

Sufficient Conditions for Asymptotic Stability and Feedback Control of Set Dynamical Systems

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Abstract—In this paper, stability properties for discrete-time dynamical systems with set-valued states are studied. We use previous results on detectability and invariance properties to present an extension of Krasovskii and Lyapunov stability results for set dynamical systems, under the assumption of outer semicontinuity of the set-valued maps that define the system’s dynamics. We also propose a formulation for closed-loop control systems with state-feedback, within the framework of set dynamical systems. Examples illustrate the results.

I. INTRODUCTION

Convergence and stability properties of dynamical systems are key topics in control systems design. The task of guaranteeing stability for closed-loop dynamics becomes particularly challenging in the presence of varying parameters, noise in the state, or general disturbances. One practical approach to model these behaviors is to represent variation ranges as sets, and to take advantage of already available set-theoretic methods to characterize their effects on the system and then specify system performance in control design [1] [2]. Formulations based on set-theoretic frameworks involve the use of techniques based on the properties of subsets of the state space, such as invariant sets theory.

A particular such approach involves the study of behavioral properties of systems, where set-valued states account for representing variables or multivalued signals. Early results in this direction include the work by Pelczar [3], where a type of generalized systems is presented and basic stability properties are studied, with follow-up developments for the study of limit sets in [4] and [5]. Notions of reachability for generalized pseudo dynamical systems are formally studied in [6], and properties of systems with set-valued states in continuous time are studied in [7]. Other contributions to describe this type of systems can be found in [8]. More recently, results associated to controlled invariant sets for systems using a set-valued state approach can be found in [9], for the case of state-feedback using set iteration, and in [10], for the case of output feedback, where set invariance results for bounded disturbances are obtained using information sets as a parameter for the calculation of the control input. For the characterization of convergence and general properties of solutions to systems with set-valued states, called *set dynamical systems*, see [11]. Detectability and invariance principles in the same framework are presented in [12].

Even though results to characterize stability and convergence of solutions for systems with set-valued states exist, tools that resemble those of classical Lyapunov based methods have not yet been fully developed, to the best knowledge of the authors. This paper presents such tools for set dynamical systems.

Set dynamical systems are discrete-time systems whose state evolution is defined in Euclidean space by a set-valued map and a constraint given in terms of a set. Solutions associated to such systems are given by sequences of sets, rather than just by sequences of single points, as in the case of classical dynamical systems. We study the stability properties of set dynamical systems and propose a formulation for state-feedback control with set-valued states. We follow similar ideas to those existing for classical dynamical systems and use recent results on convergence via invariance principles for set dynamical systems in [11], [12] to generate versions of Krasovskii and Lyapunov stability theorems for set dynamical systems. We also provide sufficient conditions for the design of stabilizing state-feedback controllers for such systems. Although results characterizing invariance for systems with set-valued states for discrete time-systems are available (see [10]), our approach provides a general framework along with tools to analyze stability using Lyapunov-like functions.

This paper is organized as follows. After basic notation is introduced, Section II presents the framework for set dynamical systems, along with basic concepts and properties that are relevant to characterize their behavior. The main results of the paper are presented in Section III and Section IV. In Section III, the Krasovskii and Lyapunov Theorems for set dynamical systems are presented, while the formulation of state-feedback control with set-valued states is presented in Section IV. Results are illustrated in examples along the paper. Complete proofs will be published elsewhere.

II. PRELIMINARIES ON SET DYNAMICAL SYSTEMS

A. Notation

The following notation is used throughout this paper. \mathbb{N} denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \dots\}$. \mathbb{R}^n denotes the n -dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers. \mathbb{B} denotes the closed unit ball around the origin in Euclidean space. $\text{dom } V$ denotes the domain of definition for the map V . Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. For a closed set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the distance $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$. The empty set is represented by \emptyset . Given a function $V : \text{dom } V \rightarrow \mathbb{R}$ and a constant $r \in \mathbb{R}$, its r -sublevel set is given by $L_V(r) := \{x \in \text{dom } V : V(x) \leq r\}$. In most

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cases, lower case letters are used to represent singletons and uppercase letters are used to refer set-valued variables.

B. Properties of sets

Some basic definitions and properties that are used to characterize set dynamical systems are given in this section.

Definition 2.1 (Distance between sets): The Hausdorff distance between two closed sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}^n$ is given by

$$d(\mathcal{A}_1, \mathcal{A}_2) = \max \left\{ \sup_{x \in \mathcal{A}_1} |x|_{\mathcal{A}_2}, \sup_{z \in \mathcal{A}_2} |z|_{\mathcal{A}_1} \right\}$$

Definition 2.2 (inner and outer limit): For a sequence of sets $\{T_i\}_{i=0}^{\infty}$ in \mathbb{R}^n :

- The inner limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted $\liminf_{i \rightarrow \infty} T_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist points $x_i \in T_i, i \in \mathbb{N}$, such that $\lim_{i \rightarrow \infty} x_i = x$.
- The outer limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted $\limsup_{i \rightarrow \infty} T_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ of $\{T_i\}_{i=0}^{\infty}$ and points $x_k \in T_{i_k}, k \in \mathbb{N}$, such that $\lim_{k \rightarrow \infty} x_k = x$.

