Incremental Graphical Asymptotic Stability for Hybrid Dynamical Systems

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Abstract This chapter introduces an incremental asymptotic stability notion for sets of hybrid trajectories $\mathscr{S}$. The elements in $\mathscr{S}$ are functions defined on hybrid time domains, which are subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ with a specific structure. For this abstract system, incremental asymptotic stability is defined as the property of the graphical distance between every pair of solutions to the system having stable behavior (incremental graphical stability) and approaching zero asymptotically (incremental graphical attractivity). Necessary conditions for $\mathscr{S}$ to have such properties are presented. When $\mathscr{S}$ is generated by hybrid systems given in terms of hybrid inclusions, that is, differential equations and difference equations with state constraints, further necessary conditions on the data are highlighted. In addition, sufficient conditions for incremental graphical asymptotic stability involving the data of the hybrid inclusion are presented. Throughout the chapter, examples illustrate the notions and results.

1 Introduction

1.1 Motivation

In contrast to asymptotic stability, which can be interpreted as a property of each system solution relative to a set, incremental stability consists of a property for every pair of solutions to the system. More precisely, for a continuous-time system of the form $\dot{x} = f(x)$, the uniform version of such a property requires every pair of solutions $t \mapsto \phi_1(t)$ and $t \mapsto \phi_2(t)$ to $\dot{x} = f(x)$ to satisfy

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$$|\phi_1(t) - \phi_2(t)| \le \beta(|\phi_1(0) - \phi_2(0)|, t) \tag{1}$$

for each t in the domain of definition of ϕ_1 and ϕ_2, where β is a class-$\mathscr{K}\mathscr{L}$ function; see, e.g., [1, 2, 3]. The bound (1) implies that the Euclidean distance between two solutions is upper bounded by a function of the difference between their initial conditions and also decreases as t gets arbitrarily large (when the domain of definition of the solutions is unbounded to the right).

Unfortunately, the incremental stability notions available in the literature (most of which are for continuous-time systems) cannot be applied directly to systems with variables that can change continuously and, at times, jump. These systems, known as *hybrid systems*, are capable of modeling a wide range of complex dynamical systems, including robotic, automotive, and power systems as well as natural processes. Hybrid systems are dynamical systems that exhibit characteristics typical of both continuous-time and discrete-time behaviors. As a set stability theory in terms of Lyapunov functions is available (see [4, 5]), the availability of an incremental stability notion for this class of systems would enable the study of similar properties for them as the current notion for continuous-time systems allows. However, as we make clear in Section 2, mismatch of jump times and length of domains of pairs of solutions starting nearby makes characterizing and guaranteeing incremental stability properties in hybrid systems difficult.

1.2 Results in this chapter

In this chapter, we introduce a notion of graphical incremental asymptotic stability for a set of hybrid trajectories, which we denote $\mathscr{S}$ and contains all trajectories that cannot be further extended (namely, they are maximal). A set of hybrid trajectories can be considered an abstract system on itself, or can be generated using hybrid inclusions. For such class of systems, we establish necessary and sufficient conditions for graphical incremental asymptotic stability. More precisely, we establish the following results:

1. The set $\mathscr{S}$ is neither graphically incrementally stable nor graphically incrementally attractive if there exists two elements in $\mathscr{S}$ with nearby initial conditions such that the amount of flow or jump is not the same, as in Proposition 1 and Proposition 2 and Proposition 3.
2. The set $\mathscr{S}$ is not incrementally graphically stable if there exists one element in $\mathscr{S}$ that is not unique, as in Proposition 4.
3. When elements in $\mathscr{S}$ are generated by all maximal solutions to a hybrid system given in terms of a hybrid inclusion with a nonempty jump set D, under mild assumptions, Theorem 1 reveals that it is necessary to have a finite-time convergence like property from points that are nearby the jump set D. Proposition 6 provides a sufficient condition to guarantee such a property.
4. In Theorem 2, sufficient conditions for a set $\mathscr{S}$ consisting of all maximal solutions to a hybrid inclusion to be incrementally graphically asymptotically stable

are given. A special case of this result (with the jump set D being discrete) is established in Corollary 1. Both results require the flow map to induce a contraction during flows.

5. An extension of the result in Theorem 2 is presented in Theorem 3, where the jump map is required to be a weak contraction mapping.

To the best of our knowledge, the notion of incremental stability and its properties for hybrid systems have not been thoroughly studied before, only discussed briefly in [6] for a class of transition systems in the context of bisimulations, and in [7] for a particular class of hybrid systems prioritizing ordinary time t; see also related definitions in [8].

1.3 Organization of the chapter

The remainder of this chapter is organized as follows. Section 2 briefly discusses notions of incremental stability for continuous-time (discrete) systems and introduces a notion of graphical incremental stability for sets of hybrid trajectories. Section 3 establishes several sufficient and necessary conditions for the proposed notion. Examples are discussed throughout the chapter to illustrate the results.

Notation: The set $\mathbb{B}$ denotes a closed unit ball in Euclidean space with appropriate dimension. Given a set $S \subset \mathbb{R}^n$, the closure of S is the intersection of all closed sets containing S, denoted by $\overline{S}$; S is said to be discrete if nonempty and there exists $\delta > 0$ such that for each $x \in S$, $(x+\delta\mathbb{B}) \cap S = \{x\}$; $\overline{\text{con}}S$ is the closure of the convex hull of the set S. $\mathbb{R}_{\geq 0} := [0,\infty)$ and $\mathbb{N} := \{0,1,2,\dots\}$. Given vectors $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $|v|$ defines the Euclidean vector norm $|v| = \sqrt{v^\top v}$, and $[v^\top\ w^\top]^\top$ is equivalent to (v,w); given a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, i.e., $P = P^\top > 0$, the weighted norm $|v|_P = \sqrt{v^\top P v}$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by dom f, i.e., dom $f := \{x \in \mathbb{R}^m :\ f(x) \text{ is defined}\}$. The range of f is denoted by rge f, i.e., rge $f := \{f(x) : x \in \operatorname{dom} f\}$. The right limit of the function f is defined as $f^+(x) := \lim_{v\to 0^+} f(x+v)$ if it exists. Given a point $y \in \mathbb{R}^n$ and a closed set $\mathscr{A} \subset \mathbb{R}^n$, $|y|_{\mathscr{A}} := \inf_{x\in\mathscr{A}} |x-y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathscr{K}_\infty$ function, also written $\alpha \in \mathscr{K}_\infty$, if α is zero at zero, continuous, strictly increasing, and unbounded; α is positive definite, also written $\alpha \in \mathscr{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathscr{KL}$ function, also written $\beta \in \mathscr{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r,s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r,s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a function $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^r$, $\nabla_x f(x,y) := \frac{\partial f}{\partial x}(x,y)$. Given a matrix $A \in \mathbb{R}^{n\times n}$, eig$(A)$ is the set of eigenvalues of A; $\overline{\lambda}(A) = \max\{\text{Re}(\lambda) : \lambda \in \text{eig}(A)\}$; $\underline{\lambda}(A) = \min\{\text{Re}(\lambda) : \lambda \in \text{eig}(A)\}$; $|A| := \max\{|\lambda|^{\frac{1}{2}} :\ \lambda \in \text{eig}(A^\top A)\}$. Given a real number $x \in \mathbb{R}$, floor(x) is the closest integer to x from below. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function with respect to a set $\mathscr{A}$ if V is continuously differentiable and such that $c_1(|x|_{\mathscr{A}}) \leq V(x) \leq c_2(|x|_{\mathscr{A}})$ for all $x \in \mathbb{R}^n$ and some functions $c_1, c_2 \in \mathscr{K}_\infty$. Given a set $\mathscr{A} \subset \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ and a metric d on $\mathbb{R}^n$, the distance $|x|^d_{\mathscr{A}} := \sup_{z\in\mathscr{A}} d(x,z)$.

2 Definition of Incremental Stability for Hybrid Systems

Informally, incremental stability is typically defined as the property of every pair of trajectories staying close when they start close (stability) and, as time gets large, converging to each other (attractivity). To formally state this notion, let the set of trajectories to a system with state in $\mathbb{R}^n$ be denoted by $\mathscr{S}$ and the time variable parameterizing such trajectories be denoted by s. The variable s parameterizes the trajectories in forward time from $s_\circ = 0$. This parameter takes values from $\mathbb{R}_{\geq 0}$ when the system is a continuous-time system, in which case $\mathscr{S}$ is a set of continuous-time trajectories and every element $\phi \in \mathscr{S}$ has a domain $\operatorname{dom}\phi$ that is a subset of $\mathbb{R}_{\geq 0}$. The parameter takes values from $\mathbb{N}$ when the system is a discrete-time system, in which case $\mathscr{S}$ is a set of discrete-time trajectories and elements in $\mathscr{S}$ have a domain that is a subset of $\mathbb{N}$. Let the function d denote a metric on $\mathbb{R}^n \times \mathbb{R}^n$ measuring the distance between pairs of elements in $\mathscr{S}$. An element $\phi \in \mathscr{S}$ is said to be maximal if there is no $\phi' \in \mathscr{S}$ such that ϕ is a proper truncation of ϕ' and complete if $\operatorname{dom}\phi$ is unbounded. Since we are interested in the behavior of maximal elements in $\mathscr{S}$, without loss of generality, from now on, it is assumed that $\mathscr{S}$ is a set of maximal hybrid trajectories.

The set of trajectories $\mathscr{S}$ is incrementally asymptotically stable with respect to a metric d if it is incrementally stable, in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\phi_1, \phi_2 \in \mathscr{S}, \quad d(\phi_1(s_\circ), \phi_2(s_\circ)) \leq \delta \\ &\qquad \Rightarrow \quad \operatorname{dom}\phi_1 = \operatorname{dom}\phi_2, \quad d(\phi_1(s), \phi_2(s)) \leq \varepsilon \quad \forall s \in \operatorname{dom}\phi_1 (= \operatorname{dom}\phi_2) \end{aligned} \tag{2}$$

and incrementally attractive, in the sense that there exists $\mu > 0$ such that

$$\begin{aligned} &\phi_1, \phi_2 \in \mathscr{S}, \quad d(\phi_1(s_\circ), \phi_2(s_\circ)) \leq \mu \\ \Rightarrow \quad &\operatorname{dom}\phi_1 = \operatorname{dom}\phi_2 \text{ unbounded}, \qquad \lim_{s\to\infty} d(\phi_1(s), \phi_2(s)) = 0 \end{aligned} \tag{3}$$

When incremental attractivity holds for any $\mu > 0$, we say that the set of trajectories $\mathscr{S}$ is globally incrementally stable.

The notion defined above captures the nominal version of [2, Definition 2.1] for continuous-time systems when the elements in $\mathscr{S}$ are generated by a nonlinear continuous-time system of the form $\dot{x} = f(x)$. It also captures the notion for discrete-time systems of the form $x^+ = g(x)$, see, e.g., [9, 10]. To assess this notion for the hybrid case, we define hybrid trajectories as functions on hybrid time domains.

Definition 1 (hybrid time domain). A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}], j\right)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T,J) \in E$, $E \cap ([0,T] \times \{0,1,\ldots,J\})$ is a compact hybrid time domain.

Given a hybrid time domain E, we define

$$\sup_t E := \sup_{(t,j)\in E} t, \qquad \sup_j E := \sup_{(t,j)\in E} j.$$

Definition 2 (hybrid trajectory). A function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is a *hybrid trajectory* (or hybrid arc) if $\operatorname{dom}\phi$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t,j)$ is locally absolutely continuous on the interval $I_j := \{t \;:\; (t,j) \in \operatorname{dom}\phi\}$.

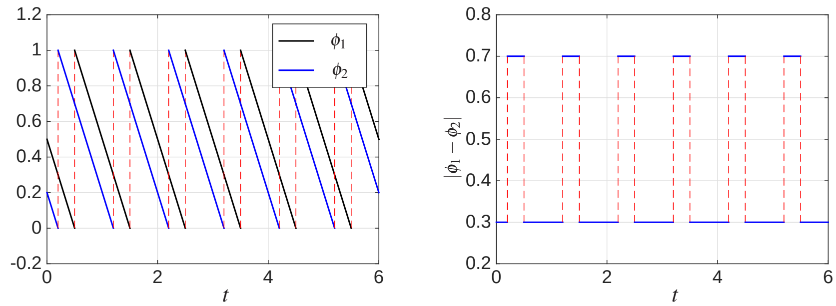
Remark 1. When every $\phi \in \mathscr{S}$ is such that $\operatorname{dom}\phi \subset \mathbb{R}_{\geq 0} \times \{0\}$ and $\operatorname{dom}\phi$ has more than one point, $\mathscr{S}$ is a set of continuous-time trajectories, while when every $\phi \in \mathscr{S}$ is such that $\operatorname{dom}\phi \subset \{0\} \times \mathbb{N}$ and $\operatorname{dom}\phi$ has more than one point, $\mathscr{S}$ is a set of discrete-time trajectories. Finally, when every $\phi \in \mathscr{S}$ is such that ϕ is either bounded or $\operatorname{dom}\phi$ is unbounded, $\mathscr{S}$ is said to be pre-forward complete.

For the case when the elements in $\mathscr{S}$ are hybrid trajectories, it is natural to consider an extension of the notion above when s takes values from $\mathbb{R}_{\geq 0} \times \mathbb{N}$ and is written as $s = (t,j)$, and $s_\circ = (0,0)$. Unfortunately, there are several subtleties that make such extension of the notion above limiting for hybrid systems, some of which we illustrate next in simple examples. The first example illustrates issues measuring the distance between a pair of trajectories for a system that one would expect to be incrementally stable (but not incrementally attractive). The second example illustrates issues in measuring such distance for pairs of trajectories with dramatically different hybrid time domains.

