u_C^{-1}(0). \tag{9}$$

The invariance principle above can be generalized to the case where the functions u_C and u_D are not necessarily computed from the function V. Given any functions $u_c, u_d : O \to [-\infty, \infty]$ such that for any solution $\xi \in \mathcal{S}_\mathcal{H}$ with $\operatorname{rge} \xi \subset U$,

$$u_c(\xi(t,j)) \leq 0, \;\; u_d(\xi(t,j)) \leq 0$$

for all $(t,j) \in \operatorname{dom} \xi$ and

$$\begin{aligned} &V(x(\overline{t}, \overline{j})) - V(x(\underline{t}, \underline{j})) \leq \\ &\qquad \int_{\underline{t}}^{\overline{t}} u_c(x(t, j(t)))\, dt + \sum_{j=\underline{j}+1}^{\overline{j}} u_d(x(t(j), j-1)) \end{aligned}$$

holds for any $(t,j), (t',j') \in \operatorname{dom} \xi$ such that $(t,j) \preceq (t',j')$, Then every bounded and complete solution $x \in \mathcal{S}_\mathcal{H}$ with $\operatorname{rge} x \subset U$ approaches the largest weakly invariant subset of

$$V^{-1}(r) \cap U \cap \left(\overline{u_c^{-1}(0)} \cup \left(u_d^{-1}(0) \cap G(u_d^{-1}(0)) \right) \right)$$

for some $r \in V(U)$.

Special cases as the ones given above also hold for general functions u_c, u_d.

Stability corollaries follow from these invariance principles. The following result that parallels Lyapunov stability theorem can be shown for the construction of the functions u_C and u_D above.

Let $\mathcal{A} \subset O$ be compact, $U \subset O$ be a neighborhood of $\mathcal{A}$, $V : O \to \mathbb{R}$ continuously differentiable and positive definite on $C \cup D$ with respect to $\mathcal{A}$, and u_C and u_D satisfy

$$u_C(z) \leq 0, u_D(z) \leq 0$$

for all $z \in U$. Then $\mathcal{A}$ is pre-stable. Suppose additionally that

$$u_C(z) < 0 \text{ and } u_D(z) < 0 \text{ for all } z \in U \setminus \mathcal{A}\,.$$

Then $\mathcal{A}$ is pre-attractive, and hence pre-asymptotically stable.

The key to these results is the said outer-semicontinuity property of solutions. More general invariance principles, connections to observability and detectability, and other consequences of asymptotic stability for hybrid systems can be found in [18].

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