Notions and Sufficient Conditions for Pointwise Asymptotic Stability in Hybrid Systems [*]

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Abstract: Pointwise asymptotic stability is a property of a set of equilibria of a dynamical system, where every equilibrium is Lyapunov stable and every solution converges to some equilibrium. Hybrid systems are dynamical systems which combine continuous-time and discrete-time dynamics. In this paper, they are modeled by a combination of differential equations or inclusions, of difference equations or inclusions, and of constraints on the resulting motions. Sufficient conditions for pointwise asymptotic stability of a closed set are given in terms of set-valued Lyapunov functions: they require that the values of the Lyapunov function shrink along solutions. Cases of strict and weak decrease are considered. Lyapunov functions, not set-valued, which imply that solutions have finite length are used in sufficient conditions and related to the set-valued Lyapunov functions. Partial pointwise asymptotic stability is also addressed.

Keywords: Hybrid Systems, Stability, Lyapunov Methods, Dynamical Systems Techniques

1. INTRODUCTION

Hybrid dynamical systems exhibit features characteristic of continuous-time dynamical systems (flow) and features characteristic of discrete-time systems (jumps). Modeling of cyber-physical systems or mechanical systems with impacts, logic-based control algorithms, uncertainty and worst-case scenarios, etc. motivate the interest in hybrid systems in the control engineering and control theory literature. This paper models hybrid systems as hybrid inclusions, as in Goebel et al. (2009), Goebel et al. (2012).

Pointwise asymptotic stability (also known as semistability) is a property of a set of equilibria in a dynamical system, where every equilibrium is Lyapunov stable and from a neighborhood of it, every solution converges to possibly another equilibrium. Motivated by applications to consensus algorithms in Hui et al. (2008), hysteresis in Oh et al. (2009), and chemical processes and thermodynamics in Haddad et al. (2010), this stability concept has been analyzed in the setting of differential equations by Bhat and Bernstein (1999), Bhat and Bernstein (2003) and in the setting of differential inclusions by Hui et al. (2009). Bhat and Bernstein (2003) proposed a sufficient condition using a standard Lyapunov function and, additionally, a requirement that solutions don't flow in directions tangent to the set of equilibria. A standard Lyapunov function, on its own, cannot give a sufficient condition except the case of a single equilibrium. Sufficient conditions related

to some results of this paper are in Bhat and Bernstein (2010), where arc-length-based Lyapunov sufficient conditions are proposed. For difference inclusions, necessary and sufficient conditions involving a set-valued Lyapunov function are in Goebel (2011), motivated by the idea of Moreau (2005): if the convex hull of positions of k agents decreases sufficiently along solutions of the system, then agents reach consensus. The sufficient condition of Moreau (2005) implies pointwise asymptotic stability of the set of consensus states, and naturally generalizes to sufficient conditions involving maps beyond the convex hull, as in Angeli and Bliman (2006), Goebel (2011). The use of general set-valued maps allows for converse Lyapunov results, i.e., necessary conditions, and enables characterizing robustness of pointwise asymptotic stability Goebel (2014). Some work on semistability for switching systems in Hui (2011) and for hybrid systems in Hui (2010) has appeared, but has not addressed necessary or sufficient conditions for pointwise asymptotic stability in the hybrid setting.

For a brief illustration of the set-valued approach, consider I agents, with states $x_i \in \mathbb{R}^l$ for $i = 1, 2, \ldots$, who communicate and agree on a target a in the convex hull of x_i's, move toward a according to $\dot{x}_i = a - x_i$ for $T > 0$ amount of time, then communicate again, agree on a new a, and repeat. One can model this in the hybrid framework of Goebel et al. (2009), Goebel et al. (2012) by

$$\dot{x}_i = a - x_i, \ \dot{a} = 0, \ \dot{\tau} = -1 \quad \text{if } \tau \geq 0,$$

which describes the flow, and

$$x_i^+ = x_i, \ a^+ \in \text{con}\{x_1, x_2, \ldots, x_I\}, \ \tau^+ = T \quad \text{if } \tau = 0,$$

which describes the jumps. That the states x_i converge to the same point, and that states where $x_1 = x_2 = \cdots =$

$x_I = a$ are Lyapunov stable, can be established by relying, for example, on the set-valued mapping given by
$$w(x, a) = \text{con}\{x_1, x_2, \ldots, x_I, a\},$$
where $x = (x_1, x_2, \ldots, x_I)$. After a period of flow, $w(x, a)$ is a subset of the initial $w(x, a)$. Similarly, after a jump, $a^+ \in \text{con}\{x_1, x_2, \ldots, x_I\}$, $x^+ = x$, and so $w(x^+, a^+) \subset w(x, a)$. These two weak decrease properties of w remain true even if, initially, a is not in the convex hull of x_i's. Based on this paper, and subject to verifying some basic properties of w, one can conclude stability. Furthermore, subject to verifying that $w(x, a)$ remains constant along flows and jumps only if $x_1 = x_2 = \cdots = x_I = a$, an invariance-based result applies and concludes consensus, i.e., convergence of x_i's to the same limit. Details are in Example 6.6, after the necessary theory is developed.

