A Hybrid Consensus Protocol for Pointwise Exponential Stability with Intermittent Information

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Abstract: We propose a solution to the problem of achieving consensus of the state of multiple systems connected over a network (over a directed graph) in which communication events are triggered stochastically. Our solution consists of a protocol design that, using intermittent information obtained over the network, asymptotically drive the values of their states to agreement, with stability, globally and with robustness to perturbations. More precisely, we propose a protocol with hybrid dynamics, namely, an algorithm with variables that jump at the communication events and evolve continuously in between such events. We design the protocols by recasting the consensus problem as a set stabilization problem and applying Lyapunov stability tools for hybrid systems. We provide sufficient conditions for exponential stability of the consensus set. Furthermore, we show that under additional conditions this set is also partially pointwise globally exponentially stable. Robustness of consensus to certain classes of perturbations is also established. Numerical examples confirm the main results.

Keywords: Consensus Protocol, Set Stability of Hybrid Systems, Networked Systems

1. INTRODUCTION

The topic of consensus has gained massive traction in recent years due to the wide range of applications science and engineering. A challenge to the design of *consensus protocols* is when information is only available at intermittent time instances. Different from consensus of continuous and discrete-time systems, which is throughly understood in Olfati-Saber and Murray (2004) and Cortés (2008), the introduction of a sampling period or impulsive information transfer for first and higher order systems has been studied in Jie and Zhong (2014); Liu et al. (2010); Guan et al. (2012); Hu et al. (2013); Wen et al. (2013). For such cases, the application of systems theory tools like Lyapunov functions, contraction theory, and incremental input-to-state stability have been proposed. Notably, recent research efforts on sample-data systems and event triggered control for the stabilization of sets provide results that can become useful for consensus, though some of the assumptions need to be carefully fit to the consensus problem under intermittent communication networks.

This article deals with the problem of consensus of first-order integrator systems communicating at stochastically determined time instances over a network. The consensus problem studied here consists of designing a protocol guaranteeing that the state of each agent converges to a common value by only using intermittent information from their neighbors. To solve this problem, we design hybrid state-feedback protocols that undergo an instantaneous change in their states when new information is available, and evolves continuously between such events. Due to the combination of continuous and impulsive dynamics, we use hybrid systems theory to model the interconnected systems, the controller, and the network topologies as well as to design the protocols, for which, we apply a Lyapunov theorem for asymptotic stability of sets for hybrid systems. Aside from asymptotic stability, we specify the point to which the consensus states converge to and, when the communications graph is strongly connected and weight balanced, write it as a function of their initial conditions. We show that a diagonal-like set is, in fact, partially pointwise globally exponentially stable, which is a stronger notion than typical notions of asymptotic stability due to the additional requirement that each point in the set is Lyapunov stable; see e.g. Goebel (2010) and Bhat and Bernstein (2003) for similar notions. Furthermore, we show that the consensus condition is robust to a class of perturbations on the information. Finally, we give some brief insight into modeling and an asymptotic stability result for the case when information may arrive at asynchronous events for each agent.

The remainder of this paper is organized as follows. Section 2 gives some preliminary background on graph theory and hybrid systems. Section 3 introduces the consensus problem, impulsive network model, and the control structure. In Section 4, we define a hybrid protocol and give the main results. The preliminary results for asynchronous update times are in Section 5.

Notation The set of natural numbers is denoted as $\mathbb{N}$, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. The set $\text{eig}(A)$ contains the eigenvalues of A. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$, notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t \geq 0} |m(t)|$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \ \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \ \lambda \in \text{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$. $\mathbf{1}_N$ is a vector of N ones.

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2. PRELIMINARIES ON GRAPH THEORY AND HYBRID SYSTEMS

2.1 Preliminaries on graph theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \ldots, N\}$ and the edges are pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two different nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can send information to agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} = (g_{ik}) \in \mathbb{R}^{N\times N}$, where $g_{ik} = 1$ if $(i, k) \in \mathcal{E}$, and $g_{ik} = 0$ otherwise. The in-degree and out-degree of agent i are defined by $d^{in}(i) = \sum_{k=1}^{N} g_{ki}$ and $d^{out}(i) = \sum_{k=1}^{N} g_{ik}$. The largest (smallest) in-degree in the digraph is given by $\overline{d} = \max_{i\in\mathcal{V}} d^{in}(i)$ ($\underline{d} = \min_{i\in\mathcal{V}} d^{in}(i)$). The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $\mathcal{D}_{ii} = d^{in}(i)$ for all $i \in \mathcal{V}$. The Laplacian matrix of the digraph Γ, denoted by $\mathcal{L}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{G}$. The Laplacian has the property that $\mathcal{L}\mathbf{1}_N = 0$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (k, i) \in \mathcal{E}\}$.

In this article, we will make varying assumptions on the complexity of the underlying graph structure corresponding to the network. For self-containedness, we summarize the needed notions and results from the literature.