Example 1 (mismatch of event times). Let $\mathscr{S}$ be the set of hybrid trajectories with (maximal and complete) elements ϕ defined as

$$\begin{aligned}\phi(t,j) =& \phi(0,0) - (t-j)\\ &\forall (t,j) \;:\; t \in \left[\max\{j-1,0\} + \operatorname{ceil}\left(\frac{j}{j+1}\right)\phi(0,0),\, j+\phi(0,0)\right], j \in \mathbb{N}\end{aligned}$$

with $\phi(0,0) \geq 0$. (This set of trajectories can be generated using the hybrid inclusion given in Example 7.) Each trajectory in $\mathscr{S}$ reaches zero in finite flow time, at which event is reset to one instantaneously and from where it periodically reaches zero and gets reset to one. Figure 1(a) shows two trajectories with initial values within $\delta = 0.3$. This figure appears to suggest that trajectories from $\mathscr{S}$ starting close stay close. However, condition (2) does not hold unless $\phi_1(0,0) = \phi_2(0,0)$. In fact, consider two such trajectories, ϕ_1 and ϕ_2, with initial values satisfying $|\phi_1(0,0) - \phi_2(0,0)| \leq$



(a) The projections of two hybrid trajectories from $\phi_1(0,0) = 0.5$ and $\phi_2(0,0) = 0.3$ on the t direction.

(b) Euclidean distance between ϕ_1 and ϕ_2.

Fig. 1 Two elements ϕ_1 and ϕ_2 from the set $\mathscr{S}$ given in Example 1. The Euclidean distance, which, precisely, is given by $|\phi_1(t, j_1(t)) - \phi_2(t, j_2(t))|$ for all $(t, j_i(t)) \in \operatorname{dom}\phi_i$, $j_i(t) = \min_{(t,j_i')\in\operatorname{dom}\phi_i} j_i'$, assumes the value 0.7 for 0.3 seconds periodically. On the other hand, the "graphical distance" from ϕ_1 to ϕ_2 is zero for $\varepsilon = 0.3$, while the "graphical distance" from ϕ_2 to ϕ_1 converges to zero in 0.3 seconds.

δ and $\phi_1(0,0) \neq \phi_2(0,0)$. First, $\operatorname{dom}\phi_1 \neq \operatorname{dom}\phi_2$ since $(\phi_1(0,0),1) \in \operatorname{dom}\phi_1$ and $(\phi_2(0,0),1) \in \operatorname{dom}\phi_2$ but $\phi_1(0,0) \neq \phi_2(0,0)$. Without loss of generality, assume $0 < \phi_1(0,0) < \phi_2(0,0)$. Then, even when the condition of equal domains is omitted, we have

$$|\phi_1(t_1,1) - \phi_2(t_1,0)| = |1 - \phi_2(t_1,0)| = |1 - \phi_2(0,0) + \phi_1(0,0)|,$$

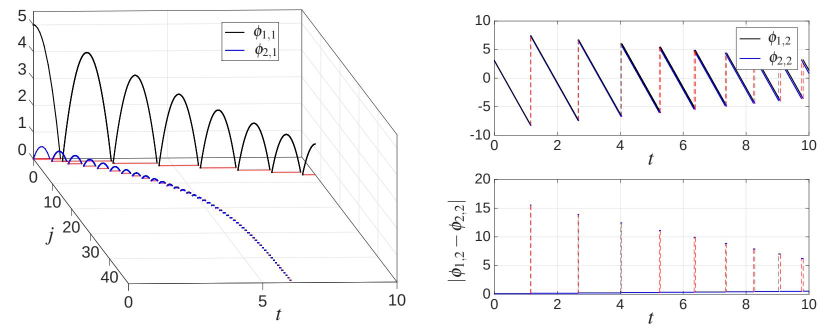
where we used the fact that $t_1 = \phi_1(0,0)$. No matter how small $\delta \in (0,1)$ is chosen, $|\phi_1(t_1,1) - \phi_2(t_1,0)| \geq 1 - \delta$. This property makes it impossible for the Euclidean distance between ϕ_1 and ϕ_2 to satisfy the ε-δ criterion in (2). In such case, the Euclidean distance (or any other metric d) may not be a good candidate of a distance function for the study of incremental properties. $\triangle$

Example 1 suggests that a notion of incremental stability for hybrid systems has to allow for a mismatch of the jump times of two hybrid trajectories. This example also highlights that the pointwise (in $s = (t, j)$) distance is not appropriate for the purposes of defining incremental stability for sets of hybrid trajectories.

Example 2 (mismatch of length of domains). Let $\mathscr{S}$ be the set of hybrid trajectories with elements ϕ defined as

$$\phi(t,j) = \begin{bmatrix} -\frac{\gamma}{2}(t-t_j)^2 + \phi_2(t_j,j)(t-t_j) + \phi_1(t_j,j) \\ -\gamma(t-t_j) + \phi_2(t_j,j) \end{bmatrix} \qquad \forall (t,j) \in \bigcup_{i\in\mathbb{N}} ([t_i,t_{i+1}] \times \{i\})$$

with $\phi(0,0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, where $t_0 = 0$, $t_1 = \frac{\phi_2(0,0)+\sqrt{\phi_2(0,0)^2+2\gamma\phi_1(0,0)}}{\gamma}$,



(a) The first component (height $\phi_{i,1}$) of hybrid trajectories starting at $\phi_1(0,0) = (5,0)$ and $\phi_2(0,0) = (0,3)$.

(b) The projection of the second component (velocity $\phi_{i,2}$) of hybrid trajectories from $\phi_1(0,0) = (3,3)$ and $\phi_2(0,0) = (3,3.1)$ on the t direction.

Fig. 2 Hybrid trajectories in Example 2. The Euclidean distance, which is $|\phi_{1,2}(t, j_1(t)) - \phi_{2,2}(t, j_2(t))|$ for all $(t, j_i(t)) \in \operatorname{dom}\phi_i$, has repetitive large peaks, where $j_i(t) = \min_{(t,j_i)\in \operatorname{dom}\phi_i} j_i$.

$$t_j = t_1 + \frac{2(\gamma t_1 - \phi_2(0,0))}{\gamma} \sum_{i=1}^{j-1} \lambda^i \qquad \forall j \in \mathbb{N} \setminus \{0,1\}$$
$$\phi_2(t_{j+1}, j+1) = -\lambda \phi_2(t_{j+1}, j) \qquad \forall j \in \mathbb{N}$$

$\gamma > 0$, and $\lambda \in (0,1)$. These trajectories capture the evolution of the height (ϕ_1) and vertical velocity (ϕ_2) of a ball bouncing on a ground at zero height, where γ represents the gravity constant and λ the restitution coefficient. A hybrid inclusion generating this set of hybrid trajectories is given in [4, Example 1.1 and 2.12]. Each element $\phi \in \mathscr{S}$ is such that

$$\sup_t \operatorname{dom}\phi = \frac{\phi_2(0,0)}{\gamma} + \frac{1+\lambda}{\gamma(1-\lambda)} \sqrt{\phi_2(0,0)^2 + 2\gamma\phi_1(0,0)} \tag{4}$$

Figure 2(a) shows the position (first) component of two hybrid trajectories ($\phi_i = (\phi_{i,1}, \phi_{i,2})$ for $i \in \{1,2\}$) from initial conditions $\phi_1(0,0) = (5,0)$ (ball starting at a positive height with zero velocity) and $\phi_2(0,0) = (0,3)$ (ball starting at the ground with a positive velocity). As Figure 2(a) shows, the jumps in ϕ_2 accumulate at about $t = 6$sec while ϕ_1 is still describing the motion of the ball bouncing.

Given two elements $\phi_1, \phi_2 \in \mathscr{S}$ with $\phi_1(0,0) \neq \phi_2(0,0)$, according to (4), $\sup_t \operatorname{dom}\phi_1 \neq \sup_t \operatorname{dom}\phi_2$. Without loss of generality, assuming that $\sup_t \operatorname{dom}\phi_2 < \sup_t \operatorname{dom}\phi_1$, then we have that ϕ_2 is not defined at points $(t', j') \in \operatorname{dom}\phi_1$ with $t' + j' \geq \sup_t \operatorname{dom}\phi_2$. Hence, at such points, it is not possible to measure the distance between ϕ_1 and ϕ_2. Note that for such points (t', j') we have that $(t', j) \notin \operatorname{dom}\phi_2$ for any $j \in \mathbb{N}$, which indicates that it is not possible to relax the incremental stability notion by instead requiring that the distance between the trajectories be small for each common t and potentially different values of the jump parameter j. Even when we omit such points, for points $(t, j) \in \operatorname{dom}\phi_2$ with t close to $\sup_t \operatorname{dom}\phi_2$ and points

$(t, j'') \in \operatorname{dom}\phi_1$, we have that j is much larger than j'' since j grows unbounded as t approaches $\sup_t \operatorname{dom}\phi_2$. This fact makes comparing trajectories using the graphical distance in this particular set of hybrid solutions very difficult. A similar situation is encountered if, instead, the pointwise distance is used. As shown in Figure 2(b), the pointwise distance between velocity (second) components of two solutions ($\phi_{i,2}$ for $i \in \{1,2\}$) has repetitive large peaks, even though they are initialized very close to each other. △

While Example 1 already has elements in $\mathscr{S}$ with different domains, Example 2 pinpoints a key difficulty in measuring the distance between solutions with jump times that accumulate, namely, Zeno solutions. In fact, when accumulation of events occur in finite time t, determining the appropriate distance function to certify incremental stability is rather difficult since, when the accumulation time depends on the initial condition as in Example 2, the distance between the trajectories may not be quantifiable over an unbounded set. On the other hand, a notion of incremental stability for a set of continuous-time trajectories or for a set of discrete-time trajectories with elements having different time domains can be formulated by only requiring the stability condition to hold over the intersection of the domains of definition of every pair of trajectories starting nearby.

Motivated by the issues mentioned above, we propose a notion of incremental asymptotic stability that employs the graphical distance between the graphs defined by the hybrid trajectories.

Definition 3 ([4, Definition 5.20]). The graph of a hybrid trajectory $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is a set in $\mathbb{R}^{n+2}$ given by

$$\operatorname{gph}\phi = \{(t, j, x) : (t, j) \in \operatorname{dom}\phi,\ x = \phi(t, j)\}. \tag{5}$$

To measure the distance between the graphs of two hybrid trajectories, given a metric d, we use the following graphical distance notion for hybrid trajectories.

Definition 4 ([4, Definition 4.11]). Given $\varepsilon > 0$, two hybrid trajectories ϕ_1 and ϕ_2 are *graphically ε-close with respect to d* if

(a) for each $(t, j) \in \operatorname{dom}\phi_1$ there exists s such that $(s, j) \in \operatorname{dom}\phi_2$, $|t-s| \le \varepsilon$, and

$$d(\phi_1(t, j), \phi_2(s, j)) \le \varepsilon,$$

(b) for each $(t, j) \in \operatorname{dom}\phi_2$ there exists s such that $(s, j) \in \operatorname{dom}\phi_1$, $|t-s| \le \varepsilon$, and

$$d(\phi_2(t, j), \phi_1(s, j)) \le \varepsilon.$$

To characterize the distance between the graphs of two hybrid arcs over a finite horizon, we use the following graphical (τ, ε)-closeness notion for hybrid trajectories.

Definition 5 ([4, Definition 5.23]). Given $\tau, \varepsilon > 0$, two hybrid trajectories ϕ_1 and ϕ_2 are *graphically (τ, ε)-close with respect to d* if

(a) for each $(t,j) \in \operatorname{dom}\phi_1$ with $t+j \le \tau$ there exists s such that $(s,j) \in \operatorname{dom}\phi_2$, $|t-s| \le \varepsilon$, and

$$d(\phi_1(t,j),\phi_2(s,j)) \le \varepsilon,$$

(b) for each $(t,j) \in \operatorname{dom}\phi_2$ with $t+j \le \tau$ there exists s such that $(s,j) \in \operatorname{dom}\phi_1$, $|t-s| \le \varepsilon$, and

$$d(\phi_2(t,j),\phi_1(s,j)) \le \varepsilon.$$

To characterize the property of hybrid trajectories graphically converging to each other, we introduce the following notion.

Definition 6. Given $\varepsilon > 0$, two hybrid trajectories ϕ_1 and ϕ_2 are *eventually graphically ε-close with respect to d* if

(a) there exists $T > 0$ such that for each $(t,j) \in \operatorname{dom}\phi_1$ and $t+j \ge T$, there exists $(s,j) \in \operatorname{dom}\phi_2$ satisfying $|t-s| \le \varepsilon$ and

$$d(\phi_1(t,j),\phi_2(s,j)) \le \varepsilon, \tag{6}$$

(b) there exists $T > 0$ such that for each $(t,j) \in \operatorname{dom}\phi_2$ and $t+j \ge T$, there exists $(s,j) \in \operatorname{dom}\phi_1$ satisfying $|t-s| \le \varepsilon$ and

$$d(\phi_2(t,j),\phi_1(s,j)) \le \varepsilon. \tag{7}$$

Remark 2. If two hybrid trajectories ϕ_1 and ϕ_2 are not complete, then, the property in Definition 6 holds for free. In particular, the property would hold vacuously for $T > \max\{T_1+J_1, T_2+J_2\}$, where $T_1 = \sup_t \operatorname{dom}\phi_1$, $J_1 = \sup_j \operatorname{dom}\phi_1$, $T_2 = \sup_t \operatorname{dom}\phi_2$ and $J_2 = \sup_j \operatorname{dom}\phi_2$.

Now, we are ready to define incremental asymptotic stability for sets of hybrid trajectories.

Definition 7 (incremental graphical asymptotic stability). The set of hybrid trajectories $\mathscr{S}$ is said to be

1. *incrementally graphically stable* (δS) *with respect to d* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\phi_1,\phi_2 \in \mathscr{S}, \quad d(\phi_1(0,0),\phi_2(0,0)) \le \delta \\ &\qquad \Rightarrow \quad \phi_1 \text{ and } \phi_2 \text{ are graphically } \varepsilon\text{-close with respect to } d \end{aligned} \tag{8}$$

2. *incrementally graphically locally attractive* (δLA) *with respect to d* if there exists $\mu > 0$ such that for every $\varepsilon > 0$

$$\begin{aligned} &\phi_1,\phi_2 \in \mathscr{S}, \quad d(\phi_1(0,0),\phi_2(0,0)) \le \mu \\ &\qquad \Rightarrow \quad \phi_1 \text{ and } \phi_2 \text{ are eventually graphically } \varepsilon\text{-close with respect to } d \end{aligned} \tag{9}$$

3. *incrementally graphically locally asymptotically stable* (δLAS) *with respect to* d if it is both δS and δLA.