2. HYBRID INCLUSIONS

This paper considers hybrid systems modeled as hybrid inclusions as described below. For further details, consult Goebel et al. (2009), Goebel et al. (2012). Below, $C, D \subset \mathbb{R}^n$ are sets, called, respectively, the *flow set* and the *jump set* and $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings, called, respectively, the *flow map* and *jump map*. A hybrid system is represented by

$$
\begin{aligned}
x \in C & \quad \dot{x} \in F(x) \\
x \in D & \quad x^+ \in G(x).
\end{aligned}
\tag{1}
$$

A special case of (1) is provided by systems where the flow and jump maps are functions, so that we have

$$
\begin{cases}
x \in C & \dot{x} = f(x) \\
x \in D & x^+ = g(x).
\end{cases}
\tag{2}
$$

A set $E \subset \mathbb{R}^2$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J} I_j \times \{j\},$$

where $J \in \mathbb{N}$ and $I_j = [t_j, t_{j+1}]$, $j = 0, 1, \ldots, J$, for some $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. A set E is a *hybrid time domain* if, for each $(T, J) \in E$, the set $\{(t, j) \in E \,|\, t \leq T, j \leq J\}$ is a compact hybrid time domain. Equivalently, a hybrid time domain is a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_1 \leq t_2 \leq \ldots$, with the last interval, if it exists, possibly of the form $[t_j, t_{j+1})$ or $[t_j, \infty)$.

A function $\phi : E \to \mathbb{R}^n$ is a *solution to the hybrid system* (1) if E is a hybrid time domain, $\phi(0, 0) \in \overline{C} \cup D$, and

- if $I^j := \{t \,|\, (t, j) \in E\}$ has nonempty interior, then $t \mapsto \phi(t, j)$ is locally absolutely continuous on I^j and
$$\phi(t, j) \in C \quad \text{for all } t \in \text{int } I^j \text{ and}$$
$$\frac{d}{dt}\phi(t, j) \in F(\phi(t, j)) \quad \text{for almost all } t \in I^j;$$
- if $(t, j) \in E$ and $(t, j + 1) \in E$ then
$$\phi(t, j) \in D \quad \text{and} \quad \phi(t, j + 1) \in G(\phi(t, j)).$$

In what follows, the domain of a solution ϕ (represented by E above) is denoted by $\text{dom } \phi$. A solution $\phi : E \to \mathbb{R}^n$ is *maximal* if it cannot be extended, and *complete* if $\text{dom } \phi$ is unbounded. Throughout the paper, the following stands.

Assumption 2.1. *Maximal solutions to* (1) *are complete.*

For conditions guaranteeing that maximal solutions are complete, see (Goebel et al., 2012, Proposition 2.10 and Proposition 6.10). In what follows, $\mathcal{S}$ denotes the set of all maximal, and hence complete, solutions to (1), $\mathcal{S}(x)$ denotes the set of maximal solutions to (1) that start from x, and for a set $K \subset \mathbb{R}^n$, $\mathcal{S}(K) := \bigcup_{x \in K} \mathcal{S}(x)$.

The set-valued analysis terms used below come from Rockafellar and Wets (1998). For further discussion, in the setting of hybrid systems, see Goebel et al. (2012).

Definition 2.2. *The hybrid system* (1) *satisfies the hybrid basic assumptions if its data,* (C, F, D, G), *satisfies the following conditions: the sets* $C, D \subset \mathbb{R}^n$ *are closed; the set-valued mappings* $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *are locally bounded and outer semicontinuous; for every* $x \in C$, $F(x)$ *is nonempty, closed, and convex; for every* $x \in D$, $G(x)$ *is nonempty and closed.*

In the special case of (2), the system satisfies the hybrid basic assumptions if $C, D \subset \mathbb{R}^n$ are closed and $f : C \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$ are continuous functions.

3. POINTWISE ASYMPTOTIC STABILITY

The definition below is global, in the sense that convergence of all solutions to (1) is required. The local case can be dealt with, but is not considered in this paper.

Definition 3.1. *A set* $A \subset \mathbb{R}^n$ *is* pointwise asymptotically stable *if*

(a) *every* $a \in A$ *is* Lyapunov stable, *that is, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that every solution* ϕ *to* (1) *with* $\|\phi(0, 0) - a\| < \delta$ *satisfies* $\|\phi(t, j) - a\| < \varepsilon$ *for every* $(t, j) \in \text{dom } \phi$, *and*

(b) *for every solution* ϕ *to* (1), $\lim_{t+j \to \infty} \phi(t, j)$ *exists and is contained in* A.