Definition 2.1. A directed graph is said to be

- *weight balanced* if, at each node $i \in \mathcal{V}$, the out-degree and in-degree are equal; i.e., for each $i \in \mathcal{V}$, $d^{out}(i) = d^{in}(i)$;
- *complete* if every pair of distinct vertices is connected by a unique edge; that is $g_{ik} = 1$ for each $i, k \in \mathcal{V}$, $i \neq k$;
- *strongly connected* if and only if any two distinct nodes of the graph can be connected via a path that traverses the directed edges of the digraph. □

Lemma 2.2. ((Olfati-Saber and Murray, 2004, Theorem 6),(Fax and Murray, 2004, Propositions 1, 3, and 4)) For an undirected graph, $\mathcal{L}$ is symmetric and positive semidefinite and each eigenvalue of $\mathcal{L}$ is real. For a directed graph, zero is a simple eigenvalue of $\mathcal{L}$ if the directed graph is strongly connected.

Lemma 2.3. (Godsil and Royle (2013)) Consider an $n \times n$ symmetric matrix $A = \{a_{ik}\}$ satisfying $\sum_{i=1}^{n} a_{ik} = 0$ for each $k \in \{1, 2, \ldots, n\}$. The following statements hold:

(i) There exists an orthogonal matrix U such that

$$U^\top A U = \begin{bmatrix} 0 & 0 \\ 0 & \star \end{bmatrix} \tag{1}$$

where $\star$ represents any nonsingular matrix with an appropriate dimension and 0 represents any zero matrix with an appropriate dimension.

(ii) The matrix A has a zero eigenvalue with eigenvector $\mathbf{1}_n \in \mathbb{R}^n$.

2.2 Preliminary on Hybrid Systems

A hybrid system $\mathcal{H}$ has data (C, f, D, G) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \qquad z \in C, \\ z^+ &\in G(z) \qquad z \in D, \end{aligned} \tag{2}$$

where $z \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The set-valued map G defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution[1] ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\operatorname{dom}\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom}\phi$, the set $\operatorname{dom}\phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\cup_{j=0}^{J}(I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ.

In this paper, we consider the following stability notions.

Definition 2.4. (global exponential stability) Let a hybrid system $\mathcal{H}$ be defined on $\mathbb{R}^n$. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be *globally exponentially stable* (GES) for $\mathcal{H}$ if there exist $\kappa, \alpha > 0$ such that every maximal solution ϕ to $\mathcal{H}$ is complete and satisfies

$$|\phi(t, j)|_\mathcal{A} \leq \kappa e^{-\alpha(t+j)} |\phi(0, 0)|_\mathcal{A}$$

for each $(t, j) \in \operatorname{dom}\phi$. □

Definition 2.5. (partial pointwise global exponential stability) Consider a hybrid system $\mathcal{H}$ with state $z = (p, q) \in \mathbb{R}^n$. The closed set $\mathcal{A} \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$ where $r \in \mathbb{N}$ and $0 < r \leq n$ is *partially pointwise global exponentially stable* with respect to the state component p for $\mathcal{H}$ if

1) every maximal solution ϕ to $\mathcal{H}$ is complete and has a limit belonging to $\mathcal{A}$;
2) $\mathcal{A}$ is exponentially attractive for $\mathcal{H}$, namely, for each $\phi \in \mathcal{S}_\mathcal{H}$ there exist $\kappa, \alpha > 0$ such that $|\phi(t, j)|_\mathcal{A} \leq \kappa e^{-\alpha(t+j)} |\phi(0, 0)|_\mathcal{A}$ for all $(t, j) \in \operatorname{dom}\phi$; and
3) for each $p^* \in \mathbb{R}^r$ such that there exists $q \in \mathbb{R}^{n-r}$ satisfying $(p^*, q) \in \mathcal{A}$, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $\phi = (\phi_p, \phi_q)$ to $\mathcal{H}$ with $\phi_p(0, 0) \in p^* + \delta\mathbb{B}$ satisfies $|\phi_p(t, j) - p^*| \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}\phi$. □

A hybrid system is said to satisfy the hybrid basic conditions if (Goebel et al., 2012, Assumption 6.5) holds. We refer the reader to Goebel et al. (2012) for more details on these notions and the hybrid systems framework. The asymptotic version of the notion in Definition 2.5 can be found in Goebel and Sanfelice (2016).

3. CONSENSUS USING INTERMITTENT INFORMATION

3.1 Problem Description

Consider a group of N agents with dynamics

$$\dot{x}_i = u_i \qquad i \in \mathcal{V} := \{1, 2, \ldots, N\} \tag{3}$$

that exchange information over a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$, where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input of the i-th agent. Our goal is to design a control protocol (or feedback controller) assigning the input u_i to drive the solutions of each agent to a common constant value. In particular, we are interested in the following asymptotic

[1] A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded.

convergence property of the states x_i, known as static consensus; see Olfati-Saber and Murray (2004, 2002).