When δLA holds for every $\mu > 0$, we say that the set of hybrid trajectories $\mathscr{S}$ is incrementally graphically globally attractive (δGA).

Remark 3. The notion in Definition 7 covers the special cases of $\mathscr{S}$ being a set of continuous-time trajectories or a set of discrete-time trajectories. In particular, when $\mathscr{S}$ is a set of complete discrete-time trajectories, condition (8) reduces to

$$\begin{aligned} \phi_1, \phi_2 \in \mathscr{S}, \quad & d(\phi_1(0,0), \phi_2(0,0)) \leq \delta \\ & \Rightarrow \quad d(\phi_1(0,j), \phi_2(0,j)) \leq \varepsilon \quad \forall j \in \mathbb{N}. \end{aligned} \tag{10}$$

Due to requiring a property for every possible pair of trajectories, incremental graphical global attractivity only holds when $\mathscr{S}$ is either a set of continuous-time trajectories or of discrete-time trajectories (see [8]). As a difference to those in [8, Definition 3], both the δS and δLA notions in Definition 7 exploit the graphically ε-closeness notion in [4, Definition 4.11], which in [4] is shown to be a structural property of solutions to well-posed hybrid systems.

Note that unboundedness of the domain of the elements in a generic set $\mathscr{S}$ is not required, but when there are elements with dramatically different domains, incremental stability may not hold – in particular, the set of solutions in Example 2 would not be δLA. The following results formalizes this fact.

Proposition 1. *Let $\mathscr{S}$ be a set of hybrid trajectories. Suppose that no matter how small $\delta' > 0$ is, there exist complete $\phi_1, \phi_2 \in \mathscr{S}$ with $|\phi_1(0,0) - \phi_2(0,0)| \leq \delta'$ such that* $\sup_t \operatorname{dom} \phi_2 < \sup_t \operatorname{dom} \phi_1 < \infty$. *Then, $\mathscr{S}$ is neither δS nor δLA with respect to any metric d.*

Proof. We proceed by contradiction. Let d be any metric, $t_1^z = \sup_t \operatorname{dom} \phi_1$, and $t_2^z = \sup_t \operatorname{dom} \phi_2$. Since $\operatorname{dom} \phi_1$ and $\operatorname{dom} \phi_2$ are unbounded and $\sup_t \operatorname{dom} \phi_2 < \sup_t \operatorname{dom} \phi_1$, there exists $T \in (t_2^z, t_1^z)$. Pick $\varepsilon \in (0, \min\{T - t_2^z, t_1^z - T\})$. By continuity of d and the fact that $d(x,x) = 0$ for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &\text{for each } \rho > 0 \text{ there exists } \delta'' > 0 \text{ such that} \\ &\qquad\qquad d(x', y') \leq \rho \text{ for all } x', y' \text{ such that } |x' - y'| \leq \delta''. \end{aligned} \tag{11}$$

Now, suppose that $\mathscr{S}$ is δS with respect to d. With ε as above, let δ be such that (8) holds. Pick $\rho \leq \delta$ and let δ'' be generated by the continuity property of d in (11). Using δ' such that $\delta' \leq \delta''$ in the assumption of the claim, in which ϕ_1 and ϕ_2 start within δ' in terms of the Euclidean distance, in particular, we have that $d(\phi_1(0,0), \phi_2(0,0)) \leq \delta$ and ϕ_1 and ϕ_2 are graphically ε-close with respect to d. However, since $\sup_t \operatorname{dom} \phi_2 < T$, there exists $(t, j) \in \operatorname{dom} \phi_1$ with $t > T$ such that $(t', j') \notin \operatorname{dom} \phi_2$ for each t' satisfying $|t - t'| < \varepsilon$ and for some $j' \in \mathbb{N}$. This fact contradicts graphical ε-closeness with respect to d guaranteed by (8). The case when $\mathscr{S}$ is δLA follows similarly. $\square$

Next, we revisit Example 1 and show that the set of hybrid trajectories therein is δS. More examples illustrating the proposed notions will be given in Section 3, in which sets of solutions $\mathscr{S}$ are generated by hybrid inclusions.

Example 3. We show that $\mathscr{S}$ given in Example 1 is δS. For a given $\varepsilon > 0$, let $0 < \delta < \varepsilon$ and assume $|\phi_1(0,0) - \phi_2(0,0)| < \delta$ and pick corresponding trajectories $\phi_1, \phi_2 \in \mathscr{S}$. Without loss of generality, we further suppose $0 \le \phi_1(0,0) \le \phi_2(0,0)$ and pick corresponding trajectories $\phi_1, \phi_2 \in \mathscr{S}$. Then, the hybrid trajectory ϕ_1 jumps before ϕ_2. For each $j \in \mathbb{N} \setminus \{0\}$, let $\bar{t}_j = \max_{(t,j-1)\in \mathrm{dom}\,\phi_1 \cap \mathrm{dom}\,\phi_2} t$ and $\bar{t}'_j = \min_{(t,j)\in \mathrm{dom}\,\phi_1 \cap \mathrm{dom}\,\phi_2} t$. Then, we have that for each $t \in [0, \bar{t}_1]$, there exists $(s,0) \in \mathrm{dom}\,\phi_2$ such that $s = t$ and

$$|\phi_1(t,0) - \phi_2(t,0)| = |\phi_1(0,0) - t - \phi_2(0,0) + t| \le \delta < \varepsilon. \tag{12}$$

For each $t \in [\bar{t}_1, \bar{t}'_1]$,

$$\begin{aligned} |\phi_1(\bar{t}_1,0) - \phi_2(t,0)| &= |\phi_2(0,0) - t| \\ &\le |\phi_2(0,0) - \bar{t}_1| = |\phi_2(0,0) - \phi_1(0,0)| \le \delta < \varepsilon, \end{aligned} \tag{13}$$

where we used the fact that $\phi_1(\bar{t}_1,0) = \phi_1(0,0) - \bar{t}_1 = 0$. Moreover, $\phi_2(\bar{t}'_1,0) = \phi_2(0,0) - \bar{t}'_1 = 0$. Then, $|\bar{t}'_1 - \bar{t}_1| = |\phi_2(0,0) - \phi_1(0,0)| \le \delta < \varepsilon$. Therefore, for each $t \in [\bar{t}_1, \bar{t}'_1]$,

$$|\phi_1(t,1) - \phi_2(\bar{t}'_1,1)| = |1 - (t - \bar{t}_1) - 1| \le \delta < \varepsilon. \tag{14}$$

Proceeding similarly and using (14), for each $t \in [\bar{t}'_{i-1}, \bar{t}_i]$, where $i \in \mathbb{N} \setminus \{0,1\}$,

$$|\phi_1(t,i-1) - \phi_2(t,i-1)| = |\phi_1(\bar{t}'_{i-1},i-1) - \phi_2(\bar{t}'_{i-1},i-1)| \le \delta < \varepsilon.$$

Moreover, since $\phi_1(\bar{t}_i,i-1) = 0$, for each $t \in [\bar{t}_i, \bar{t}'_i]$, where $i \in \mathbb{N} \setminus \{0,1\}$,

$$\begin{aligned} |\phi_1(\bar{t}_i,i-1) - \phi_2(t,i-1)| &= |\phi_1(\bar{t}_i,i-1) - \phi_2(\bar{t}_i,i-1) + (t - \bar{t}_i)| \\ &\le |\phi_2(\bar{t}_i,i-1) - \phi_1(\bar{t}_i,i-1)| \le \delta < \varepsilon, \end{aligned}$$

and $|\phi_1(t,i) - \phi_2(\bar{t}'_i,i)| = |1 - (t - \bar{t}_i) - 1| \le \delta < \varepsilon$. Therefore, the set $\mathscr{S}$ is δS[1]. On the other hand, since the distance between ϕ_1 and ϕ_2 does not converge to zero, ϕ_1, ϕ_2 are not eventually ε-close and thus the set $\mathscr{S}$ is neither δLA nor δGA. $\triangle$

To further illustrate the notion in Definition 7, the following example shows that a set $\mathscr{S}$ is δLAS.

Example 4. Let $\mathscr{S}$ be the set of hybrid trajectories with elements ϕ

$$\phi(t,j) = \left(\phi(t_j,j) - \mathrm{ceil}\left(\tfrac{j}{j+1}\right)\right)\exp(-t + t_j) \tag{15}$$

[1] Using the ideas in [11], it may be possible to construct an alternative distance function that is decreasing along trajectories.

for all $(t,j) \in \bigcup_{i\in\mathbb{N}, i<J} ([t_i, t_{i+1}] \times \{i\}) \bigcup ([t_J, \infty) \times \{J\})$ with $\phi(0,0) \subset \bigcup_{i\in\{2k: k\in\mathbb{N}\}} [i, i+1]$, where $J = \frac{1}{2}\,\mathrm{floor}(\phi(0,0))$, $t_0 = 0$, and, for $J > 0$, $t_J = t_{J-1}$ and

$$\begin{aligned} t_j &= \ln(\phi(0,0)) - \ln(\mathrm{floor}(\phi(0,0))) \\ &\quad + \sum_{k=1}^{j-1} (\ln(\mathrm{floor}(\phi(0,0)) - k) - \ln(\mathrm{floor}(\phi(0,0)) - k - 1)) \quad \forall j \in \mathbb{N} \setminus \{0\}, j \le J. \end{aligned}$$

(This set of trajectories can be generated using the hybrid inclusion given in Example 6.) Given $\varepsilon > 0$, consider two elements $\phi_1, \phi_2 \in \mathscr{S}$ such that $|\phi_1(0,0) - \phi_2(0,0)| \le \delta$, where $0 \le \delta < \min\{1, \varepsilon\}$. Then, it is guaranteed that $\bar{J} := \sup_j \mathrm{dom}\,\phi = \sup_j \mathrm{dom}\,\phi_2 < \infty$ since $\mathrm{floor}(\phi_1(0,0)) = \mathrm{floor}(\phi_2(0,0))$. For each $j \in \mathbb{N} \setminus \{0\}$, let $\bar{t}_j = \max_{(t,j-1)\in \mathrm{dom}\,\phi_1 \cap \mathrm{dom}\,\phi_2} t$ and $\bar{t}'_j = \min_{(t,j)\in \mathrm{dom}\,\phi_1 \cap \mathrm{dom}\,\phi_2} t$. Without loss of generality, assume $\phi_2(0,0) > \phi_1(0,0) \ge 2$, then ϕ_1 jumps first. Then, we have that for each $t \in [0, \bar{t}_1]$, there exists $(s,0) \in \mathrm{dom}\,\phi_2$ such that $s = t$ and

$$|\phi_1(t,0) - \phi_2(t,0)| = |\phi_1(0,0)\exp(-t) - \phi_2(0,0)\exp(-t)| \le \delta < \varepsilon. \tag{16}$$

For each $t \in [\bar{t}_1, \bar{t}'_1]$,

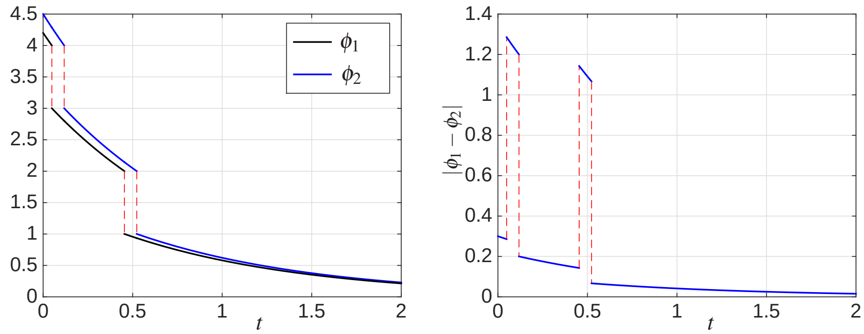
$$\begin{aligned} |\phi_1(\bar{t}_1, 0) - \phi_2(t,0)| &= |\exp(-\bar{t}_1)\phi_1(0,0) - \exp(-t)\phi_2(0,0)| \\ &\le |\exp(-\bar{t}_1)\phi_1(0,0) - \exp(-\bar{t}_1)\phi_2(0,0)| \le \delta < \varepsilon, \end{aligned} \tag{17}$$

where we used the property $\exp(-\bar{t}_1)\phi_1(0,0) = \mathrm{floor}(\phi_1(0,0)) = \mathrm{floor}(\phi_2(0,0)) = \exp(-\bar{t}'_1)\phi_2(0,0)$. Note that $\bar{t}_1 = \ln(\phi_1(0,0)) - \ln(\mathrm{floor}(\phi_1(0,0)))$ and $\bar{t}'_1 = \ln(\phi_2(0,0)) - \ln(\mathrm{floor}(\phi_2(0,0)))$. Therefore, $\bar{t}'_1 - \bar{t}_1 = \ln(\phi_2(0,0)) - \ln(\phi_1(0,0))$. Furthermore, by the mean value theorem, there exists $\phi_0^\star \in [\phi_1(0,0), \phi_2(0,0)]$ such that $|\bar{t}'_1 - \bar{t}_1| = \frac{1}{\phi_0^\star}|\phi_1(0,0) - \phi_2(0,0)| \le |\phi_1(0,0) - \phi_2(0,0)| \le \delta < \varepsilon$. Similarly, for each $t \in [\bar{t}_1, \bar{t}'_1]$,

$$\begin{aligned} |\phi_1(t,1) - \phi_2(\bar{t}'_1, 1)| &= |\exp(-t + \bar{t}_1)\phi_1(\bar{t}_1, 1) - \phi_2(\bar{t}'_1, 1)| \\ &= \phi_2(\bar{t}'_1, 1) - \exp(-t + \bar{t}_1)\phi_1(\bar{t}_1, 1) \\ &\le \phi_2(\bar{t}'_1, 1) - \exp(-\bar{t}'_1 + \bar{t}_1)\phi_1(\bar{t}_1, 1) \\ &\le \phi_2(\bar{t}'_1, 1) - \exp(-\ln(\phi_2(0,0)) + \ln(\phi_1(0,0)))\phi_1(\bar{t}_1, 1) \\ &\le \mathrm{floor}(\phi_2(0,0))\left(1 - \exp\left(\ln\frac{\phi_1(0,0)}{\phi_2(0,0)}\right)\right) \\ &\le \frac{\mathrm{floor}(\phi_2(0,0))}{\phi_2(0,0)}(\phi_2(0,0) - \phi_1(0,0)) \\ &\le \phi_2(0,0) - \phi_1(0,0) \le \delta. \end{aligned} \tag{18}$$

Note that the derivation in (18) can be repeated for $\bar{J}$ times.