3.1 Structural properties of solutions

In absence of pointwise asymptotic stability, or a similar property, limits of solutions to a hybrid system (and, in fact, to a simple differential equation), even if they exist, may depend irregularly on initial conditions. For example, for $\dot{x} = -x(x - 1)^2$, limits equal 0 for solutions from $(-\infty, 1)$ and equal 1 otherwise. Limits of solutions depend discontinuously on initial conditions at $x = 1$. Note that the smallest globally asymptotically stable set here is $[0, 1]$, and the discontinuity occurs at a point in A. In fact, the limits may depend discontinuously on initial conditions even when the hybrid inclusion has linear flow and jump maps, due to the geometry of flow and jump sets.

As described below, pointwise asymptotic stability leads to reasonable dependence of solutions and their limits on initial conditions, and then to regularity of reachable sets etc. For an exposition of graphical convergence of hybrid arcs, featured in the next result, see Goebel et al. (2012).

Theorem 3.2. *Suppose that* (1) *satisfies the hybrid basic assumptions. For every sequence* $\phi_i \in \mathcal{S}$ *with convergent* $\phi_i(0, 0)$, *there exists a graphically convergent subsequence, which is not relabeled, such that*

(a) the graphical limit ϕ of the graphically convergent subsequence ϕ_i is a complete solution to (1).

If, additionally, the closed set $A \subset \mathbb{R}^n$ is pointwise asymptotically stable, then

(b) $\lim\limits_{i \to \infty} \lim\limits_{t+j \to \infty} \phi_i(t,j) = \lim\limits_{t+j \to \infty} \phi(t,j)$;

(c) convergence of ϕ_i to ϕ is uniform in the following sense: for every $\varepsilon > 0$ there exists $\tau > 0$ such that, for every large enough i, ϕ_i and ϕ are ε-close to τ-truncations of one another, in the sense that

   (i) for every $(t,j) \in \operatorname{dom} \phi$ there exists $(t',j') \in \operatorname{dom} \phi_i$, $t' + j' < \tau$ with $\|\phi(t,j) - \phi_i(t',j)\| < \varepsilon$;
   (ii) for every $(t',j') \in \operatorname{dom} \phi_i$ there exists $(t,j) \in \operatorname{dom} \phi$, $t + j < \tau$ with $\|\phi_i(t',j') - \phi(t,j')\| < \varepsilon$.

Proof. Take a sequence of solutions $\phi_i \in \mathcal{S}$ with $\phi_i(0,0)$ convergent. By (Goebel et al., 2012, Theorem 6.1), there exists a graphically convergent subsequence. From now on, let ϕ_i be that subsequence. Under Assumption 2.1, (Goebel et al., 2012, Definition 6.2) and (Goebel et al., 2012, Theorem 6.8) imply (a). Let ϕ be the graphical limit of ϕ_i. If A is pointwise asymptotically stable, $\lim_{t+j \to \infty} \phi(t,j) =: a$ exists and belongs to A.

Given $\varepsilon > 0$, pick $\delta > 0$ using Lyapunov stability of a. For some $(t_0, j_0) \in \operatorname{dom} \phi$, $\|\phi(t_0, j_0) - a\| < \delta/2$. Graphical convergence of ϕ_i to ϕ yields $(t_i, j_i) \in \operatorname{dom} \phi_i$ so that $(t_i, j_i, \phi(t_i, j_i))$ converge to $(t_0, j_0, \phi(t_0, j_0))$. Then, for all large enough i, $\|\phi_i(t_i, j_i) - a\| < \delta$ and hence $\|\phi_i(t,j) - a\| < \varepsilon$ for all $(t,j) \in \operatorname{dom} \phi_i$, $t + j > t_i + j_i$. In particular, for all large enough i, $\|\lim_{t+j \to \infty} \phi_i(t,j) - a\| \le \varepsilon$. As $\varepsilon > 0$ is arbitrary, $\lim\limits_{i \to \infty} \lim\limits_{t+j \to \infty} \phi_i(t,j) = a$. This shows (b).

Similarly, given $\varepsilon > 0$, pick $\delta \in (0, \varepsilon/2)$ using stability of a, so $\|\psi(t,j) - a\| < \varepsilon/2$ for every $(t,j) \in \operatorname{dom} \psi$ if $\|\psi(0,0) - a\| < \delta$. Pick $(t_0, j_0) \in \operatorname{dom} \phi$ so that $\|\phi(t_0, j_0) - a\| < \delta/2$ and let $\tau = t_0 + j_0 + 1$. Then, by graphical convergence, for large enough i and every $(t,j) \in \operatorname{dom} \phi_i$ with $t + j < \tau$ there exists $(t',j') \in \operatorname{dom} \phi$, $t' + j' < \tau$ so that $\|\phi_i(t,j) - \phi(t',j')\| < \varepsilon$ and for every $(t,j) \in \operatorname{dom} \phi$ with $t + j < \tau$ there exists $(t',j') \in \operatorname{dom} \phi_i$, $t' + j' < \tau$ so that $\|\phi(t,j) - \phi_i(t',j')\| < \varepsilon$. Thus, for all large enough i, there exist $(t_i, j_i) \in \operatorname{dom} \phi_i$, $(t_i, j_i) \to (t_0, j_0)$, and hence $t_i + j_i < \tau$, so that $\|\phi_i(t_i, j_i) a\| < \delta$. By choice of δ, for all large enough i, for all $(t,j) \in \operatorname{dom} \phi_i$ with $t + j \ge \tau$, $\|\phi_i(t,j) - a\| < \varepsilon/2$. Hence, for each such (t,j), $\|\phi_i(t,j) - \phi(t_0, j_0)\| < \varepsilon$. Similarly, for all $(t,j) \in \operatorname{dom} \phi$ with $t + j \ge \tau$, $\|\phi(t,j) - \phi_i(t_i, j_i)\| < \varepsilon$. This verifies (c). $\blacksquare$