Definition 3.1. (static consensus). Given the agents in (3) over a digraph Γ, a control protocol u_i is said to solve the consensus problem if every resulting maximal solution with $u = (u_1, u_2, \ldots, u_N)$ is complete and its x component $t \mapsto (x_1(t), x_2(t), \ldots, x_N(t))$ satisfies

$$\lim_{t\to\infty} |x_i(t) - x_k(t)| = 0$$

for each $i, k \in \mathcal{V}$, $i \neq k$. □

We consider the scenario where the state of each system is available to each other system only at isolated time instances. Namely, the i-th agent receives information from its neighbors at time instances t_s, where $s \in \mathbb{N} \setminus \{0\}$ is the communication event index; specifically, agent i receives

$$y_{ki}(t_s) = x_k(t_s) \quad \forall k \in \mathcal{N}(i)$$

at each t_s. Given positive numbers $T_2 \geq T_1$, we assume that the time between these events is governed by a discrete random variable with some bounded probability distribution. Namely, for each $i \in \mathcal{V}$, the random variable $\Omega_s \in [T_1, T_2]$ determines the time elapsed between such communication events for each i-th system, i.e.,

$$t_{s+1} - t_s = \Omega_s \quad \forall\ s \in \mathbb{N} \setminus \{0\}. \tag{4}$$

The scalar values T_1 and T_2 define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive transmission instances. In this way, the random variable Ω_s may take values only on the bounded interval $[T_1, T_2]$, while the probability density function governing its distribution can be arbitrary as long as it assigns values to Ω_s that are in $[T_1, T_2]$.

3.2 Proposed Controller Design

We propose a hybrid control protocol and design procedure for consensus of (3) over networks with intermittent transmission of information as defined in the previous section. For the i-th agent, the proposed control protocol assigns a value to u_i based on the measured outputs of the neighboring agents obtained at communication events. In particular, the controller assigns u_i to a control variable η_i which is allowed to be impulsively updated via $\eta_i^+ = \sum_{k\in\mathcal{N}(i)} G_c^{ki}(x_i, y_{ki}, \eta_i)$ at each communication event, where $G_c^{ki} : \mathbb{R}^3 \to \mathbb{R}$ is the update law using information from the k-th neighbor. Furthermore, the controller state η_i is allowed to evolve continuously between such events. Due to the nonperiodic arrival of information and impulsive dynamics, classical analysis tools (for continuous-time or discrete-time systems) do not apply to the design of the proposed controller. This motivates us to design the proposed controller by recasting the interconnected systems, the impulsive network, and such a control protocol in a hybrid system framework; specifically, the one given in Goebel et al. (2012).

4. MAIN RESULTS

4.1 Hybrid Modeling

A timer state τ is introduced to model the network communication times given by (4). We design its hybrid dynamics as follows: from positive values it decreases to zero as ordinary time increases and, whenever it reaches zero, it is reset to an arbitrary value in the interval $[T_1, T_2]$. Such dynamics lead to the hybrid system

$$\begin{aligned} \dot{\tau} &= -1 & \tau &\in [0, T_2] \\ \tau^+ &\in [T_1, T_2] & \tau &= 0 \end{aligned} \tag{5}$$

Due to its set-valued jump map in particular, this system effectively generates any sequence of communication events at times satisfying (4) with Ω determined by any bounded probability distribution function that assigns Ω to a value in $[T_1, T_2]$.

At communication times, each system shares its state information to its neighboring agents. A control protocol using this impulsive information is proposed next. As we show in Section 5, multiple timers can be used to trigger communication events for each agent asynchronously.

Protocol 4.1. Given parameter T_2 of the network, the i-th hybrid controller has state η_i with the following dynamics:

$$\begin{aligned} u_i &= \eta_i \\ \dot{\eta}_i &= 0 & \tau &\in [0, T_2] \\ \eta_i^+ &= -\gamma \sum_{k\in\mathcal{N}(i)} (x_i - x_k) & \tau &= 0 \end{aligned} \tag{6}$$

where $\gamma > 0$ is the controller gain parameter.

Using Protocol 4.1, we employ the hybrid system framework outlined in Section 2 and the properties of the Laplacian matrix to build the interconnected state-feedback network system, which we denote by $\mathcal{H}$. The state of $\mathcal{H}$ is given by $\xi = (x, \eta, \tau) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T_2] =: \mathcal{X}$, where $x = (x_1, x_2, \ldots, x_N)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ comprise the agents' system states and controller states, respectively. By combining the agents' continuous dynamics in (3), the timer's hybrid dynamics in (5), and the protocol in (6), we arrive to the hybrid system $\mathcal{H}$ given by

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \eta \\ 0 \\ -1 \end{bmatrix} =: f(\xi) & \xi &\in C := \mathcal{X}, \\ \xi^+ &\in \begin{bmatrix} x \\ -\gamma\mathcal{L}x \\ [T_1, T_2] \end{bmatrix} =: G(\xi) & \xi &\in D := \mathbb{R}^N \times \mathbb{R}^N \times \{0\}. \end{aligned} \tag{7}$$