If $\phi_1(0,0), \phi_2(0,0) \in [0,1]$, we have that $|\phi_1(t,0) - \phi_2(t,0)| \le \exp(-t)|\phi_1(0,0) - \phi_2(0,0)| \le \delta$ for all $(t,0) \in \mathrm{dom}\,\phi_1 = \mathrm{dom}\,\phi_2$. In fact, $\lim_{t\to\infty, (t,0)\in \mathrm{dom}\,\phi_1} |\phi_1(t,0) - \phi_2(t,0)| = 0$. Therefore, the set $\mathscr{S}$ is δLAS.



(a) The projections of two elements from $\phi_1(0,0) = 4.2$ and $\phi_2(0,0) = 4.5$ on the t direction.

(b) Comparison between graphical distance and Euclidean distance between ϕ_1 and ϕ_2.

Fig. 3 Two maximal elements ϕ_1 and ϕ_2. Unlike the Euclidean distance, which is $|\phi_1(t, j_1(t)) - \phi_2(t, j_2(t))|$ for all $(t, j_i(t)) \in \operatorname{dom} \phi_i$ and $j_i(t) = \min_{(t,j_i)\in \operatorname{dom} \phi_i} j_i$, which does not decrease along hybrid trajectories, the "graphical distance" from ϕ_1 to ϕ_2 is zero for $\varepsilon = 0.3$ and the "graphical distance" from ϕ_2 to ϕ_1 converges to zero.

As shown in Figure 3(a), the domains of two elements in the set $\mathscr{S}$ may be different from each other. The Euclidean distance between ϕ_1 and ϕ_2 has peaks during the mismatch part of the hybrid time domain, i.e., the time instances (t) when two solutions have different values of j, as shown in Figure 3(b). $\triangle$

While the notion introduced in Definition 7 appears to be suitable for the study of incremental stability properties of sets of hybrid trajectories, in particular, for those generated using hybrid inclusions, conditions guaranteeing it are not obvious due to the noncausality nature of the notion. Necessary and sufficient conditions for this notion are proposed in the next section, both for sets of hybrid trajectories as well as hybrid inclusions.

3 Necessary and Sufficient Conditions for Incremental Graphical Stability Notions

In this section, we explore several necessary and sufficient conditions of incremental graphical stability properties for hybrid systems that satisfy certain assumptions. In particular, Proposition 2 implies a basic necessary condition for two hybrid arcs to be ε-close and eventually ε-close, respectively. Proposition 4 shows that maximal elements in a set $\mathscr{S}$ are unique if $\mathscr{S}$ is δS. In Theorem 2, a sufficient condition for $\mathscr{H}$ to be δLAS is presented for a hybrid system with generic jump sets. When D is a discrete set, Corollary 1 provides sufficient conditions for $\mathscr{H}$ to be δLAS. Moreover, Proposition 6 establishes Lyapunov-like sufficient conditions for item 2) of Corollary 1. Then, a finite-time stability property is shown to be necessary for

$\mathscr{H}$ to be δS or δLA in Theorem 1. Furthermore, Theorem 3 studies conditions for which $\mathscr{H}$ is δLAS when the jump map is Lipschitz.

For them to be constructive, some of the necessary and sufficient conditions are stated for sets of hybrid trajectories generated by hybrid system given by hybrid inclusions. A hybrid system $\mathscr{H}$ has data (C, f, D, g) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &= g(z) \quad z \in D, \end{aligned} \tag{19}$$

where $z \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map g defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathscr{H}$ is hybrid trajectory that satisfies the dynamics of (19). A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set of hybrid trajectories $\mathscr{S}_{\mathscr{H}}$ contains all maximal solutions to $\mathscr{H}$, and the set $\mathscr{S}_{\mathscr{H}}(\xi)$ contains all maximal solutions to $\mathscr{H}$ from ξ. Note the use of single-valued maps f and g in (19) are necessary when studying incremental stability; see Proposition 4.

Definition 8. A hybrid system $\mathscr{H} = (C, f, D, g)$ is said to satisfy the hybrid basic conditions if

(a) the sets C and D are closed;
(b) the functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

We refer the reader to [4] and [5] for more details on these notions and the hybrid systems framework.

3.1 Necessary Conditions

The following result highlights a necessary property of the hybrid time domains of two hybrid arcs that are graphically close. In particular, it holds for every pair of elements in a set $\mathscr{S}$ that is δS, δLA, or δGA.

Proposition 2. *Given $\varepsilon > 0$ and two elements $\phi_1, \phi_2 \in \mathscr{S}$, the following hold:*

1. *if ϕ_1 and ϕ_2 are graphically ε-close, or*
2. *if ϕ_1 and ϕ_2 are complete and graphically eventually ε-close,*

then

$$\sup_j \operatorname{dom} \phi_1 = \sup_j \operatorname{dom} \phi_2. \tag{20}$$

Proof. We proceed by contradiction. Given $\varepsilon > 0$, consider two hybrid arcs ϕ_1, ϕ_2 that are graphically ε-close. Suppose that $J_1 = \sup_j \operatorname{dom} \phi_1$, $J_2 = \sup_j \operatorname{dom} \phi_2$ and

$J_1 \neq J_2$. Moreover, without loss of generality, assume that J_1 and J_2 are both finite and $J_1 > J_2$. Then, $J_1 > 0$. Let $(t_{J_1}, J_1) \in \operatorname{dom}\phi_1$ be such that $(t_{J_1}, J_1 - 1) \in \operatorname{dom}\phi_1$. Then, $(t, J_1) \notin \operatorname{dom}\phi_2$ for any $t \in \mathbb{R}_{\geq 0}$, which implies that there does not exists $(t, J_1) \in \operatorname{dom}\phi_2$ such that $|t - t_{J_1}| \leq \varepsilon$ and $d(\phi_1(t_{J_1}, J_1), \phi_2(t, J_1)) \leq \varepsilon$. This contradicts the fact that ϕ_1 and ϕ_2 are graphically ε-close. The situation where either J_1 or J_2 is ∞ follows similarly.

When ϕ_1 and ϕ_2 are complete and eventually graphically ε-close, given $\varepsilon > 0$, there exists $T > 0$ such that ϕ_1 and ϕ_2 satisfy (6) and (7) for all $(t_1, j_1) \in \operatorname{dom}\phi_1$ and $(t_2, j_2) \in \operatorname{dom}\phi_2$ such that $t_1 + j_1 > T$ and $t_2 + j_2 > T$. Proceeding by contradiction, suppose that $J_1 = \sup_j \operatorname{dom}\phi_1$, $J_2 = \sup_j \operatorname{dom}\phi_2$ and $J_1 \neq J_2$. Moreover, without loss of generality, assume that J_1 and J_2 are both finite and $J_1 > J_2$. Then, $J_1 > 0$. Let $(t_{J_1}, J_1) \in \operatorname{dom}\phi_1$ be such that $(t_{J_1}, J_1 - 1) \in \operatorname{dom}\phi_1$. Pick $(t, J_1) \in \operatorname{dom}\phi_1$ and $t + J_1 > T$, which is always possible since ϕ_1 is complete. Then, $(t, J_1) \notin \operatorname{dom}\phi_2$ which implies that there does not exists $(t, J_1) \in \operatorname{dom}\phi_2$ such that $|t - t_{J_1}| \leq \varepsilon$ and $d(\phi_1(t_{J_1}, J_1), \phi_2(t, J_1)) \leq \varepsilon$. This contradicts the fact that ϕ_1 and ϕ_2 are eventually graphically ε-close. The situation where either J_1 or J_2 is ∞ follows similarly. □

Example 5. Consider the set $\mathscr{S}$ given in Example 4, and two elements ϕ_1 and ϕ_2 with $\phi_1(0,0) = 4.5$ and $\phi_2(0,0) = 1$, respectively. The hybrid trajectory ϕ_1 jumps twice while the hybrid trajectory ϕ_2 never jumps. Therefore, ϕ_1 and ϕ_2 are not graphically eventually ε-close according to Proposition 2. This property prevents the set $\mathscr{S}$ from being δGA while Example 4 shows that this set is δLAS. △

Proposition 3. *Let $\mathscr{S}$ be a set of hybrid trajectories.*

1. *If $\mathscr{S}$ is δS or δLA with respect to a metric d, there exists $\delta > 0$ such that*

$$\phi_1, \phi_2 \in \mathscr{S} \text{ complete}, \quad d(\phi_1(0,0), \phi_2(0,0)) \leq \delta \quad \Rightarrow \sup_t \operatorname{dom}\phi_1 = \sup_t \operatorname{dom}\phi_2. \tag{21}$$

2. *If $\mathscr{S}$ is δGA with respect to a metric d, for each $\phi_1, \phi_2 \in \mathscr{S}$ complete*

$$\sup_t \operatorname{dom}\phi_1 = \sup_t \operatorname{dom}\phi_2. \tag{22}$$

Proof. Proceeding by contradiction, suppose $\mathscr{S}$ is δS and, no matter how small $\delta > 0$ is chosen, there exist $\phi_1, \phi_2 \in \mathscr{S}$ such that $d(\phi_1(0,0), \phi_2(0,0)) < \delta$ and $\sup_t \operatorname{dom}\phi_1 \neq \sup_t \operatorname{dom}\phi_2$. Then, by Proposition 1, $\mathscr{S}$ is neither δS nor δLA. The argument follows similarly when $\mathscr{S}$ is δGA. □

The following result establishes that uniqueness is a necessary condition for δS. In turn, according to Proposition 4, it justifies the choice of using single-valued flow and jump maps in the definition of $\mathscr{H}$ in (19).

Proposition 4. *(uniqueness of elements in $\mathscr{S}$) Let $\mathscr{S}$ be a set of hybrid trajectories. Suppose $\mathscr{S}$ is δS with respect to a metric d. Then, every complete element of $\mathscr{S}$ is unique.*

Proof. We proceed by contradiction. Assume that there exist two elements $\phi_1, \phi_2 \in \mathscr{S}$ such that $\phi_1(0,0) = \phi_2(0,0)$ but $\phi_1 \not\equiv \phi_2$. We have the following cases:

1. $\operatorname{dom}\phi_1 \neq \operatorname{dom}\phi_2$. If $\sup_j \operatorname{dom}\phi_1 \neq \sup_j \operatorname{dom}\phi_2$, by Proposition 2, ϕ_1 and ϕ_2 cannot be graphically ε-close, which contradicts that $\mathscr{S}$ is δS. While if

$$\sup_t \operatorname{dom}\phi_1 \neq \sup_t \operatorname{dom}\phi_2,$$

according to Proposition 3, ϕ_1 and ϕ_2 cannot be graphically ε-close, which contradicts that $\mathscr{S}$ is δS. If $\sup_j \operatorname{dom}\phi_1 = \sup_j \operatorname{dom}\phi_2$ and $\sup_t \operatorname{dom}\phi_1 = \sup_t \operatorname{dom}\phi_2$, since $\operatorname{dom}\phi_1 \neq \operatorname{dom}\phi_2$, there exists $(t^\star, j^\star) \in \operatorname{dom}\phi_1$ such that $(t^\star, j^\star) \notin \operatorname{dom}\phi_2$. Without loss of generality, assume the ϕ_1 and ϕ_2 have their domains of definition unbounded in the t direction. It must be one of the following cases:

   a. $(t^\star, \bar{j}) \in \operatorname{dom}\phi_2$ for some $j^\star \neq \bar{j} \in \mathbb{N}$. Then,
      i. if $\bar{j} < j^\star$, it follows that there exists $\bar{t} > t^\star$ such that $(\bar{t}, j^\star) \in \operatorname{dom}\phi_2$. Moreover, $(t, j^\star) \notin \operatorname{dom}\phi_2$ for all $t \in [t^\star - \frac{1}{2}(\bar{t} - t^\star), t^\star + \frac{1}{2}(\bar{t} - t^\star)]$. Then, for $\varepsilon = \frac{1}{2}(\bar{t} - t^\star)$, there does not exists $(t, j^\star) \in \operatorname{dom}\phi_2$ such that $|t - t^\star| \leq \varepsilon$ and $d(\phi_1(t^\star, j^\star), \phi_2(t, j^\star)) \leq \varepsilon$. This contradicts the fact that ϕ_1 and ϕ_2 are graphically ε-close due to the set $\mathscr{S}$ being δS.
      ii. the case when $\bar{j} > j^\star$ follows similarly.
   b. $(\bar{t}, j^\star) \in \operatorname{dom}\phi_2$ for some $\bar{t} \neq t^\star$ and $\bar{t} \in \mathbb{R}_{\geq 0}$. Then,
      i. if $\bar{t} < t^\star$, let $\bar{t}' = \max\{t : (t, j^\star) \in \operatorname{dom}\phi_2, t \leq t^\star\}$. Then, $\bar{t}' < t^\star$. Furthermore, either $j^\star = \sup_j \operatorname{dom}\phi_2$ or $(\bar{t}', j^\star + 1) \in \operatorname{dom}\phi_2$. In either case, pick $\varepsilon = \frac{1}{2}(t^\star - \bar{t}')$, and note it is not possible to find $(t, j^\star) \in \operatorname{dom}\phi_2$ such that $|t - t^\star| \leq \varepsilon$ and $d(\phi_2(t, j^\star), \phi_1(t^\star, j^\star)) \leq \varepsilon$. This contradicts the fact that ϕ_1 and ϕ_2 are graphically ε-close.
      ii. the case when $\bar{t} > t^\star$ follows similarly.