Corollary 3.3. *Suppose that* (1) *satisfies the hybrid basic assumptions and the closed set* $A \subset \mathbb{R}^n$ *is pointwise asymptotically stable. Then:*

(a) *the set-valued mapping* $\mathcal{L} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *defined by*

$$\mathcal{L}(x) = \bigcup_{\phi \in \mathcal{S}(x)} \lim_{t+j \to \infty} \phi(t,j)$$

is outer semicontinuous and locally bounded;

(b) *the set-valued mapping* $\overline{\mathcal{R}}_\infty : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *defined by*

$$\overline{\mathcal{R}}_\infty(x) = \overline{\mathcal{R}_\infty(x)}$$

is outer semicontinuous and locally bounded and

$$\overline{\mathcal{R}}_\infty(x) = \mathcal{R}_\infty(x) \cup \mathcal{L}(x)$$

for every $x \in \mathbb{R}^n$, *where the set-valued mapping* $\mathcal{R}_\infty : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is defined by*

$$\mathcal{R}_\infty(x) = \bigcup_{\phi \in \mathcal{S}(x), (t,j) \in \operatorname{dom} \phi} \phi(t,j);$$

(c) *for every compact set* $K \subset \mathbb{R}^n$ *and every* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $\tau \ge 0$ *with the following property: for every* $\phi \in \mathcal{S}(K + \delta \mathbb{B})$ *there exists* $\phi' \in \mathcal{S}(K)$ *such that* ϕ *and* ϕ' *are* ε-*close to* τ-*truncations of one another.*

Above, $\mathbb{B}$ is the closed unit ball, of appropriate dimension, centered at 0 and so $K + \delta \mathbb{B}$ is the closed neighborhood of K of size δ: $\{x \mid \exists k \in K$ so that $\|x - k\| \le \delta\}$.

The mappings $\mathcal{L}$ and $\overline{\mathcal{R}}_\infty$ can have empty values: for x from which there is no solutions, including $x \notin \overline{C} \cup D$, $\mathcal{L}(x) = \overline{\mathcal{R}}_\infty(x) = \emptyset$. The proof of (a), (b) above is similar to that of (Goebel, 2014, Proposition 2.10) for the discrete-time case; the proof of (c) is similar to (Goebel et al., 2012, Proposition 6.14), with (d) of Theorem 3.2 replacing the closeness concept used in Goebel et al. (2012).

4. SET-VALUED LYAPUNOV FUNCTIONS

This section defines a set-valued Lyapunov function for a hybrid system and uses it in strict Lyapunov conditions, sufficient for pointwise asymptotic stability. This is in Section 4.1, where no regularity assumptions are placed on the hybrid system. Weakened Lyapunov conditions turn out sufficient via invariance-like arguments, as stated in Section 4.2, which requires that the data satisfy the hybrid basic assumptions. Finally, a combination of weak and strict conditions is possible if solutions exhibit enough flow or enough jumps, as shown in Section 4.3.

A function $\gamma : \mathbb{R}^n \to \mathbb{R}$ is *positive definite with respect to* A if $\gamma(x) \ge 0$ for every $x \in \mathbb{R}^n$ and $\gamma(x) = 0$ iff $x \in A$.

Definition 4.1. *A set-valued mapping* $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is a set-valued Lyapunov function for a nonempty* $A \subset \mathbb{R}^n$ *if*

(a) $W(x) = \{x\}$ *for every* $x \in A$ *and* $x \in W(x)$ *for every* $x \in \overline{C} \cup D \cup G(D)$;

(b) W *is locally bounded and, at every* $x \in A$, *it is outer semicontinuous;*

there exist continuous $c, d : \mathbb{R}^n \to [0, \infty)$, *positive definite with respect to* A, *so that*

(c) *for every solution* $\phi : [0, T] \to \mathbb{R}^n$ *to* $\dot{x} \in F(x)$ *such that* $\phi(t) \in C$ *for every* $t \in (0, T)$,

$$W(\phi(t)) + \int_0^t c(\phi(s)) \, ds \mathbb{B} \subset W(\phi(0)) \quad \forall t \in [0, T];$$