Remark 4.2. Due to the fact that the timer variable being zero is the only trigger of the jumps, some properties of the domain of solutions can easily be characterized. In particular, a solution ϕ to the hybrid system $\mathcal{H}$ is such that, with $t_0 = 0$, the assumption that $T_1 \leq t_{j+1} - t_j \leq T_2$ for all $j \geq 2$, and $0 \leq t_1 \leq T_2$, leads to the hybrid time domain having the following property for the flow time t:

$$(j-1)T_1 \leq t \leq (j+1)T_2 \qquad \forall\ j \geq 1$$

for all $(t, j) \in \operatorname{dom}\phi$. Moreover, due to the assumption that $T_1 > 0$, every $\phi \in \mathcal{S}_\mathcal{H}$ is complete and the hybrid time domain is unbounded in both t and j. □

Our goal is to show that Protocol 4.1 not only guarantees the static consensus property in Definition 3.1 with an exponential decay rate, but also renders Lyapunov stable the set of points such that $x_i = x_k$ for all $i, k \in \mathcal{V}$. To this end, we define the set to exponentially stabilize as

$$\mathcal{A} := \{\xi \in \mathcal{X} : x_i = x_k, \eta_i = 0\ \forall i, k \in \mathcal{V}, \tau \in [0, T_2]\}. \tag{8}$$

We establish exponential stability by changing to coordinates obtained through a key property of the Laplacian matrix. More precisely, let Γ be a strongly connected digraph. Using Lemma 2.2 and Lemma 2.3, its associated Laplacian $\mathcal{L}$ is such that there exists a nonsingular matrix $T = [\mathbf{1}_N, T_1]$ such that $T^{-1}\mathcal{L}T = \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathcal{L}} \end{bmatrix}$, which is a diagonal matrix containing the eigenvalues of $\mathcal{L}$, where $\bar{\mathcal{L}}$ is a diagonal matrix with diagonal elements $(\lambda_2, \lambda_3, \ldots, \lambda_N)$ with λ_i's being the positive eigenvalues of $\mathcal{L}$. Then, we change the coordinates ξ of $\mathcal{H}$ to the new coordinates χ defined using $\bar{x} = T^{-1}x$ and $\bar{\eta} = T^{-1}\eta$. By applying the transformation T^{-1} to both sides of the continuous dynamics of the state x and η of $\mathcal{H}$ in (7), we have $\dot{\bar{x}} = \bar{\eta}$ and $\dot{\bar{\eta}} = 0$. During jumps, the difference equations of the states x and η of $\mathcal{H}$ in (7) become $\bar{x}^+ = \bar{x}$ and $\bar{\eta}^+ = -\gamma \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathcal{L}} \end{bmatrix} \bar{x}$. Then, the new coordinates denoted as χ are defined by collecting the scalar states $\bar{x}_1$ and $\bar{\eta}_1$ into $\bar{z}_1 = (\bar{x}_1, \bar{\eta}_1)$ and the remaining states of $\bar{x}$ and $\bar{\eta}$ into $\bar{z}_2 = (\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_N, \bar{\eta}_2, \bar{\eta}_3, \ldots, \bar{\eta}_N)$, so as to write χ as $\chi = (\bar{z}_1, \bar{z}_2, \tau) \in \mathcal{X}$. The new coordinates lead to a hybrid system denoted as $\widetilde{\mathcal{H}}$ with the following data:

$$\begin{aligned} \tilde{f}(\chi) &:= \begin{bmatrix} A_{f1}\bar{z}_1 \\ A_{f2}\bar{z}_2 \\ -1 \end{bmatrix} \qquad \forall \chi \in \widetilde{C} := \mathcal{X} \\ \widetilde{G}(\chi) &:= \begin{bmatrix} A_{g1}\bar{z}_1 \\ A_{g2}\bar{z}_2 \\ [T_1, T_2] \end{bmatrix} \qquad \forall \chi \in \widetilde{D} := \{\chi \in \mathcal{X} : \tau = 0\} \end{aligned} \tag{9}$$

where

$$A_{f1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{g1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_{f2} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, A_{g2} = \begin{bmatrix} I & 0 \\ -\gamma\bar{\mathcal{L}} & 0 \end{bmatrix} \tag{10}$$

and $\gamma > 0$. Moreover, in the new coordinates, the set to stabilize for the hybrid system $\widetilde{\mathcal{H}}$ in (9) is defined as

$$\widetilde{\mathcal{A}} := \{(\bar{z}_1, \bar{z}_2, \tau) \in \mathcal{X} : \bar{z}_1 = (x^*, 0), x^* \in \mathbb{R}, \bar{z}_2 = 0\}. \tag{11}$$

4.2 Global Exponential Stability Results

Inspired by Ferrante et al. (2015), we have the following stability results for $\mathcal{H}$.