The *limit* of the sequence exists if the outer and the inner limit sets are equal, namely

$$\lim_{i \rightarrow \infty} T_i = \liminf_{i \rightarrow \infty} T_i = \limsup_{i \rightarrow \infty} T_i$$

The inner and outer limit of a sequence always exist and are closed, although the limit itself might not exist.

Definition 2.3: [13, Convergence of a sequence of sets] When the limit of the sequence $\{T_i\}_{i=0}^{\infty}$ in \mathbb{R}^n exists in the sense of Definition 2.2, and is equal to T , the sequence of sets $\{T_i\}_{i=0}^{\infty}$ is said to converge to the set T .

C. Set Dynamical Systems with outputs

We consider set dynamical systems defined by

$$\begin{aligned} \mathfrak{X}^+ &= G(\mathfrak{X}) \\ \mathcal{Y} &= H(\mathfrak{X}) \\ \mathfrak{X} &\subset D \end{aligned} \quad (1)$$

where \mathfrak{X} is the set-valued state, \mathcal{Y} is the system's output, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are set-valued maps defining the right-hand side and the output map, respectively, and $D \subset \mathbb{R}^n$ defines a constraint that solutions to the system must satisfy. A solution to the system in (1) is defined as the sequence of nonempty sets $\{\mathfrak{X}_j\}_{j=0}^J$ and its associated output is defined by the sequence $\{\mathcal{Y}_j\}_{j=0}^J, J \in \mathbb{N} \cup \{\infty\}$, satisfying¹

$$\begin{aligned} \mathfrak{X}_{j+1} &= G(\mathfrak{X}_j) \\ \mathcal{Y}_j &= H(\mathfrak{X}_j) \\ \mathfrak{X}_j &\subset D \end{aligned}$$

for all $j \in \text{dom } \mathfrak{X}_J$, where $\text{dom } \mathfrak{X}_J$ is the domain of definition of the solution $\{\mathfrak{X}_j\}_{j=0}^J$, which is given by the collection $\{0, 1, 2, \dots, J\} \cap \mathbb{N}$. The first entry of the solution, \mathfrak{X}_0 , is the initial set. We assume \mathfrak{X}_0 to be compact. If a solution has $J = 0$ then we say that it is *trivial*, and if it

¹The solution is defined as a collection of sets, but its elements for each j are used to evaluate G and H .

has $J > 0$ we say that it is *nontrivial*. If it has $J = \infty$, we say that it is *complete*. A solution $\{\mathfrak{X}_j\}_{j=0}^J$ is said to be maximal if it cannot be further extended. Given an initial set $\mathfrak{X}_0 \subset \mathbb{R}^n$, $S(\mathfrak{X}_0)$ denotes the set of maximal solutions to (1) from \mathfrak{X}_0 ².

To make notation easier to follow, at times, the collection of sets given by the sequence $\{\mathfrak{X}_j\}_{j=0}^J$ is represented as \mathfrak{X}_j (or even just \mathfrak{X}). Notation $\{\mathfrak{X}_j^i\}_{i=0}^{\infty}$ refers to the sequence of solutions \mathfrak{X}_j^i , indexed by i , where j is the associated discrete time. We make the same notational simplification when referring to the output \mathcal{Y} . Also, the term solution-output pair $\{\mathfrak{X}, \mathcal{Y}\}$ is used to represent a solution \mathfrak{X} and its associated output $\mathcal{Y} = H(\mathfrak{X})$.

Now we provide some basic definitions and assumptions for set dynamical systems that will be used in the following sections.

Definition 2.4 (outer semicontinuity): The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=0}^{\infty}$ converging to $x \in \mathbb{R}^n$ and each sequence $\{y_i\}_{i=0}^{\infty}$ such that $y_i \in G(x_i)$ for each i , converging to y , it holds that $y \in G(x)$. It is *outer semicontinuous* if G is outer semicontinuous at each $x \in \mathbb{R}^n$.

Definition 2.5 (locally bounded): The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded if, for each compact set $K \subset \mathbb{R}^n$, there exists a compact set $K' \subset \mathbb{R}^n$ such that $G(K) \subset K'$.