(d)

$$W(G(x)) + d(x) \mathbb{B} \subset W(x) \quad \forall x \in D.$$

Example 4.2. Let $x_i \in \mathbb{R}^l$, $i = 1, 2, \ldots, I$ represent states of I agents. Let $x = (x_1, x_2, \ldots, x_I) \in \mathbb{R}^n$, $n = Il$. Let

$$\operatorname{ave} x = \frac{1}{I} \sum_{i=1}^{I} x_i, \quad a(x) = \underbrace{(\operatorname{ave} x, \operatorname{ave} x, \ldots, \operatorname{ave} x)}_{I \text{ copies}}.$$

Let $\alpha > 0$ and consider the continuous-time system

$$\dot{x}_i = \alpha(\operatorname{ave} x - x_i), \quad i = 1, 2, \ldots, I. \tag{3}$$

It is immediate that $\operatorname{ave}\dot{x} = 0$. Pointwise asymptotic stability of the diagonal set of consensus states
$$A := \{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_I\} \qquad (4)$$
can be shown using the fact that for every solution $\phi : [0,\infty) \to \mathbb{R}^n$, $\operatorname{ave}\phi(t) = \operatorname{ave}\phi(0)$ and $\|\phi_i(t) - \operatorname{ave}\phi(t)\| = e^{-\alpha t}\|\phi_i(0) - \operatorname{ave}\phi(0)\|$. Alternatively, let
$$V(x) = \|x - a(x)\| = \sqrt{\sum_{i=1}^{I} \|x_i - \operatorname{ave} x\|^2},$$
note that for every solution $\phi : [0,\infty) \to \mathbb{R}^n$ one has $V(\phi(t)) = e^{-\alpha t}V(\phi(0))$, and define $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by
$$W(x) = a(x) + V(x)\mathbb{B}.$$
By construction, $x \in W(x)$, and since $x \in A$ is equivalent to $x = a(x)$, $W(x) = \{x\}$ for $x \in A$. Because V is a continuous function, W is continuous as a set-valued mapping and is locally bounded. Furthermore, one has
$$W(\phi(t)) = a(\phi(t)) + e^{-\alpha t}V(\phi(0))\mathbb{B}$$
and hence, remembering that $\operatorname{ave}\phi(t) = \operatorname{ave}\phi(0)$ and so $a(\phi(t)) = a(\phi(0))$,
$$W(\phi(t)) + (1 - e^{-\alpha t})V(\phi(0))\mathbb{B}$$
$$\subset a(\phi(t)) + V(\phi(0))\mathbb{B} = W(\phi(0)).$$
Thus (c) of Definition 4.1 holds with $c(x) = \alpha V(x)$.

Now, take $\beta \in (0,1)$ and consider the discrete-time system
$$x_i^+ = \beta x_i + (1 - \beta)\operatorname{ave} x, \quad i = 1, 2, \ldots, I. \qquad (5)$$
Similarly as above, $\operatorname{ave} x^+ = \operatorname{ave} x$, and for every solution $\phi : \mathbb{N} \to \mathbb{R}^n$, $\|\phi_i(j) - \operatorname{ave}\phi(j)\| = \beta^j \|\phi_i(0) - \operatorname{ave}\phi(0)\|$. Pointwise asymptotic stability can of course be argued directly. Alternatively, for W as above, because $V(x^+) = \beta V(x)$, one has
$$W(x^+) + (1 - \beta)V(x)\mathbb{B} \subset W(x),$$
and (d) of Definition 4.1 holds with $d(x) = (1 - \beta)V(x)$.

One can further conclude that W above is a set-valued Lyapunov function for the diagonal A in (4) and the hybrid system given by the flow map (3), the jump map (5), and by arbitrarily chosen C and D.

4.1 Strict Lyapunov sufficient condition

Theorem 4.3. *If there exist a set-valued Lyapunov function W for (1) and a closed set $A \subset \mathbb{R}^n$, then A is pointwise asymptotically stable.*

Proof. Take any complete solution ϕ. Pick any $(T, J) \in \operatorname{dom}\phi$ and suppose that
$$\operatorname{dom}\phi \cap [0,T] \times [0,J] = \bigcup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\}. \qquad (6)$$
Then, (c) and (d) of Definition 4.1 imply that
$$W(\phi(T,J)) + \left(\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} c(\phi(s))\, ds + \sum_{j=1}^{J} d(\phi(t_j, j)) \right) \mathbb{B}$$
$$\subset W(\phi(0,0)) \qquad (7)$$
With this established, the proof of stability of every $a \in A$ is a special case of what is done in Proposition 6.3.

Local boundedness of W, in (b) of Definition 4.1, and $\phi(t,j) \in W(\phi(0,0))$ for every $(t,j) \in \operatorname{dom}\phi$ imply that every $\phi \in \mathcal{S}(\operatorname{dom} W)$ is bounded. To prove convergence of such ϕ to a point in A, fix ϕ and note that by (7),
$$\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} c(\phi(s))\, ds + \sum_{j=1}^{J} d(\phi(t_j, j))$$
is bounded over all $(T, J) \in \operatorname{dom}\phi$. Then, there exists a sequence $(t_k, j_k) \in \operatorname{dom}\phi$, with $t_k + j_k \to \infty$ as $k \to \infty$, such that either $c(\phi(t_k, j_k))$ or $d(\phi(t_k, j_k))$ converges to 0 as $k \to \infty$. As ϕ is bounded, one may assume that the sequence $\phi(t_k, j_k)$ converges. Denote the limit by x. Positive definiteness of c and d with respect to A implies that $x \in A$. Lyapunov stability of every $x \in A$, established before, implies that $\lim_{t+j \to \infty} \phi(t,j) = a$.