Proposition 4.3. Let T_1 and T_2 be two positive scalars such that $T_1 \leq T_2$. The set $\mathcal{A}$ is GES for the hybrid system $\mathcal{H}$ if either one of the following properties hold:

(1) the digraph is strongly connected, and there exist a positive scalar γ and a positive definite symmetric matrix P satisfying

$$A_{g2}^\top e^{A_{f2}^\top \nu} P e^{A_{f2}\nu} A_{g2} - P < 0 \quad \forall \nu \in [T_1, T_2], \tag{12}$$

where the matrices A_{g2} and A_{f2} are given in (10).

(2) the digraph is completely connected, and there exist a positive scalar γ and a positive definite symmetric matrix P satisfying

$$A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < 0 \quad \forall \nu \in [T_1, T_2], \tag{13}$$

where $A_g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $A_f = \begin{bmatrix} 1 & 0 \\ -\gamma & 0 \end{bmatrix}$.

Furthermore, if the digraph Γ is weight balanced, then every solution $\phi = (\phi_x, \phi_\eta, \phi_\tau) \in \mathcal{S}_{\mathcal{H}}(\phi(0,0))$ is complete and satisfies $\lim_{t+j\to\infty} \phi_\eta(t,j) = 0$ and $\lim_{t+j\to\infty} \phi_x(t,j) = \rho(\phi(0,0))$, where

$$\rho(\phi(0,0)) := \left(\frac{1}{N}\left(\mathbf{1}_N^\top \phi_x(0,0) + \mathbf{1}_N^\top \phi_\eta(0,0)\phi_\tau(0,0)\right)\right)\mathbf{1}_N. \tag{14}$$

The property that $\mathcal{A}$ being GES under condition *(1)* can be established using the Lyapunov function $V(\chi) = V_1(\chi) + V_2(\chi)$, where $V_1(\chi) = \bar{z}_1^\top e^{A_{f1}^\top \tau} e^{A_{f1}\tau} \bar{z}_1$ and $V_2(\chi) = \bar{z}_2^\top e^{A_{f2}^\top \tau} P e^{A_{f2}\tau} \bar{z}_2$. Note that there exist positive scalars α_1, α_2 such that $\alpha_1 |\chi|^2_{\widetilde{\mathcal{A}}} \leq V(\chi) \leq \alpha_2 |\chi|^2_{\widetilde{\mathcal{A}}}$. For each $\chi \in \widetilde{C}$, $\langle \nabla V(\chi), \tilde{f}(\chi) \rangle = 0$ and, in light of (12) and the update law for $\bar{z}_1$ and $\bar{z}_2$, for each $\chi \in \widetilde{D} \backslash \widetilde{\mathcal{A}}$ and each $g \in \widetilde{G}(\chi)$, $V(g) - V(\chi) < 0$. By continuity of condition (12), there exists a positive scalar β such that $V(g) - V(\chi) \leq -\frac{\beta}{\alpha_2} V(\chi)$. Let $\lambda_d = \ln\left(1 - \frac{\beta}{\alpha_2}\right)$. Pick $\alpha \in \left(0, \frac{|\lambda_d|}{1+T_2}\right]$ and $R \in \left[\frac{T_2|\lambda_d|}{1+T_2}, \infty\right)$ and $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}$. From Remark 4.2, we have that $\lambda_d j \leq R - \alpha(t+j)\ |\phi(t,j)|_{\widetilde{\mathcal{A}}} \leq e^{R/2}\sqrt{\alpha_2/\alpha_1} e^{-\alpha/2(t+j)} |\phi(t,j)|_{\widetilde{\mathcal{A}}}$ for all $(t,j) \in \operatorname{dom}\phi$. Since all maximal solutions are complete, we have that $\widetilde{\mathcal{A}}$ is GES for $\widetilde{\mathcal{H}}$. The proof is completed by noticing that GES of $\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{H}}$ is equivalent to GES of $\mathcal{A}$ for $\mathcal{H}$.

Remark 4.4. Conditions (12) and (13) may be difficult to satisfy numerically. In fact, these conditions are not convex in γ and P, and need to be verified for infinitely many values of ν. In Ferrante et al. (2014), the authors use a polytopic embedding strategy to arrive to a linear matrix inequality in which one needs to find some matrices X_i such that the exponential matrix is an element in the convex hull of the X_i matrices. Those results can be adapted to our setting. □