2. $\operatorname{dom}\phi_1 = \operatorname{dom}\phi_2$ but there exists $(t^\star, j^\star) \in \operatorname{dom}\phi_1$ such that $\phi_1(t^\star, j^\star) \neq \phi_2(t^\star, j^\star)$. Suppose $(t^\star, j^\star)$ is not an "end point", i.e., $(t^\star, j^\star - 1) \notin \operatorname{dom}\phi_1$ and $(t^\star, j^\star + 1) \notin \operatorname{dom}\phi_1$. Denote $\bar{\varepsilon} = d(\phi_1(t^\star, j^\star), \phi_2(t^\star, j^\star)) > 0$. Since $t \mapsto \phi_1(t, j^\star)$ and $t \mapsto \phi_2(t, j^\star)$ are locally absolutely continuous for all t such that $(t, j^\star) \in \operatorname{dom}\phi_1$ and $(t, j^\star) \in \operatorname{dom}\phi_2$ according to Definition 2, there exists $\delta > 0$ such that $|t - t^\star| \leq \delta$ implies that $d(\phi_1(t, j^\star), \phi_1(t^\star, j^\star)) \leq \frac{1}{2}\bar{\varepsilon}$ and $d(\phi_2(t, j^\star), \phi_2(t^\star, j^\star)) \leq \frac{1}{2}\bar{\varepsilon}$. Therefore, by triangle inequality,

$$\begin{aligned} d(\phi_1(t, j^\star), \phi_2(t^\star, j^\star)) &\geq d(\phi_1(t^\star, j^\star), \phi_2(t^\star, j^\star)) - d(\phi_1(t, j^\star), \phi_1(t^\star, j^\star)) \\ &\geq \bar{\varepsilon} - \frac{1}{2}\bar{\varepsilon} = \frac{1}{2}\bar{\varepsilon}. \end{aligned}$$

Thus, for $\varepsilon = \frac{1}{4}\bar{\varepsilon}$, no matter how small δ is chosen, we have that

$$d(\phi_1(0,0), \phi_2(0,0)) = 0 < \delta$$

and ϕ_1 and ϕ_2 are not graphically ε-close which contradicts the assumption that the set $\mathscr{S}$ is δS with respect to d. The situation where $(t^\star, j^\star)$ is an "end point" can be proved similarly. $\square$

When the set $\mathscr{S}$ is generated by solutions to a hybrid system $\mathscr{H} = (C, f, D, g)$, a sufficient condition for guaranteeing uniqueness of maximal solutions requires f to be locally Lipschitz and no flow from $C \cap D$ – a rigorous statement is given in [4, Proposition 2.11]. According to Proposition 4, assuming uniqueness of solutions to $\mathscr{H}$ is not at all restrictive, in fact, when studying incremental graphical stability, it is necessary. Hence, in the following results we impose the following uniqueness of solutions assumption.

Assumption 3.1 *The hybrid system $\mathscr{H} = (C, f, D, g)$ is such that each maximal solution ϕ to $\mathscr{H}$ is unique.*

Next, we show that, to have δS or δLA, a finite-time convergence property within a neighborhood of the jump set D is a necessary condition for a set of hybrid trajectories generated by hybrid system $\mathscr{H}$. Indeed, without the finite-time convergence property nearby D and $g(D)$, the graphs of the solutions would not be graphically close.

Theorem 1. *(necessary condition for δS and δLA) Consider a hybrid system $\mathscr{H} = (C, f, D, g)$ with state $z \in \mathbb{R}^n$ satisfying Assumption 3.1 and the hybrid basic conditions. Suppose $D \neq \emptyset$ and $g(D) \subset C \cup D$. If $\mathscr{S}_{\mathscr{H}}$ is δS or δLA with respect to a metric d, then there exists $\delta_0 > 0$ such that each maximal solution ϕ to $\mathscr{H}$ from $\phi(0,0)$ satisfying $|\phi(0,0)|_D^d \leq \delta_0$ and $\phi(0,0) \in C$ converges to D within finite time, i.e., there exists $s > 0$ such that $|\phi(s,0)|_D^d = 0$.*

Proof. Let $\varepsilon > 0$ be given. Proceeding by contradiction, for all $\delta_0 > 0$, there exists $\phi \in \mathscr{S}_{\mathscr{H}}$ satisfying

$$\phi(0,0) \in C, \quad |\phi(0,0)|_D^d \leq \delta_0 \tag{23}$$

and $|\phi(t,0)|_D^d > 0$ for all $(t,0) \in \operatorname{dom}\phi$. Let $z^\star \in D$ be such that $|\phi(0,0)|_D^d = d(\phi(0,0), z^\star) = \delta_0$. Consider a solution $\phi_1 \in \mathscr{S}_{\mathscr{H}}$ from $z^\star$. Then, we have that $d(\phi_1(0,0), \phi(0,0)) \leq \delta_0$ which implies that ϕ_1 and ϕ are graphically ε-close due to $\mathscr{S}$ being δS with respect to d (using $\delta = \delta_0$ in the definition). Since each maximal solution to $\mathscr{H}$ is unique under Assumption 3.1, $(0,1) \in \operatorname{dom}\phi_1$. Then, since $\phi(t,0) \notin D$ for all $(t,0) \in \operatorname{dom}\phi$, there does not exist $(s,1) \in \operatorname{dom}\phi$ such that $d(\phi_1(0,1), \phi(s,1)) < \varepsilon$ with $|s| \leq \varepsilon$. This contradicts the assumption that ϕ and ϕ_1 are graphically ε-close. Now suppose $\mathscr{S}$ is δLA. For any $T > 0$, $t + j \geq T$ and $(t, j) \in \operatorname{dom}\phi_1$, there does not exists $(s, j) \in \operatorname{dom}\phi$ with $|s - t| \leq \varepsilon$ such that $d(\phi_1(t, j), \phi(s, j)) \leq \varepsilon$. This contradicts the fact that ϕ_1 and ϕ are graphically eventually ε-close. $\square$

The δS property leads to the following necessary condition pertaining to dependence of solutions with respect to initial conditions.

Proposition 5. *(necessary condition for δS) Consider a hybrid system $\mathscr{H} = (C, f, D, g)$ with state $z \in \mathbb{R}^n$ satisfying Assumption 3.1. Suppose $\mathscr{S}_{\mathscr{H}}$ is δS with respect to a metric d. Then, $\mathscr{S}_{\mathscr{H}}$ satisfies the following property: for every $\phi \in \mathscr{S}_{\mathscr{H}}$, and for* *every $\varepsilon > 0$, there exists $\delta > 0$ such that for every solution $\bar{\phi} \in \mathscr{S}_{\mathscr{H}}(\phi(0,0)+\delta\mathbb{B})$, $\bar{\phi}$ and ϕ are graphically ε-close with respect to d.*

Proof. Since the set $\mathscr{S}_{\mathscr{H}}$ is δS, for a given $\varepsilon > 0$, there exists $\bar{\bar{\delta}} > 0$ such that for $\phi_1, \phi_2 \in \mathscr{S}_{\mathscr{H}}$,

$$d(\phi_1(0,0), \phi_2(0,0)) \leq \bar{\bar{\delta}} \Longrightarrow \phi_1, \phi_2 \text{ are graphically } \varepsilon\text{-close.}$$

Let $\delta > 0$ be small enough such that $|\phi_1(0,0) - \phi_2(0,0)| \leq \delta$ implies that

$$d(\phi_1(0,0), \phi_2(0,0)) \leq \bar{\bar{\delta}}.$$

Then, for any ϕ and $\bar{\phi}$ picked as in the theorem, $|\phi(0,0) - \bar{\phi}(0,0)| \leq \delta$ implies that $d(\bar{\phi}(0,0), \phi(0,0)) \leq \bar{\bar{\delta}}$. Therefore, by using the δS property of the set $\mathscr{S}_{\mathscr{H}}$, $\bar{\phi}$ and ϕ are graphically ε-close. $\square$

3.2 Sufficient Conditions

To establish sufficient conditions for δLAS, we impose the following assumptions. The first assumption is that each maximal solution to $\mathscr{H}$ has its domain of definition unbounded in the t direction. The second assumption enables each maximal solution to $\mathscr{H}$ to flow for sufficient amount of time in between jumps. A sufficient condition for Assumption 3.3 can be found in [12, Lemma 2.7].

Assumption 3.2 *The hybrid system $\mathscr{H} = (C, f, D, g)$ is such that every $\phi \in \mathscr{S}_{\mathscr{H}}$ satisfies $\sup_t \operatorname{dom} \phi = \infty$.*

Assumption 3.3 *The hybrid system $\mathscr{H} = (C, f, D, g)$ is such that there exists $\gamma > 0$ such that for each $\phi \in \mathscr{S}_{\mathscr{H}}$, the flow time between two consecutive jumps is lower bounded by γ.*

Moreover, we will use the following forward invariance notion.

Definition 9. (forward invariance from away of D) A set $\mathscr{A} \subset \mathbb{R}^n$ is said to be forward invariant for $\mathscr{H}$ from away of D if for each solution ϕ to $\mathscr{H}$ from $\phi(0,0) \in \mathscr{A} \setminus D$, $\phi(t,0) \in \mathscr{A}$ for all $(t,0) \in \operatorname{dom} \phi$.

Remark 4. Note that the standard forward invariance notion for a set captures the property that every solution from the set stays within the set for all time, see, e.g., [4, Definition 6.25].

Now, we are ready to present the sufficient condition.

Theorem 2. *(δLAS through flow for generic D) Consider a hybrid system $\mathscr{H} = (C, f, D, g)$ with state $z \in \mathbb{R}^n$. Suppose $\mathscr{H}$ satisfies Assumption 3.1, Assumption 3.2, Assumption 3.3, and the hybrid basic conditions. Let γ be generated from Assumption 3.3. If there exist $P = P^\top > 0$, $\beta > 0$, and $\delta_0 > 0$ such that $\mathscr{H}$ satisfies*

1) $\nabla f^\top(z)P + P\nabla f(z) \leq -2\beta P$ *for all* $z \in \overline{\mathrm{con}}C$*;*
2) for each $\delta \in [0, \delta_0]$*, each* $\phi \in \mathscr{S}_{\mathscr{H}}$ *from* $\phi(0,0)$ *satisfying*

$$\phi(0,0) \in C, \quad |\phi(0,0)|_D = \delta \tag{24}$$

is such that there exists $s \in [0, \delta]$ *for which we have*

$$|\phi(s,0)|_D = 0 \tag{25}$$

and the set $\phi(s,0) + \delta\mathbb{B}$ *is forward invariant from away of D, and each* $\bar{\phi} \in \mathscr{S}_{\mathscr{H}}(g(\phi(s,0)) + \delta\mathbb{B})$ *satisfies*

$$\bar{\phi}(t,0) \in g(\phi(s,0)) + \delta\mathbb{B} \tag{26}$$

for all $t \in [0, s]$*;*
3) the jump map g is locally Lipschitz on D with Lipschitz constant $L_1 \in [0,1]$*,* [2] *i.e.,* $|g(z_1) - g(z_2)| \leq L_1|z_1 - z_2|$ *for all* $z_1, z_2 \in D$ *such that* $|z_1 - z_2| \leq \delta_0$*; and*
4) $c < \exp(\beta\gamma)$*, where* $c = \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}$*;*

then, the set $\mathscr{S}_{\mathscr{H}}$ *is* δ*LAS with d being the Euclidean distance.*

Proof. Given $\varepsilon > 0$, and using δ_0, γ as in the assumption, consider $\phi_1, \phi_2 \in \mathscr{S}_{\mathscr{H}}$ such that $|\phi_1(0,0) - \phi_2(0,0)| < \delta$, where δ is chosen such that

$$0 < \delta \leq \min\left\{\frac{\varepsilon}{c}, \frac{\delta_0}{c}, \gamma - \frac{1}{\beta}\ln c\right\}.$$

First, we show that $\mathscr{S}_{\mathscr{H}}$ is δS for the case when $\phi_1(0,0), \phi_2(0,0) \in C$ and $\sup_j \mathrm{dom}\,\phi_1 = \sup_j \mathrm{dom}\,\phi_2 = 0$, i.e., no jump occurs to either ϕ_1 or ϕ_2. By the generalized mean value theorem (for vector-valued functions), for almost all $(t,0) \in \mathrm{dom}\,\phi_1 (= \mathrm{dom}\,\phi_2 = [0,\infty) \times \{0\})$, we have that

$$\begin{aligned}\dot{\phi}_1(t,0) - \dot{\phi}_2(t,0) &= f(\phi_1(t,0)) - f(\phi_2(t,0)) \\ &= \int_0^1 \nabla f(\eta(t,s))ds(\phi_1(t,0) - \phi_2(t,0)),\end{aligned}$$

where $\eta(t,s) = \phi_1(t,0) + s(\phi_2(t,0) - \phi_1(t,0))$. Then, using item 1), for almost all $t \in [0,\infty)$, we have

[2] Such g is also known as a weak contraction map.