4.2 Invariance-based sufficient condition

Definition 4.4. *A set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a* weak set-valued Lyapunov function *for a nonempty $A \subset \mathbb{R}^n$ if*

(a) *$W(x) = \{x\}$ for every $x \in A$ and $x \in W(x)$ for every $x \in \overline{C} \cup D \cup G(D)$;*
(b) *W is locally bounded and, at every $x \in A$, it is outer semicontinuous;*
(c) *for every solution $\phi : [0,T] \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0,T)$,*
$$W(\phi(t)) \subset W(\phi(0)) \quad \forall t \in [0,T];$$
(d)
$$W(G(x)) \subset W(x) \quad \forall x \in D.$$

Theorem 4.5. *Suppose that (1) satisfies the hybrid basic assumptions. If there exist a weak set-valued Lyapunov function W for a closed $A \subset \mathbb{R}^n$, which is continuous as a set-valued mapping, and every weakly invariant set on which W is constant is contained in A, then A is pointwise asymptotically stable.*

4.3 Weakened Lyapunov sufficient conditions

Below, $\sup_t \operatorname{dom}\phi := \sup\{t \mid \exists j \text{ so that } (t,j) \in \operatorname{dom}\phi\}$ and $\sup_j \operatorname{dom}\phi$ has a symmetric definition.

Theorem 4.6. *If there exist a weak set-valued Lyapunov function W for a closed $A \subset \mathbb{R}^n$, and either (a) or (b) holds:*

(a) *the hybrid system (1) is forward complete in the t direction, in the sense that every maximal solution ϕ to (1) is such that $\sup_t \operatorname{dom}\phi = \infty$, and there exists a continuous function $c : \operatorname{dom} W \to [0,\infty)$, positive definite with respect to A, such that (c) in Definition 4.1 holds;*
(b) *the hybrid system (1) is forward complete in the j direction, in the sense that every maximal solution ϕ to (1) is such that $\sup_j \operatorname{dom}\phi = \infty$, and there exists a continuous function $c : \operatorname{dom} W \to [0,\infty)$, positive definite with respect to A, such that (d) in Definition 4.1 holds;*

then A is pointwise asymptotically stable.

Proof. Suppose (a) holds. As in the proof of Theorem 4.3, take $\phi \in \mathcal{S}(U \cap \operatorname{dom} W)$. Then ϕ is complete, $\sup_t \operatorname{dom} \phi = \infty$, and $\phi(t,j) \in \operatorname{dom} W$ for all $(t,j) \in \operatorname{dom} \phi$. Pick any $(T,J) \in \operatorname{dom} \phi$ and suppose that (6) holds. Then

$$W(\phi(T,J)) + \left(\sum_{j=0}^{J} \int_{t_j}^{\min\{t_{j+1},T\}} c(\phi(s)) \, ds \right) \mathbb{B}$$

is a subset of $W(\phi(0,0))$. Lyapunov stability of each $a \in A$ follows as in the proof of Theorem 4.3. To prove convergence of a solution ϕ to a point in A, note that

$$\sum_{j=0}^{J} \int_{t_j}^{\min\{t_{j+1},T\}} c(\phi(s)) \, ds$$

is bounded over all $(T,J) \in \operatorname{dom} \phi$. Thus, since ϕ is complete in the t direction, there exists a sequence $(t_k, j_k) \in \operatorname{dom} \phi$, with $t_k \to \infty$ as $k \to \infty$, such that $c(\phi(t_k, j_k))$ converges to 0 as $k \to \infty$. Arguments as in Theorem 4.3 finish the proof. The case of (b) is similar.

5. FINITE-LENGTH LYAPUNOV FUNCTIONS

This section proposes Lyapunov inequalities which not only ensure the decrease of a Lyapunov function, but also imply that the length of solutions is finite. The implication is by inspired by Bhat and Bernstein (2010), where it was used for semistability in differential equations.

Definition 5.1. *A continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$ is a finite-length Lyapunov function for a closed $A \subset \mathbb{R}^n$ if V is positive definite with respect to A, and there exist continuous $c, d : \mathbb{R}^n \to [0, \infty)$, positive definite with respect to A, such that the following hold:*

(c) for every $x \in C$, $f \in F(x)$,
$$\nabla V(x) \cdot f \leq -c(x) - \|f\|,$$
(d) for every $x \in D$, $g \in G(x)$,
$$V(g) - V(x) \leq -d(x) - \|g - x\|.$$

The name "finite-length Lyapunov function" comes from the fact that V as in Definition 5.1 implies that for every solution ϕ, for every $(T,J) \in \operatorname{dom} \phi$ and with the representation (6), the quantity that can be considered the length of ϕ from $(0,0)$ to (T,J), namely

$$\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} \|\dot{\phi}(t,j)\| \, ds + \sum_{j=1}^{J} \|\phi(t_j, j) - \phi(t_j, j-1)\|,$$

is bounded above by $V(\phi(0,0)) - V(\phi(T,J))$, and in particular it is finite.