Remark 4.5. Condition (12) has a form that is similar to the discrete Lyapunov equation $A^\top PA - P < 0$. Due to this, condition (12) is satisfied if $|e^{A_{f2}\nu} A_{g2}| \leq 1$ for every $\nu \in [T_1, T_2]$. This property can be exploited using the specific forms of A_{f2} and A_{g2}. In fact, it can be checked that $e^{A_{f2}\nu} A_{g2} = \begin{bmatrix} (I - \gamma\bar{\mathcal{L}}\nu) & 0 \\ -\gamma\bar{\mathcal{L}} & 0 \end{bmatrix}$, so its eigenvalues are those of the matrix $I - \bar{\mathcal{L}}\gamma\nu$ and 0, the latter with multiplicity N. If $|I - \gamma\bar{\mathcal{L}}\nu| < 1$ for each $\nu \in [T_1, T_2]$ then condition (12) is satisfied. For a strongly connected digraph, $\bar{\mathcal{L}}$ is a diagonal matrix containing the eigenvalues $\lambda_i > 0$ for each $i \in \{2, 3, \ldots, N\}$, we have that the matrix $I - \gamma\bar{\mathcal{L}}\nu$ is also diagonal where each diagonal element is given by $1 - \gamma\lambda_i\nu$. In light of the above and since the eigenvalues λ_i are fixed by the digraph, $\gamma > 0$ must satisfy $|1 - \gamma\lambda_i\nu| < 1$ for each $i \in \{2, 3, \ldots, N\}$ and $\nu \in [T_1, T_2]$. Therefore, given a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$, if one picks γ such that $\gamma < 2/(\overline{\lambda}(\bar{\mathcal{L}})T_2)$, then condition (12) holds. □

Using the above results, we can now give sufficient conditions for the set $\mathcal{A}$ to be partially pointwise globally exponentially stable with respect to (x, η) for $\mathcal{H}$. Along with GES established in Proposition 4.3, partially pointwise globally exponentially stable requires that each point in the diagonal-like set $\mathcal{A}$ is stable with respect to (x, η).

Theorem 4.6. Given $0 < T_1 \leq T_2$ and a weight balanced digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. Suppose either (1) or (2) in Proposition 4.3 hold. Then, the set $\mathcal{A}$ is partially pointwise globally exponentially stable with respect to (x, η) for the hybrid system $\mathcal{H}$.

Example 4.7. Consider five agents with dynamics as in (3) over the strongly connected graph with adjacency matrix

$$\mathcal{G} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}. \tag{15}$$

Let $T_1 = 0.5$ and $T_2 = 1.5$. If $\gamma = 0.3$, then a matrix P can be found such that condition (12) is satisfied. Figure 1 shows the x_i components $i \in \{1, 2, 3, 4, 5\}$ of a

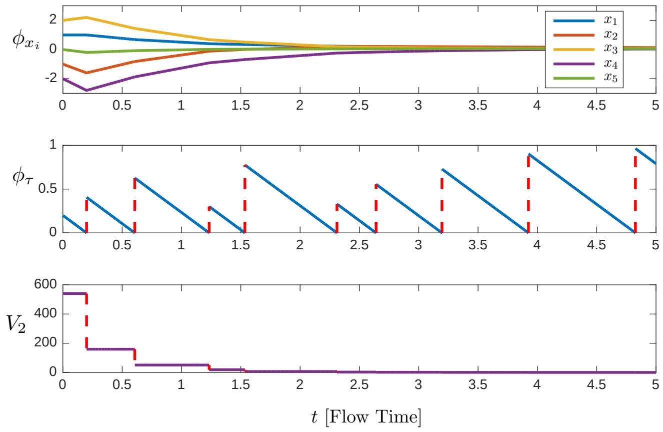

Fig. 1. (top) The x and τ components of a solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ to $\mathcal{H}$ with $\mathcal{G}$ in (15) using Protocol 4.1 which satisfies Proposition 4.3. (bottom) Note that since $V_2(\chi)$ for $\widetilde{\mathcal{H}}$ deceases to zero with respect to flow time, it indicates that the solution reaches consensus.

solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ from initial conditions given by $\phi_x(0,0) = (1,-1,2,-2,0)$, $\phi_\eta(0,0) = (0,-3,1,-4,-1)$, and $\phi_\tau(0,0) = 0.2$ as well as the the function $V_2(\chi)$ below Proposition 4.3 evaluated along ϕ projected onto the ordinary time domain. [2] $\triangle$

4.3 Robustness to Perturbations on Communication Noise

In a realistic setting, the information transmitted is affected by communication noise. In this section, we consider the systems under the effect of communication noise m_i when agent i sends out information. Specifically, if the k-th agent receives information of the i-th agent perturbed by $m_i \in \mathbb{R}$, $i \in \mathcal{V}$, we have that when communication occurs, [3] the output of each agent is given by $y_i = x_i + m_i$. In such a case, the controller from Protocol 4.1 becomes

$$\begin{aligned} \dot{\eta}_i &= 0 & \tau &\in [0, T_2] \\ \eta_i^+ &= -\gamma \sum_{k \in \mathcal{N}(i)} (y_i - y_k) & \tau &= 0 \end{aligned}$$

which, different than (6), leads to an update law with noise given by $\eta_i^+ = -\gamma \sum_{k\in\mathcal{N}(i)}(x_i - x_k) - \gamma \sum_{k\in\mathcal{N}(i)}(m_i - m_k)$. We show that the hybrid system $\mathcal{H}$ in (7) is input-to-state stable (ISS) [4] with respect to such measurement noise.