$$
\begin{aligned}
&\frac{d}{dt}\,|\phi_1(t,0)-\phi_2(t,0)|_P^2\\
&\quad=(\phi_1(t,0)-\phi_2(t,0))^\top\left(\int_0^1\left(\nabla f^\top(\eta(t,s))P+P\nabla f(\eta(t,s))\right)ds\right)(\phi_1(t,0)-\phi_2(t,0))\\
&\quad\le-\int_0^1 2\beta(\phi_1(t,0)-\phi_2(t,0))^\top P(\phi_1(t,0)-\phi_2(t,0))ds\\
&\quad\le-2\beta|\phi_1(t,0)-\phi_2(t,0)|_P^2,
\end{aligned}\tag{27}
$$

where we used the property that $\eta(t,s)\in\overline{\text{con}}C$ for all $t\in[0,\infty)$ and $s\in[0,1]$. Therefore, by the comparison lemma, we have, for all $t\in[0,\infty)$,

$$|\phi_1(t,0)-\phi_2(t,0)|_P\le\exp(-\beta t)|\phi_1(0,0)-\phi_2(0,0)|_P.\tag{28}$$

Then, using the property

$$\underline{\lambda}(P)|z|^2\le|z|_P^2=z^\top Pz\le\overline{\lambda}(P)|z|^2\quad\forall z\in\mathbb{R}^n\tag{29}$$

and the choice of δ, we obtain

$$
\begin{aligned}
|\phi_1(t,0)-\phi_2(t,0)|&\le\frac{1}{\sqrt{\underline{\lambda}(P)}}|\phi_1(t,0)-\phi_2(t,0)|_P\\
&\le c\exp(-\beta t)|\phi_1(0,0)-\phi_2(0,0)|\le\varepsilon.
\end{aligned}\tag{30}
$$

Next, we show $\mathscr{S}_{\mathscr{H}}$ is δS for the case when either ϕ_1 or ϕ_2 jump. By the choice of δ and item 2), $\sup_j\operatorname{dom}\phi_1=\sup_j\operatorname{dom}\phi_2=:J$. Without loss of generality, assume ϕ_1 jumps first and $J=\infty$. Furthermore, for each $j\in\mathbb{N}\setminus\{0\}$, let $\bar t_j=\max_{(t,j-1)\in\operatorname{dom}\phi_1\cap\operatorname{dom}\phi_2}t$ and $\bar t'_j=\min_{(t,j)\in\operatorname{dom}\phi_1\cap\operatorname{dom}\phi_2}t$, and $\bar t'_0=0$, where $\bar t_j$ denotes the minimum time when one of the two solutions ϕ_1,ϕ_2 jumps j times, while $\bar t'_j$ denotes the minimum time when both solutions have jumped j times. In fact, $[\bar t'_j,\bar t_{j+1}]\times\{j\}\subset\operatorname{dom}\phi_1\cap\operatorname{dom}\phi_2$ for all $j\in\mathbb{N}$. For simplicity, assume that the time when j-th jump occurs to ϕ_1 is always smaller than or equal to that of ϕ_2 for $j\in\mathbb{N}$.

(I) If $\phi_1(0,0),\phi_2(0,0)\in C$. Similarly as in (30), for all $t\in[0,\bar t_1]$, we have that

$$|\phi_1(t,0)-\phi_2(t,0)|\le c\exp(-\beta t)|\phi_1(0,0)-\phi_2(0,0)|\le\varepsilon.\tag{31}$$

When $t=\bar t_1$, since ϕ_1 jumps first, $\phi_1(\bar t_1,0)\in D$ and $\phi_1(\bar t_1,1)=g(\phi_1(\bar t_1,0))$. Note that under item 3) of Assumption 3.3, $g(D)\cap D=\emptyset$. Then,

a. if $\phi_2(\bar t_1,0)\in D$, i.e., $\bar t_1=\bar t'_1$, by (31), $|\phi_1(\bar t_1,0)-\phi_2(\bar t_1,0)|\le\delta$ and

$$\phi_1(\bar t_1,0),\phi_2(\bar t_1,0)\in D.$$

Then, we can apply the argument in item (II);

b. If $\phi_2(\bar t_1,0)\notin D$, i.e., $\bar t_1<\bar t'_1$, by (31), it follows that $\phi_2(\bar t_1,0)\in(D+\delta\mathbb{B})\setminus D$. For each $t\in[\bar t_1,\bar t'_1]$, since, $|\phi_2(\bar t_1,0)|_D\le|\phi_2(\bar t_1,0)-\phi_1(\bar t_1,0)|=\bar\delta_1$ for some

$\bar{\delta}_1 \in [0,\delta]$, by (24) and (25) in item 2), it follows that $\bar{t}'_1 - \bar{t}_1 \leq \delta$. Since the set $\phi(\bar{t}_1,0) + \bar{\delta}_1\mathbb{B}$ is forward invariant from away of D according to item 2), we obtain, for each $t \in [\bar{t}_1, \bar{t}'_1]$,

$$|\phi_2(t,0) - \phi_1(\bar{t}_1,0)| \leq |\phi_2(\bar{t}_1,0) - \phi_1(\bar{t}_1,0)|. \tag{32}$$

Furthermore, since $\phi_1(\bar{t}_1,0), \phi_2(\bar{t}'_1,0) \in D$, by item 3),

$$\begin{aligned}|\phi_2(\bar{t}'_1,1) - \phi_1(\bar{t}_1,1)| &\leq |\phi_2(\bar{t}'_1,0) - g(\phi_1(\bar{t}_1,0))|\\ &\leq |\phi_2(\bar{t}'_1,0) - \phi_1(\bar{t}_1,0)|.\end{aligned}$$

Then, since $\phi_1(\bar{t}_1,1) \in \phi_2(\bar{t}'_1,1) + \bar{\delta}_1\mathbb{B}$ according to (32), by item 2), for each $t \in [\bar{t}_1, \bar{t}'_1]$,

$$\begin{aligned}|\phi_1(t,1) - \phi_2(\bar{t}'_1,1)| &\leq |\phi_1(\bar{t}_1,1) - \phi_2(\bar{t}'_1,1)|\\ &\leq |\phi_1(\bar{t}_1,0) - \phi_2(\bar{t}'_1,0)|.\end{aligned} \tag{33}$$

In general, for each $j \in \mathbb{N}$ and $t \in [\bar{t}'_j, \bar{t}_{j+1}]$, since $\phi_1(\bar{t}'_j,j), \phi_2(\bar{t}'_j,j) \in C$, similarly as for (31), we have

$$|\phi_1(t,j) - \phi_2(t,j)| \leq c\exp(-\beta(t-\bar{t}'_j))|\phi_1(\bar{t}'_j,j) - \phi_2(\bar{t}'_j,j)|. \tag{34}$$

While for $j \in \mathbb{N}\setminus\{0\}$ and $t \in [\bar{t}_j, \bar{t}'_j]$, we have $[\bar{t}_j, \bar{t}'_j] \times \{j\} \subset \operatorname{dom}\phi_1$, $[\bar{t}_j, \bar{t}'_j] \times \{j-1\} \subset \operatorname{dom}\phi_2$ and $|\bar{t}'_j - \bar{t}_j| \leq \delta$. Then, similarly as for (32) and (33), we obtain

i. for each $j \in \mathbb{N}\setminus\{0\}$ and each $t \in [\bar{t}_j, \bar{t}'_j]$:

$$|\phi_2(t,j-1) - \phi_1(\bar{t}_j,j-1)| \leq |\phi_2(\bar{t}_j,j-1) - \phi_1(\bar{t}_j,j-1)|, \tag{35}$$

ii. for each $j \in \mathbb{N}\setminus\{0\}$ and each $t \in [\bar{t}_j, \bar{t}'_j]$:

$$|\phi_1(t,j) - \phi_2(\bar{t}'_j,j)| \leq |\phi_1(\bar{t}_j,j-1) - \phi_2(\bar{t}_j,j-1)|, \tag{36}$$

Therefore, by using (34), (35), (36) and $|\phi_1(0,0) - \phi_2(0,0)| \leq \delta$, it follows that

i. for each $j \in \mathbb{N}\setminus\{0\}$ and each $t \in [\bar{t}'_j, \bar{t}_{j+1}]$:

$$\begin{aligned}|\phi_1(t,j) - \phi_2(t,j)| &\leq c\exp(-\beta(t-\bar{t}'_j))|\phi_1(\bar{t}'_j,j) - \phi_2(\bar{t}'_j,j)|\\ &\leq c\exp(-\beta(t-\bar{t}'_j))|\phi_1(\bar{t}_j,j-1) - \phi_2(\bar{t}_j,j-1)|\\ &\leq c^2\exp(-\beta(t-\bar{t}'_j))\exp(-\beta(\bar{t}_j-\bar{t}'_{j-1}))\\ &\quad\times|\phi_1(\bar{t}'_{j-1},j-1) - \phi_2(\bar{t}'_{j-1},j-1)|\\ &\ \ \vdots\\ &\leq c^{j+1}\exp(-\beta(t-\bar{t}'_{j-1}+\Delta_j))|\phi_1(0,0) - \phi_2(0,0)| \leq \varepsilon,\end{aligned} \tag{37}$$

where $\Delta_j := \sum_{k=1}^{j}(\bar{t}_k - \bar{t}'_{k-1})$. In particular, the first inequality in (37) uses (34) with $t \in [\bar{t}'_j, \bar{t}_{j+1}]$, the second inequality in (37) uses (36) with $t = \bar{t}'_j$, and the third inequality in (37) uses (34) with $t = \bar{t}_j$.

ii. for each $j \in \mathbb{N} \setminus \{0\}$ and each $t \in [\bar{t}_j, \bar{t}'_j]$:

$$\begin{aligned}
|\phi_2(t, j-1) - \phi_1(\bar{t}_j, j-1)| &\le |\phi_2(\bar{t}_j, j-1) - \phi_1(\bar{t}_j, j-1)| \\
&\le c\exp(-\beta(\bar{t}_j - \bar{t}'_{j-1}))|\phi_1(\bar{t}'_{j-1}, j-1) - \phi_2(\bar{t}'_{j-1}, j-1)| \\
&\ \ \vdots \\
&\le c^j \exp(-\beta\Delta_j)|\phi_1(0,0) - \phi_2(0,0)| \\
&\le \exp(-(\beta(\gamma-\delta) - \ln c)j)|\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon,
\end{aligned} \tag{38}$$

where the first inequality follows from (35) with $t \in [\bar{t}_j, \bar{t}'_j]$, and the second inequality follows from (34) with $t = \bar{t}_j$.

iii. for each $j \in \mathbb{N} \setminus \{0\}$ and each $t \in [\bar{t}_j, \bar{t}'_j]$:

$$\begin{aligned}
|\phi_1(t, j) - \phi_2(\bar{t}'_j, j)| &\le |\phi_1(\bar{t}_j, j-1) - \phi_2(\bar{t}_j, j-1)| \\
&\le c\exp(-\beta(\bar{t}_j - \bar{t}'_{j-1}))|\phi_1(\bar{t}'_{j-1}, j-1) - \phi_2(\bar{t}'_{j-1}, j-1)| \\
&\ \ \vdots \\
&\le c^j \exp(-\beta\Delta_j)|\phi_1(0,0) - \phi_2(0,0)| \\
&\le \exp(-(\beta(\gamma-\delta) - \ln c)j)|\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon.
\end{aligned} \tag{39}$$

In particular, the first inequality in (39) uses (36) with $t \in [\bar{t}_j, \bar{t}'_j]$, and the second inequality in (39) uses (34) with $t = \bar{t}_j$.

Therefore, ϕ_1 and ϕ_2 are ε-close.

(II) If $\phi_1(0,0), \phi_2(0,0) \in D$, by condition 3) that the jump map is locally Lipschitz on D with Lipschitz constant $L_1 \le 1$, we obtain

$$|\phi_1(1,0) - \phi_2(1,0)| = |g(\phi_1(0,0)) - g(\phi_2(0,0))| \le |\phi_1(0,0) - \phi_2(0,0)|. \tag{40}$$

Note that after the jump, $\phi_1(1,0), \phi_2(1,0) \in C$, we can apply the arguments in item (I).

(III) If $\phi_1(0,0) \in C, \phi_2(0,0) \in D$, the arguments follows similarly as in item (I).

Therefore, by combining arguments in items (I), (II), (III), it is proved that ϕ_1 and ϕ_2 are ε-close. Note that the case when $J < \infty$ follows similarly. Therefore, $\mathscr{S}_{\mathscr{H}}$ is δS with respect to Euclidean distance.

Now, we show that $\mathscr{S}_{\mathscr{H}}$ is δLA. consider the case in item (I) (the other cases follow similarly). Note that $\bigcup_{j=1}^{\infty}[\bar{t}'_j, \bar{t}_{j+1}] = \infty$ if $J = \infty$ (or $\bigcup_{j=1}^{J-1}[\bar{t}'_j, \bar{t}_{j+1}] \bigcup [\bar{t}'_J, \infty) = \infty$ if $J < \infty$ respectively). Moreover, since $[\bar{t}'_j, \bar{t}_{j+1}] \times \{j\} \subset \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$ for all $j \in \mathbb{N}$. Then, on each interval $[\bar{t}'_j, \bar{t}_{j+1}]$, we have that $|\phi_1(t, j+1) - \phi_2(t, j+1)| \le \exp(-\beta(t - \bar{t}'_j))|\phi_1(\bar{t}'_j, j) - \phi_2(\bar{t}'_j, j)|$ for all $t \in [\bar{t}'_j, \bar{t}_{j+1}]$. In particular, pick

$$\mu = \delta < \min\left\{\frac{\delta_0}{c}, \gamma - \frac{1}{\beta}\ln c\right\},$$

for a given $\varepsilon' > 0$, pick

$$T = -\frac{1}{\beta(\gamma-\delta)}\ln\left(\min\left\{1, \frac{\varepsilon'}{c\delta}\right\}\right).$$

Then, by using (37), (38) and (39), we obtain

1. for (t, j) such that $j \geq T$ and $t \in [\bar{t}'_j, \bar{t}_{j+1}]$:

$$\begin{aligned}|\phi_1(t,j) - \phi_2(t,j)| &\leq c\exp(-(\beta(\gamma-\delta) - \ln c)j)|\phi_1(0,0) - \phi_2(0,0)| \\ &\leq c\exp(-(\beta(\gamma-\delta) - \ln c)T)|\phi_1(0,0) - \phi_2(0,0)| \\ &\leq \min\left\{1, \frac{\varepsilon'}{c\delta}\right\} c|\phi_1(0,0) - \phi_2(0,0)| \leq \varepsilon',\end{aligned}$$

2. for (t, j) such that $j \geq T$ and $t \in [\bar{t}_j, \bar{t}'_j]$:

$$|\phi_2(t, j-1) - \phi_1(\bar{t}_j, j-1)| \leq \exp(-(\beta(\gamma-\delta) - \ln c)j)|\phi_1(0,0) - \phi_2(0,0)| \leq \varepsilon',$$

3. for (t, j) such that $j \geq T$ and $t \in [\bar{t}_j, \bar{t}'_j]$:

$$|\phi_1(t,j) - \phi_2(\bar{t}'_j, j)| \leq \exp(-(\beta(\gamma-\delta) - \ln c)j)|\phi_1(0,0) - \phi_2(0,0)| \leq \varepsilon'.$$

Therefore, for ϕ_1, ϕ_2 such that $|\phi_1(0,0) - \phi_2(0,0)| \leq \mu$, ϕ_1, ϕ_2 are eventually ε-close and $\mathscr{S}_{\mathscr{H}}$ is δLA. □

When the jump set D is discrete, the conditions in Theorem 2 simplify and we obtain the following result.