In Bhat and Bernstein (2010), for differential equations, the condition $\dot{V} + \|f\| \leq 0$ was used. The stricter conditions (c), (d) above let one relate V as above to a set-valued Lyapunov function. Indeed, a corollary follows from Theorem 4.3 by showing that $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, given by

$$W(x) = x + V(x)\mathbb{B} \qquad \forall x \in \mathbb{R}^n,$$

is a set-valued Lyapunov function for A.

Corollary 5.2. *If there exists a finite-length Lyapunov function $V : \mathbb{R}^n \to [0, \infty)$ for a closed $A \subset \mathbb{R}^n$, then A is pointwise asymptotically stable.*

6. PARTIAL POINTWISE ASYMPTOTIC STABILITY

This sections extends the previous results to the case of pointwise asymptotic stability of only a part of the state. Let $n = n_1 + n_2$, write $x \in \mathbb{R}^n$ as $x = (x_1, x_2)$ with $x_i \in \mathbb{R}^{n_i}$. Similar notation will apply to solutions ϕ of (1). Let $A \in \mathbb{R}^n$ be given by $A = A_1 \times \mathbb{R}^{n_2}$, where $A_1 \subset \mathbb{R}^{n_1}$ is nonempty and closed.

Definition 6.1. *A set $A \subset \mathbb{R}^n$ is partially pointwise asymptotically stable if*

(a) every $a \in A$ is partially Lyapunov stable, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to (1) with $\|\phi_1(0,0) - a_1\| < \delta$ satisfies $\|\phi_1(t,j) - a_1\| < \varepsilon$ for every $(t,j) \in \operatorname{dom} \phi$, and
(b) for every maximal solution ϕ to (1), $\lim_{t+j \to \infty} \phi_1(t,j)$ exists and is contained in A_1.

Definition 6.2. *A set-valued map $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n_1}$ is a weak partial set-valued Lyapunov function for A if*

(a) $W(x) = \{x_1\}$ for every $x \in A$ and $x_1 \in W(x)$ for every $x \in \overline{C} \cup D \cup G(D)$;
(b) W is locally bounded and, at every $x \in A$, it is outer semicontinuous in x_1 uniformly in x_2;

and (c) and (d) of Definition 4.4 hold.

Under the local boundedness assumption, the outer semicontinuity of W in x_1 uniform in x_2 at $x \in A$ means that for every $x_1 \in A_1$, every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x_2 \in \mathbb{R}^{n_2}$, $W(x_1 + \delta \mathbb{B}, x_2) \subset W(x) + \varepsilon \mathbb{B}$. This property holds if W does not depend on x_2, i.e., $W(x) = W(x_1)$, and W, understood as a mapping from $\mathbb{R}^{n_1}$, is outer semicontinuous at every $x_1 \in A_1$.

Proposition 6.3. *If there exist a weak partial set-valued Lyapunov function W for A, then every $a \in A$ is partially Lyapunov stable.*

Proof. As before, from (c) and (d) of Definition 4.4 it follows that for every solution ϕ one has

$$W(\phi(t,j)) \subset W(\phi(0,0)) \quad \forall (t,j) \in \operatorname{dom} \phi.$$

Pick $a \in A$ and $\varepsilon > 0$. By (a) of Definition 6.2, $W(a) = \{a_1\}$ and by (b) of Definition 6.2, there exists $\delta > 0$ such that for every $x_1 \in a_1 + \delta \mathbb{B}$, every $a_2 \in \mathbb{R}^{n_2}$, $W(x_1, a_2) \subset W(a) + \varepsilon \mathbb{B} = a_1 + \varepsilon \mathbb{B}$. For any $\phi \in \mathcal{S}(a_1 + \delta \mathbb{B}, a_2)$,

$$\phi_1(t,j) \in W(\phi(t,j)) \subset W(\phi(0,0)) \subset a_1 + \varepsilon \mathbb{B}$$

for every $(t,j) \in \operatorname{dom} \phi$. Lyapunov stability of a is shown.