The perturbed hybrid system, which we denote by $\mathcal{H}_m$, can be written in the compact form

$$\left.\begin{aligned} \dot{x} &= \eta \\ \dot{\eta} &= 0 \\ \dot{\tau} &= -1 \end{aligned}\right\} \tau \in [0, T_2], \qquad \left.\begin{aligned} x^+ &= x \\ \eta^+ &= -\gamma \mathcal{L} x - \gamma \mathcal{L} m \\ \tau &\in [T_1, T_2] \end{aligned}\right\} \tau = 0. \tag{16}$$

where $m = (m_1, m_2, \ldots, m_N)$. Then, using the change of coordinates as in Section 4.1, we can show that global exponential stability of $\mathcal{A}$ in (11) for $\mathcal{H}_m$ in (9) is robust to communication noise.

Theorem 4.8. Let $0 < T_1 \le T_2$ be given. Suppose the digraph $\Gamma = (\mathcal{E}, \mathcal{V}, \mathcal{G})$ is strongly connected. If there exists $\gamma > 0$ and a positive definite symmetric matrix P such that (12) holds for all $\nu \in [T_1, T_2]$, then the hybrid system $\mathcal{H}_m$ with input $\bar{m}$ is ISS with respect to $\mathcal{A}$ as in (11).

5. ON ASYNCHRONOUS EVENT TIMES

In this section, we present results for the scenario where each agent receives information asynchronously. To model such events, we attach a local timer to each agent so that when it reaches zero it triggers the transmission of information of its connected neighbors.

Protocol 5.1. Given parameter T_2 of the network, the i-th hybrid controller has state η_i with the following dynamics:

$$\begin{aligned} u_i &= \eta_i \\ \dot{\eta}_i &= h \eta_i & \tau_i &\in [0, T_2] \\ \eta_i^+ &= \gamma \sum_{k \in \mathcal{N}(i)} (x_i - x_k) & \tau_i &= 0 \end{aligned} \tag{17}$$

where $h, \gamma \in \mathbb{R}$ are the controller's parameters.

Define $\bar{x}_i = x_i - \frac{1}{N}\sum_{k=1}^N x_k$ and $\theta_i = \gamma \sum_{k\in\mathcal{N}(i)}(x_i - x_k) - \eta_i$. From the interconnection between (3), (17), and each τ_i with dynamics as in (5), the continuous dynamics of $\bar{x}_i$ and θ_i are given by $\dot{\bar{x}}_i = \eta_i - \frac{1}{N}\sum_{k=1}^N \eta_k$ and $\dot{\theta}_i = \gamma \sum_{k\in\mathcal{N}(i)}(\eta_i - \eta_k) - h\eta_i$. Denote $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$. From the definition of θ_i and noting that $\bar{x}_i - \bar{x}_k = x_i - x_k$, it follows that

$$\theta = \gamma \mathcal{L} \bar{x} - \eta.$$

With the definitions of $\bar{x}_i$ and θ_i and the assumption of Γ being weight balanced [5], we have that $\dot{\bar{x}} = \gamma \mathcal{L} \bar{x} - \left(I - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^\top\right)\theta$ and $\dot{\theta} = \bar{x} - (\gamma\mathcal{L} - hI)\theta$ for each $\tau \in [0, T_2]^N$. At each jump, say, there exists $i \in \mathcal{V}$ such that $\tau_i = 0$, $\bar{x}_i$ and θ_i are updated as $\bar{x}_i^+ = \bar{x}_i$ and $\theta_i^+ = 0$, while all other states are updated by the identity.

In the coordinates $\bar{x}$ and θ, we define the hybrid system $\mathcal{H}_a$ with state $\xi_a = (\psi, \tau) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T_2]^N =: \mathcal{X}_a$ where $\psi = (\bar{x}, \theta)$ and data (C_a, f_a, D_a, G_a) given by

$$f_a(\xi_a) := \begin{bmatrix} A\psi \\ -\mathbf{1}_N \end{bmatrix}, A := \begin{bmatrix} \gamma \mathcal{L} & -\left(I - \dfrac{1}{N}\mathbf{1}_N \mathbf{1}_N^\top\right) \\ \gamma^2 \mathcal{L}\mathcal{L} - h\gamma\mathcal{L} & -(\gamma\mathcal{L} - hI) \end{bmatrix}$$

for each $\xi_a \in C_a := \mathcal{X}_a$, and $G_a(\xi_a) := \{G_i(\xi_a) : \xi_a \in D_i, i \in \mathcal{V}\}$ for each $\xi_a \in D_a := \bigcup_{i\in\mathcal{V}} D_i$, where $D_i := \{\xi_a \in C_a : \tau_i = 0\}$ and

$$G_i(\xi_a) := \begin{bmatrix} \bar{x} \\ (\theta_1, \theta_2, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \tau_2, \ldots, \tau_{i-1}, [T_1, T_2], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix}.$$

The definition of G_i is such that the i-th component of θ and τ are updated only when $\tau_i = 0$.