Corollary 1. *(δLAS through flow with D being a discrete set) Consider a hybrid system $\mathscr{H} = (C, f, D, g)$ with state $z \in \mathbb{R}^n$ and D being a discrete set. Suppose $\mathscr{H}$ satisfies Assumption 3.1, Assumption 3.2, Assumption 3.3, and the hybrid basic conditions. Let γ be generated from Assumption 3.3. If there exist $P = P^\top > 0$, $\beta > 0$, and $\delta_0 > 0$ such that $\mathscr{H}$ satisfies*

1) $\nabla f^\top(z)P + P\nabla f(z) \leq -2\beta P$ for all $z \in \overline{\mathrm{con}}C$;
2) for each $\delta \in [0, \delta_0]$, each $\phi \in \mathscr{S}_{\mathscr{H}}$ from $\phi(0,0)$ satisfying

$$\phi(0,0) \in C, \quad |\phi(0,0)|_D = \delta \tag{41}$$

is such that there exists $s \in [0, \delta]$ for which we have

$$|\phi(s,0)|_D = 0, \quad |\phi(t,0)|_D \leq \delta \quad \forall t \in [0, s], \tag{42}$$

and

$$|\bar{\phi}(t,0) - g(\phi(s,0))| \leq \delta \quad \forall t \in [0,s], \tag{43}$$

where $\bar{\phi} \in \mathcal{S}_{\mathcal{H}}(g(\phi(s,0)))$*; and*

3) $c < \exp(\beta\gamma)$*, where* $c := \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}$*;*

then, the set $\mathcal{S}_{\mathcal{H}}$ *is* δ*LAS with* d *being the Euclidean distance.*

The proof of this corollary is in Section 5.1.

Remark 5. Item 1) in Corollary 1 guarantees strict decrease of the distance between every pair of maximal solutions to $\mathcal{H}$ on the intersections of their hybrid time domains. In fact, these conditions guarantee a contraction property of the nonlinear system with right-hand side given by f; see, e.g., [9]. The second item in Corollary 1 implies that, over the mismatch parts of their hybrid time domains, the graphical distance between them does not grow. The third item in Corollary 1 ensures that every pair of maximal solutions can flow for enough time to overcome the possible overshoot on the distance between them. When $P = I$, the third condition is satisfied for free.

The necessity of item 2) of Corollary 1 is justified in Theorem 1. This condition is guaranteed by the following sufficient condition.

Proposition 6. *Consider a hybrid system* $\mathcal{H} = (C, f, D, g)$ *with state* $z \in \mathbb{R}^n$ *and* D *being a discrete set. Suppose* $\mathcal{H}$ *satisfies the hybrid basic conditions. Then, item* 2) *of Corollary 1 holds if there exists* $\delta_0 > 0$ *such that, for any* $z^\star \in D$*, the following hold: there exist* $c_1, c_2 > 0$*,* $c_2 \in (0, c_1]$*, and* $\alpha \in (0,1)$ *such that*

1) *the function* $V_1(z) := |z - z^\star|^2$ *satisfies* $\langle \nabla V_1(z), f(z)\rangle + c_1 V_1^\alpha(z) \leq 0$ *and* $|z - z^\star|^{1-2\alpha} \leq c_1(1-\alpha)$ *for all* $z \in C \bigcap ((z^\star + \delta_0\mathbb{B}) \setminus D)$*,*
2) *the function* $V_2(z) := |z - g(z^\star)|^2$ *satisfies* $\langle \nabla V_2(z), f(z)\rangle - c_2 V_2^\alpha(z) \leq 0$ *for all* $z \in C \bigcap (g(z^\star) + \delta_0\mathbb{B})$*.*

Proof. Let δ be such that $0 < \delta \leq \delta_0$ and for each $z \in D$, $(z + \delta\mathbb{B}) \cap D = \{z\}$. Given $z^\star \in D$, consider $\phi \in \mathcal{S}_{\mathcal{H}}(C \bigcap ((z^\star + \delta\mathbb{B}) \setminus D))$. By item 1) in Proposition 6 and by integrating $t \mapsto \frac{dV_1^{1-\alpha}}{dt}(\phi(t,0))$ over $[0,t_1] \times \{0\} \subset \operatorname{dom}\phi$, it follows that

$$V_1(\phi(t,0))^{1-\alpha} \leq -c_1(1-\alpha)t + V_1(\phi(0,0))^{1-\alpha} \quad \forall (t,0) \in \operatorname{dom}\phi. \tag{44}$$

Note that, since V_1 is a positive definite function with respect to $z^\star$, by using the property that $|z - z^\star|^{1-2\alpha} \leq c_1(1-\alpha)$ for all $z \in C \bigcap ((z^\star + \delta_0\mathbb{B}) \setminus D)$, ϕ converges to $z^\star$ within $t^\star$ seconds, where

$$t^\star = \frac{V_1(\phi(0,0))^{1-\alpha}}{c_1(1-\alpha)} = \frac{|\phi(0,0) - z^\star|^{2-2\alpha}}{c_1(1-\alpha)} \tag{45}$$

$$\implies \quad t^\star \leq \frac{c_1(1-\alpha)|\phi(0,0) - z^\star|}{c_1(1-\alpha)} = |\phi(0,0) - z^\star|. \tag{46}$$

Moreover, by (44) and the fact that $V_1(\phi(t,0)) = |\phi(t,0) - z^\star|^2$,

$$|\phi(t,0) - z^\star| = \sqrt{V_1(\phi(t,0))} \leq \sqrt{V_1(\phi(0,0))} = |\phi(0,0) - z^\star| \quad \forall (t,0) \in \operatorname{dom}\phi. \tag{47}$$

It is implied from (46) that there exists $s \in [0, |\phi(0,0) - z^\star|]$ such that $\phi(s,0) = z^\star$ and, from (47), $|\phi(t,0)|_D \leq \delta$ for all $t \in [0,s]$. Now using item 2) of the assumptions and proceeding similarly to arrive to (44), the maximal solution $\bar{\phi} \in \mathcal{S}_{\mathcal{H}}(g(\phi(s,0)))$ satisfies

$$V_2(\bar{\phi}(t,0))^{1-\alpha} \leq c_2(1-\alpha)t + V_2(\bar{\phi}(0,0))^{1-\alpha} \quad \forall (t,0) \in \operatorname{dom}\bar{\phi}. \tag{48}$$

Since $V_2(\bar{\phi}(0,0)) = |g(\phi(s,0)) - g(z^\star)|^2 = 0$, by using (45), (47), and (48), we obtain that for all $t \in [0,s]$,

$$\begin{aligned}
|\bar{\phi}(t,0) - g(\phi(s,0))| = |\bar{\phi}(t,0) - g(z^\star)| = \sqrt{V_2(\bar{\phi}(t,0))} &\leq \sqrt{(c_2(1-\alpha)t)^{\frac{1}{1-\alpha}}} \\
&\leq \sqrt{\left(c_2(1-\alpha)\frac{V_1(\phi(0,0))^{1-\alpha}}{c_1(1-\alpha)}\right)^{\frac{1}{1-\alpha}}} \\
&\leq \sqrt{\left(\frac{c_2}{c_1}\right)^{\frac{1}{1-\alpha}}}|\phi(0,0) - z^\star| \leq |\phi(0,0) - z^\star| \leq \delta,
\end{aligned}$$

where we used the property $0 < c_2 \leq c_1$. □

Remark 6. In item 1) of Proposition 6, if c_1 and α can be chosen as $c_1 \geq 2$ and $\alpha = \frac{1}{2}$, then, for any $z^\star \in D$, the condition $|z - z^\star|^{1-2\alpha} \leq c_1(1-\alpha)$ is true for any $z \in \mathbb{R}^n$ since $|z - z^\star|^{1-2\alpha} = |z - z^\star|^0 = 1 \leq \frac{1}{2}c_1$.

The following example illustrates the sufficient condition in Corollary 1, for which Proposition 6 is used to guarantee that item 2) in Corollary 1 holds.

Example 6. Consider the following hybrid system $\mathcal{H} = (C, f, D, g)$ with state $z \in \mathbb{R}$ and data given by

$$\begin{aligned}
f(z) &= -z & \forall\, z \in \mathbb{R} \\
C &:= \bigcup_{i \in \{2k : k \in \mathbb{N}\}} [i, i+1] \\
g(z) &= z - 1 & \forall\, z \in D := \{2k : k \in \mathbb{N} \setminus \{0\}\},
\end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{N}$. The conditions in Corollary 1 can be verified as follows. Each $\phi \in \mathcal{S}_{\mathcal{H}}$ is complete and its domain is unbounded in the t direction. Moreover, the flow map is continuously differentiable on $\overline{\text{con}}C$. Furthermore, for any $\phi \in \mathcal{S}_{\mathcal{H}}$ from $\phi(0,0) \in (C \cup D)$, denote $\rho^\star := \max\{x : x \in C, x \leq \phi(0,0)\}$. If $\rho^\star \leq 1$, then ϕ never jumps and the jump time between two consecutive jumps is bounded below by ∞. If $\rho^\star \geq 2$, the flow time between two consecutive jumps of

ϕ is bounded below by $\bar{\rho} := \ln \frac{\rho^\star}{\rho^\star - 1}$. For all $z \in \overline{\mathrm{con}}C$, $\nabla f(z) + \nabla f(z)^\top = -2$, so item 1) in Corollary 1 is satisfied with $\beta = 1$ and $P = I$. Moreover, given $z^\star \in D$, the function $V_1(z) = |z - z^\star|^2$ satisfies $\langle \nabla V_1(z), f(z) \rangle = 2(z - z^\star)(-z) \le -2z^\star(z - z^\star) = -2z^\star V_1^{\frac{1}{2}}(z)$ for $z \in C \bigcap ((z^\star + \bar{\rho}\mathbb{B}) \setminus D)$, where we used the property that $z \ge z^\star$ for all $z \in C \bigcap ((z^\star + \bar{\rho}\mathbb{B}) \setminus D)$. Furthermore, the function $V_2(z) = |z - g(z^\star)|^2$ satisfies $\langle \nabla V_2(z), f(z) \rangle = 2(z - g(z^\star))(-z) \le 2z^\star(g(z^\star) - z) = 2g(z^\star)V_2^{\frac{1}{2}}(z)$ for $z \in (g(z^\star) + \bar{\rho}\mathbb{B}) \bigcap C$, where we used the property that $z \le g(z^\star)$ for all $z \in (g(z^\star) + \bar{\rho}\mathbb{B}) \bigcap C$ and $g(z^\star) = z^\star - 1 < z^\star$. Then, Proposition 6 is satisfied with $c_1 = 2z^\star$, $\alpha = 1/2$, and $c_2 = 2(z^\star - 1) \in (0, c_1]$. Thus, the condition 2) in Corollary 1 is verified. Note that the condition 3) in Corollary 1 holds for free since $\beta = 1$, $c = 1$ and $\gamma = \bar{\rho} > 0$. Then, by Corollary 1, we have that $\mathscr{H}$ is δLAS. △

The following result establishes a sufficient condition for a set $\mathscr{S}_\mathscr{H}$ to be δLAS "through jumps." In particular, under such conditions, the graphical distance between any two maximal solutions to a hybrid system $\mathscr{H}$ strictly decreases during jumps. Due to such requirement, we need to impose the following assumption to guarantee that every maximal solution to $\mathscr{H}$ jumps infinitely many times.

Assumption 3.4 *The hybrid system $\mathscr{H} = (C, f, D, g)$ is such that every $\phi \in \mathscr{S}_\mathscr{H}$ satisfies $\sup_j \operatorname{dom} \phi = \infty$.*

Theorem 3. *(δLAS through jump for generic D) Consider a hybrid system $\mathscr{H} = (C, f, D, g)$ with state $z \in \mathbb{R}^n$. Suppose $\mathscr{H}$ satisfies Assumption 3.1, Assumption 3.3, and the hybrid basic conditions. If there exist $\delta_0, L_1, L_2 > 0, P = P^\top > 0$ such that*

1) $\nabla f(z)^\top P + P \nabla f(z) \le 0$ for all $z \in \overline{\mathrm{con}}C$;
2) for each $\delta \in [0, \delta_0]$, each maximal solution ϕ to $\mathscr{H}$ from $\phi(0,0)$ satisfying

$$\phi(0,0) \in C, \quad |\phi(0,0)|_D = \delta$$

satisfies $|\phi(s,0)|_D = 0$ for some $s \in [0, \delta]$;
3) for each $z \in D$ and each $\delta \in [0, \delta_0]$, the set $z + \delta\mathbb{B}$ is forward invariant for $\mathscr{H}$ from away of D;
4) the jump map g is locally Lipschitz on D with Lipschitz constant L_1, i.e., $|g(z_1) - g(z_2)| \le L_1|z_1 - z_2|$ for all $z_1, z_2 \in D$ such that $|z_1 - z_2| \le \delta_0$;
5) f is bounded on $\overline{\mathrm{con}}C$ with bound L_2, i.e., $|f(z)| \le L_2$ for all $z \in \overline{\mathrm{con}}C$;
6) $c(L_1 + L_2) \le 1$ where $c = \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}$;

then, the set $\mathscr{S}_\mathscr{H}$ is δS with d being the Euclidean distance. Furthermore, if L_1 and L_2 can be chosen such that $c(L_1 + L_2) < 1$ and $\mathscr{H}$ satisfies Assumption 3.4, then, $\mathscr{S}_\mathscr{H}$ is δLAS with d being the Euclidean distance.