Theorem 6.4. *Suppose (1) satisfies the hybrid basic assumptions. If there exist a weak partial set-valued Lyapunov function W for A, which is continuous as a set-valued mapping, and every weakly invariant set on which W is constant is contained in A, then for every bounded solution ϕ, $\lim_{t+j \to \infty} \phi_1(t,j)$ exists and belongs to A_1.*

Proof. Pick a bounded $\phi \in \mathcal{S}$. The set-valued mapping $(t,j) \mapsto W(\phi(t,j))$ is nonincreasing, and hence has a limit, denoted by $K \subset \mathbb{R}^{n_1}$, and given by $K = \bigcap_{(t,j) \in \operatorname{dom} \phi} W(\phi(t,j))$. Let $\Omega(\phi)$ be the nonempty and compact ω-limit of ϕ, which, thanks to hybrid basic assumptions, is weakly invariant. Continuity of W and convergence of $W(\phi(t,j))$ imply that for every $x \in \Omega(\phi)$,

$W(x) = K$. Hence, by assumption, $\Omega(\phi) \subset A$. Fix $a = (a_1, a_2) \in \Omega(\phi) \subset A$ and $\varepsilon > 0$. Partial Lyapunov stability of a was established in Proposition 6.3. Let $\delta > 0$ come from partial Lyapunov stability of this a. By the definition of $\Omega(\phi)$, there exists $(t, j) \in \operatorname{dom}\phi$ so that $\|\phi(t, j) - a\| < \delta$, and thus also $\|\phi_1(t, j) - a_1\| < \delta$. Partial Lyapunov stability implies that $\|\phi_1(t', j') - a_1\| < \varepsilon$ for every $(t', j') \in \operatorname{dom}\phi$ with $t' + j' > t + j$. Thus, for every $a' \in \Omega(\phi)$ it must be that $\|a_1' - a_1\| \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, $a_1' = a_1$. Hence $\phi_1(t, j)$ converges to a_1.

Example 6.5. Consider the hybrid system mentioned in Example 4.2, but with a constraint on the timing of jumps. The notation below is borrowed from 4.2 and z replaces x.

$$\dot{z}_i = \alpha(\operatorname{ave} z - z_i), i = 1, 2, \ldots, I, \ \dot{\tau} = -1 \ \text{if} \ \ \tau \geq 0$$
$$z_i^+ = \beta z_i + (1 - \beta) \operatorname{ave} z, i = 1, 2, \ldots, I, \ \tau^+ = T \ \text{if} \ \ \tau = 0$$

Let $A = A_1 \times \mathbb{R}$, where A_1 is the diagonal (4). Then $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n-1}$ ($n - 1$ is the dimension of z) given by

$$W(z, T) = a(z) + V(z)\mathbb{B}$$

is a weak partial set-valued Lyapunov function for A. It is continuous, uniformly in τ as it does not depend on τ. One can verify directly that every solution to this system is bounded. Hence, Proposition 6.3 and Theorem 6.4 can be used to deduce partial pointwise asymptotic stability of A. The only work to be done is then to establish that if W is constant along a complete solution ϕ, then $\phi_1(t, j) \in A_1$ for every $(t, j) \in \operatorname{dom}\phi$. This is clear from Example 4.2.

Example 6.6. Recall the example informally discussed in the introduction:

$$\dot{z}_i = a - z_i, \ \dot{a} = 0, \ \dot{\tau} = -1 \ \ \text{if} \ \ \tau \geq 0,$$
$$z_i^+ = z_i, \ a^+ \in \operatorname{con}\{z_1, z_2, \ldots, z_I\}, \ \tau^+ = T \ \ \text{if} \ \ \tau = 0,$$

with state $x = (x_1, x_2)$ where $x_1 = (z_1, z_2, \ldots, z_I, a)$ and $x_2 = \tau$. Recall the set-valued mapping

$$w(z, a) = \operatorname{con}\{z_1, z_2, \ldots, z_I, a\},$$

and define

$$W(x) = W(x_1) = \underbrace{w(z, a) \times w(z, a) \times \cdots \times w(z, a)}_{I+1 \text{ copies}}.$$

Then W is a continuous weak partial set-valued Lyapunov function for $A = A_1 \times \mathbb{R}$, where

$$A_1 = \{x_1 \mid z_1 = z_2 = \cdots = z_I = a\},$$

and partial Lyapunov stability of $A_1 \times \mathbb{R}$ follows from Proposition 6.3. This hybrid system satisfies the hybrid basic assumptions, and to apply Theorem 6.4, one only needs to argue that if W is constant along a solution, then the solution is contained in A_1.

A different approach to the hybrid system in Example 6.5 is possible through a partial set-valued Lyapunov function which decreases strictly. The general definition and result are below. That the function W in Example 6.5 meets the definition below follows from Example 4.2.

Definition 6.7. *A set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a* partial set-valued Lyapunov function *for (1) and A if it is a weak partial set-valued Lyapunov function and there exist continuous functions $c, d : \mathbb{R}^{n_1} \to [0, \infty)$, positive definite with respect to A_1, such that the following conditions hold:*

(c') for every solution $\phi : [0, T] \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,

$$W(\phi(t)) + \int_0^t c(\phi_1(s)) \, ds\mathbb{B} \subset W(\phi(0)) \quad \forall t \in [0, T].$$

(d')

$$W(G(x)) + d(x_1)\mathbb{B} \subset W(x) \quad \forall x \in D.$$

Theorem 6.8. *If there exist a partial set-valued Lyapunov function W for (1) and A, then A is partially pointwise asymptotically stable.*

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