It follows that the set to stabilize is given by

$$\mathcal{A}_a := \{\xi_a \in \mathcal{X}_a : \xi_a = (\bar{x}^* \mathbf{1}_N, 0, \nu), \bar{x}^* \in \mathbb{R}, \nu \in [0, T_2]^N\} \tag{18}$$

for the hybrid system $\mathcal{H}_a$ with data (C_a, f_a, D_a, G_a). We have the following stability result for $\mathcal{H}_a$.

Proposition 5.2. Let T_1 and T_2 be two positive scalars such that $T_1 \le T_2$ and a digraph Γ be strongly connected and weight balanced. If there exist scalars $\gamma, h \in \mathbb{R}$, and $\sigma > 0$, positive definite diagonal matrices P and Q such that

[2] Code at `https://github.com/HybridSystemsLab/ConsSyncTimes`

[3] In this way, m_i can account for errors local to the i-th agent as well as communication noise.

[4] We use the ISS definition in Cai and Teel (2009).

[5] For a weight balanced digraph, $\mathbf{1}_N^\top \mathcal{L} = \mathbf{0}^\top$.

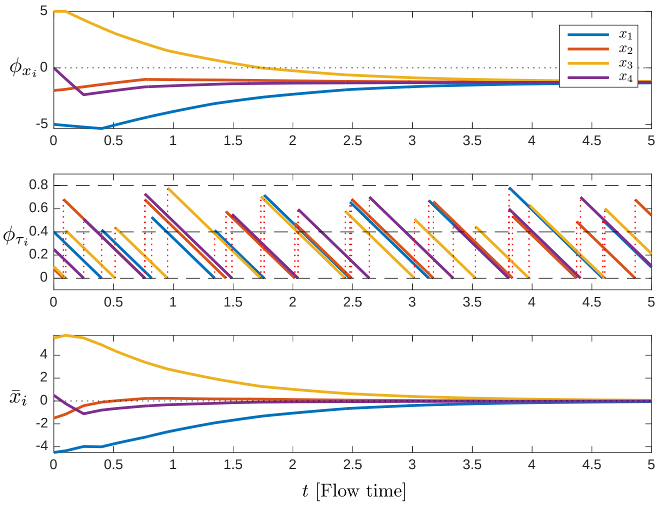

Fig. 2. The x, τ components of a solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ to a hybrid system with with agent dynamics (3), communication governed by local timers τ_i with dynamics in (5), and Protocol 5.1. The coordinate $\bar{x}$ is also plotted over flow time.

$$\begin{bmatrix} \gamma \text{He}(P, \mathcal{L}) & -P\Pi + \mathcal{K}_1^\top QE(\tau) \\ \star & -\sigma QE(\tau) - \text{He}(QE(\tau), \mathcal{K}_2) \end{bmatrix} \leq 0 \qquad (19)$$

for each $\tau \in [0, T_2]^N$*, where* $\Pi = I - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$, $\mathcal{K}_1 = \gamma\mathcal{K}_2\mathcal{L}$, $\mathcal{K}_2 = \gamma\mathcal{L} - hI$*, and* $E(\tau) = \text{diag}(e^{\sigma\tau_1}, e^{\sigma\tau_2}, \ldots, e^{\sigma\tau_N})$*, then the set* $\mathcal{A}_a$ *in* (18) *is globally asymptotically stable*[6] *for the hybrid system* $\mathcal{H}_a$*.*

Example 5.3. Consider four agents with dynamics as in (3) over a digraph with adjacency matrix $\mathcal{G} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ which is strongly connected and weight balanced. Let $T_1 = 0.4$. For $T_2 = 0.8$, and parameters $\gamma = -0.4$, $h = -0.4$, and $\sigma = 2$, we find the matrices $P = 5.06I$ and $Q = 1.58I$ satisfy condition (19) in Proposition 5.2. Figure 2 shows a solution ϕ to a hybrid system with agent dynamics (3), communication governed by local timers τ_i with dynamics in (5), and Protocol 5.1 from initial conditions $\phi_x(0,0) = (-5, -2, 5, 0)$, $\phi_\eta(0,0) = (-1, 1, 0, -10)$, and $\phi_\tau(0,0) = (.4, 1, 0.1, 0.25)$.[7] Furthermore, as indicated by Figure 2, it follows that the new states $\bar{x}_i$ asymptotically converge to zero over time indicating that the x_i's achieve consensus. $\triangle$

6. CONCLUSION

We showed that hybrid consensus protocols are viable algorithms for the consensus of first order systems with stochastically determined communication events over a general graph. Using a hybrid systems framework, we defined the communication events between the systems using a hybrid decreasing timer. Recasting consensus as a set stability problem, we took advantage of several properties of the graph structure and employed a Lyapunov based approach to certify that this set is partially pointwise globally exponentially stable. We further showed that global exponential stability of the consensus set is robust to communication noise. Lastly, we presented a protocol for reaching state consensus where agents receive local updates asynchronously. The results in this paper can be used to design large-scale networked systems that communicate at stochastic time instants over general communication graphs.

[6] Global asymptotic stability is defined in Goebel et al. (2012).

[7] Code at `https://github.com/HybridSystemsLab/ConsAsyncTimes`

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