Proof. Given $\varepsilon > 0$ and using δ_0 as in the item 2)-5) of assumption and γ as in Assumption 3.3, consider $\phi_1, \phi_2 \in \mathscr{S}_\mathscr{H}$ such that $|\phi_1(0,0) - \phi_2(0,0)| < \delta$, where δ is chosen such that

$$0 < \delta \le \min\left\{\frac{\varepsilon}{c}, \frac{\delta_0}{c}, \gamma\right\}.$$

First, we show that $\mathscr{S}_{\mathscr{H}}$ is δS for the case when $\phi_1(0,0), \phi_2(0,0) \in C$ and $\sup_j \phi_1 = \sup_j \operatorname{dom} \phi_2 = 0$. Similarly as derived in (27), by using item 1) and the comparison lemma, we have, for all $t \in [0, \infty)$,

$$|\phi_1(t,0) - \phi_2(t,0)| \le c|\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon.$$

Next, we show $\mathscr{S}_{\mathscr{H}}$ is δS for the case when either ϕ_1 or ϕ_2 jump. By the choice of δ and item 2), $\sup_j \operatorname{dom} \phi_1 = \sup_j \operatorname{dom} \phi_2 =: J$. Without loss of generality, assume ϕ_1 jumps first and $J = \infty$. Furthermore, for each $j \in \mathbb{N} \setminus \{0\}$, let $\bar{t}_j = \max_{(t,j-1) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$ and $\bar{t}'_j = \min_{(t,j) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$, and $\bar{t}'_0 = 0$. For simplicity, assume that the time when j-th jump occurs to ϕ_1 is always smaller than or equal to that of ϕ_2 for $j \in \mathbb{N}$.

(I) If $\phi_1(0,0), \phi_2(0,0) \in C$, by using item 1) and similar derivations in Theorem 2, we obtain for all $t \in [0, \bar{t}_1]$,

$$|\phi_1(t,0) - \phi_2(t,0)| \le c|\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon. \tag{49}$$

When $t = \bar{t}_1$, since ϕ_1 jumps first, $\phi_1(\bar{t}_1, 0) \in D$ and $\phi_1(\bar{t}_1, 1) = g(\phi_1(\bar{t}_1, 0))$. Note that under Assumption 3.3, $g(D) \cap D = \emptyset$. Then,

a. if $\phi_2(\bar{t}_1, 0) \in D$, i.e., $\bar{t}_1 = \bar{t}'_1$, by (49) and the choice of δ, $|\phi_1(\bar{t}_1, 0) - \phi_2(\bar{t}_1, 0)| \le \delta$ and $\phi_1(\bar{t}_1, 0), \phi_2(\bar{t}_1, 0) \in D$. By condition 4) and (49),

$$|\phi_1(\bar{t}_1, 1) - \phi_2(\bar{t}_1, 1)| \le L_1 |\phi_1(\bar{t}_1, 0) - \phi_2(\bar{t}_1, 0)| \le \varepsilon. \tag{50}$$

Since $\phi_1(\bar{t}_1, 1), \phi_2(\bar{t}_1, 1) \in C$, we can recursively apply the arguments in (I)

b. If $\phi_2(\bar{t}_1, 0) \notin D$, i.e., $\bar{t}_1 < \bar{t}'_1$, by (49), it follows that $\phi_2(\bar{t}_1, 0) \in (D + \delta\mathbb{B}) \setminus D$. For each $t \in [\bar{t}_1, \bar{t}'_1]$, since, $\phi_1(\bar{t}_1, 0) \in D$ and $|\phi_2(\bar{t}_1, 0) - \phi_1(\bar{t}_1, 0)| = \bar{\delta}_1$ for some $\bar{\delta}_1 \in [0, \delta]$, by item 2) and item 3), we obtain
   i. for each $j \in \mathbb{N} \setminus \{0\}$ and each $t \in [\bar{t}'_j, \bar{t}_{j+1}]$:

$$|\phi_1(t,j) - \phi_2(t,j)| \le (L_1 + L_2)^j c^{j+1} |\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon. \tag{51}$$

   ii. for each $j \in \mathbb{N} \setminus \{0\}$ and each $t \in [\bar{t}_j, \bar{t}'_j]$:

$$|\phi_2(t, j-1) - \phi_1(\bar{t}_j, j-1)| \le (L_1 + L_2)^{j-1} c^j |\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon. \tag{52}$$

   iii. for each $j \in \mathbb{N} \setminus \{0\}$ and each $t \in [\bar{t}_j, \bar{t}'_j]$:

$$|\phi_1(t,j) - \phi_2(\bar{t}'_j, j)| \le (L_1 + L_2)^{j-1} c^j |\phi_1(0,0) - \phi_2(0,0)| \le \varepsilon. \tag{53}$$

Therefore, ϕ_1 and ϕ_2 are ε-close.

The other cases follow similarly. Therefore, $\mathcal{S}_{\mathcal{H}}$ is δS with respect to Euclidean distance.

When the domain of each $\phi \in \mathcal{S}_{\mathcal{H}}$ is unbounded in the j direction and $c(L_1 + L_2) < 1$, the δLA property can be established by picking $0 < \mu \leq \min\left\{\frac{\delta_0}{c}, \gamma\right\}$, for a given $\varepsilon' > 0$, pick $T = \max\left\{1, \log_{c(L_1+L_2)} \frac{\varepsilon'}{c\mu}\right\} + 1$. □

The following example illustrates the conditions in Theorem 3.

Example 7. Consider a timer system $\mathcal{H}$ with state $z \in \mathbb{R}$ and data given by

$$\begin{aligned} \dot{z} &= -1 \qquad z \geq 0, \\ z^+ &= 1 \qquad z = 0. \end{aligned}$$

Each maximal solution ϕ to it has a domain that is unbounded in the t and j direction. Moreover, the flow time between two consecutive jumps of ϕ is lower bounded by 1. The condition in item 1) of Theorem 3 can be verified with $P = I$ as $\nabla f(z) + \nabla f(z)^\top = 0$ for all $z \in \overline{\mathrm{con}}C$. The condition in item 2) can be verified according to Proposition 6. Consider $\delta_0 \in (0,1)$ and the function $V(z) = |z|_D^2 = z^2$. For each $z \in (D + \delta_0\mathbb{B}) \cap C \setminus D$, we have $\langle \nabla V(z), f(z) \rangle = -2z = -2V^{\frac{1}{2}}(z)$, where we used the property that $z \in [0,1]$. Item 3) of Theorem 3 follows from the fact $D = \{0\}$ is a singleton and $\langle \nabla V(z), f(z) \rangle = -2z < 0$ for all $z \in (D + \delta_0\mathbb{B}) \cap C \setminus D$. Item 4) of Theorem 3 is satisfied with $L_1 = 0$, and item 5) of Theorem 3 is satisfied with $L_2 = 1$. Item 6) of Theorem 3 holds for free since $c = 1$. Therefore, the set $\mathcal{S}_{\mathcal{H}}$ is δS with d being the Euclidean distance. △

4 Final Remarks

In this chapter, we introduced and studied several notions of graphical incremental stability for hybrid systems. When compared to the pointwise distance, the proposed graphical notion can be applied to systems with "peaking phenomenon," which is a typical behavior in tracking and observer design for hybrid systems. Graphical incremental stability involves a convergence property where solutions converge to each other. Several sufficient and necessary conditions for a hybrid system to be graphically incrementally stable and graphically incrementally attractive were provided and illustrated in examples.

An alternative approach to using the graphical distance is to prioritize ordinary time. When one prioritizes ordinary time t, i.e., studying the incremental property of solutions' projection to the t direction, it leads to the result as in [7]. Note that the notion defined therein imposes the incremental stability property in some of the state components. This is due to the fact that when studying the incremental stability for certain hybrid systems, such as mechanical systems and dynamical systems that are dominated by continuous-time behavior, one may not be interested in having state components pertaining to variables such as timers, logic variables, and memory states to have the incremental stability property.

The results in [7] cover results for continuous-time system as in [2]. In [2], several sufficient and necessary conditions for continuous-time systems to be incrementally stable are provided. For continuous-time systems, incremental stability has also been studied in more general spaces and using general distance notions, such as the Riemannian distance in the context of contraction theory; see, e.g., the study of contracting and nonexpansive flows in [13, 14], the local arguments in [9], and the regional results in [15] in the context of observer design. Due to often being misinterpreted as a property of convergent systems [16], the authors in [17] provide a rigorous comparison between incremental stability and the property of convergent systems, and conclude that neither implies the other.

Following the ideas in [7, 18], one could alternatively define a notion that prioritizes jumps and mimics the case of purely discrete-time systems. Unfortunately, such a notion would only apply to a narrow class of hybrid system due to the general aforementioned difficulty. For instance, for the rather elementary set of hybrid trajectories in Example 1, the pointwise distance between every pair of trajectories with different initial conditions is clearly nondrecreasing as a function of t, while the graphical distance between them is small and, as shown in Example 3, the system is graphically incrementally stable. As argued in this chapter, for hybrid systems that exhibit a "peaking phenomenon," see, e.g., [19, 20], approaches that prioritize ordinary time t or jump time j in the incremental stability notion do not have broad applicability in the analysis of incremental stability for hybrid dynamical systems.

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5 Appendix

5.1 Proof of Corollary 1

Given $\varepsilon > 0$, and using δ_0, γ as in the assumption, consider $\phi_1, \phi_2 \in \mathscr{S}_{\mathscr{H}}$ such that $|\phi_1(0,0) - \phi_2(0,0)| < \delta$, where δ is chosen such that

$$0 < \delta \leq \min\left\{\frac{\varepsilon}{c}, \frac{\delta_0}{c}, \gamma - \frac{1}{\beta}\ln c\right\}$$

and, for each $z \in D$, $(z + \delta\mathbb{B}) \cap D = \{z\}$; namely $z + \delta\mathbb{B}$ is a small neighborhood around z that does not intersect D. Note that from condition 3), $\gamma - \frac{1}{\beta}\ln c > 0$.

First, we show that $\mathscr{S}_{\mathscr{H}}$ is δS for the case when $\phi_1(0,0), \phi_2(0,0) \in C$ and $\sup_j \operatorname{dom} \phi_1 = \sup_j \operatorname{dom} \phi_2 = 0$, i.e., no jump occurs to either ϕ_1 or ϕ_2. Similarly as derived in (27), by using condition 1) and the comparison lemma, we obtain for all $(t,0) \in \operatorname{dom} \phi_1 (= \operatorname{dom} \phi_2 = [0,\infty) \times \{0\})$,

$$|\phi_1(t,0) - \phi_2(t,0)|_P \leq c \exp(-\beta t) |\phi_1(0,0) - \phi_2(0,0)| \leq \varepsilon. \tag{54}$$

Next, we show $\mathscr{S}_{\mathscr{H}}$ is δS for the case when either ϕ_1 or ϕ_2 jump. By the choice of δ and item 2), $\sup_j \operatorname{dom} \phi_1 = \sup_j \operatorname{dom} \phi_2 =: J$. This can be verified as follows. When ϕ_1 flows to the jump set D, ϕ_2 is within the δ neighborhood of ϕ_1, then, by item 2), ϕ_2 flows into the jump set D within δ time. Furthermore, since $\gamma \geq \delta$, therefore, ϕ_1 will not jump again before ϕ_2 jumps. Without loss of generality, assume ϕ_1 jumps first and $J = \infty$ (Alternatively, we could pick J large enough, but ∞ suffices). Furthermore, for each $j \in \mathbb{N} \setminus \{0\}$, let $\bar{t}_j = \max_{(t,j-1) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$ and $\bar{t}'_j = \min_{(t,j) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$, and $\bar{t}'_0 = 0$, where $\bar{t}_j$ denotes the minimum time in $\operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2$ when at least one of the two solutions ϕ_1, ϕ_2 has jumped j times (note that $\bar{t}_j$ and $\bar{t}'_j$ are not necessarily jump times of both solutions), while $\bar{t}'_j$ denotes the minimum time when both solutions have jumped j times. In fact, $[\bar{t}'_j, \bar{t}_{j+1}] \times \{j\} \subset \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2$ for all $j \in \mathbb{N}$. Note that with Assumption 3.3 and the choice of δ, $\phi_1(\bar{t}_1, 0) = \phi_2(\bar{t}'_1, 0)$ and $\bar{t}'_1 > \bar{t}_1$. By the uniqueness of solutions, $\phi_1(\bar{t}_1, 1) = \phi_2(\bar{t}'_1, 1)$ and ϕ_2 is "following" the trajectory of ϕ_1 after that, which implies that ϕ_1 and ϕ_2 jumps one after another. In particular, after the j-th jump occurs to ϕ_1, the j-th jump occurs to ϕ_2 before the $(j+1)$-th jump occurs to ϕ_1. The derivation follows the steps as in Theorem 2. The main difference is that in the derivation of (32) and (33), instead of using the condition 2) in Theorem 2, we use condition 2) of Corollary 1.

The proof for δLA follows similarly as that in Theorem 2 with $\mu > 0$ and $\mu < \min\left\{ \frac{\delta_0}{c}, \gamma - \frac{1}{\beta} \ln c \right\}$.