We consider the following concept of invariance and omega limit set for set dynamical systems

Definition 2.6 (forward and backward invariance):

A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *forward invariant* for (1) if for every set $T \subset \mathcal{M} \cap D$ we have $G(T) \subset \mathcal{M} \cap D$. A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *backward invariant* for (1) if for every set $T' \subset \mathcal{M} \cap D$ for which there exists a set T with the property $T' = G(T)$, we have $T \subset \mathcal{M} \cap D$ for every such set T . A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *invariant* if it is both forward and backward invariant.

Definition 2.7 (ω -limit set): The ω -limit set of a solution $\{\mathfrak{X}_j\}_{j=0}^J$ to (1) is given by

$$\tilde{\omega}(\mathfrak{X}_j) = \{Y \subset \mathbb{R}^n : \exists \{j_i\}_{i=0}^{\infty}, \lim_{i \rightarrow \infty} j_i = \infty, Y = \lim_{i \rightarrow \infty} \mathfrak{X}_{j_i}\}$$

Note that $\tilde{\omega}(\mathfrak{X}_j)$ is a collection of sets.

In the following sections, some results for set dynamical systems are presented under the following assumption on their data.

Assumption 2.8: The set dynamical system defined in (1), with data (D, G, H) satisfies the following properties:

- (A0) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded, and, for each $x \in D$, $G(x)$ is a nonempty subset of \mathbb{R}^n .
- (A1) The set $D \subset \mathbb{R}^n$ is closed.
- (A2) The set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous, locally bounded, and, for each $x \in D$, $H(x)$ is a nonempty subset of \mathbb{R}^m .

²By [11, Lemma 4.5], maximal solutions to (1) are unique.

III. STABILITY PROPERTIES OF SET DYNAMICAL SYSTEMS

This section pertains to the formulation of sufficient conditions guaranteeing stability properties of set dynamical systems.

Definition 3.1 (stability of a set): A closed set $\mathcal{A} \subset \mathbb{R}^n$ is stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution \mathfrak{X} to (1) with $d(\mathfrak{X}_0, \mathcal{A}) \leq \delta$ satisfies $d(\mathfrak{X}_j, \mathcal{A}) \leq \epsilon$ for all $j \in \text{dom } \mathfrak{X}$.

Definition 3.2 (attractivity): A closed set $\mathcal{A} \subset \mathbb{R}^n$ is locally attractive for (1) if there is $\rho > 0$ such that for any compact set $\mathfrak{X}_0 \subset \mathcal{A} + \rho\mathbb{B}$, $\mathfrak{X} \in S(\mathfrak{X}_0)$ is complete and satisfies

$$\lim_{j \rightarrow \infty} d(\mathfrak{X}_j, \mathcal{A}) = 0$$

Definition 3.3 (asymptotic stability of a set): The compact set $\mathcal{A} \subset \mathbb{R}^n$ is asymptotically stable if it is stable and locally attractive.

Next, we propose conditions that resemble those in Krasovskii and Lyapunov stability theorems, which are based on the existence of a Lyapunov-like function (see e.g. [11] [14]) for set dynamical systems.

Theorem 3.4: (Krasovskii-type sufficient conditions for set dynamical systems) Suppose the data of the set dynamical system in (1) satisfies Assumption 2.8. Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set and $\mathcal{M} \subset \mathbb{R}^n$ contain a neighborhood of \mathcal{A} . If

(\star) There exists a function $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ that is continuous on \mathcal{M} and positive definite on $\mathcal{M} \cap D$ with respect to \mathcal{A} , and a map $u_D : \mathcal{M} \rightarrow [-\infty, +\infty]$ defined as

$$u_D(\mathfrak{X}) = \begin{cases} \sup_{\eta \in G(\mathfrak{X})} V(\eta) - \sup_{x \in \mathfrak{X}} V(x) & \text{if } \mathfrak{X} \subset D \\ -\infty & \text{otherwise} \end{cases}$$

that satisfies $u_D(\mathfrak{X}) \leq 0$ for all $\mathfrak{X} \subset \mathcal{M}$,

then \mathcal{A} is stable.

Suppose additionally that

($\star\star$) there exists $r^* > 0$ such that for all $r \in (0, r^*)$, the largest invariant set in

$$E \cap L_V(r) \cap D \quad (2)$$

is empty, where

$$E = \{X \subset D : \sup_{\eta \in G(X)} V(\eta) = \sup_{x \in X} V(x)\}$$

$$L_V(r) = \{x \in D : V(x) \leq r\}$$

Then \mathcal{A} is asymptotically stable.

Proof Sketch: Assume (\star) and let $\epsilon > 0$ be small enough so that $\mathcal{A} + 2\epsilon\mathbb{B} \subset \mathcal{M}$. We claim there exist $r_\epsilon > 0$ such that for $\mathfrak{X} \subset (\mathcal{A} + 2\epsilon\mathbb{B}) \cap D$, if $\sup_{x \in \mathfrak{X}} V(x) \leq r_\epsilon$ then

$$\mathfrak{X} \subset (\mathcal{A} + \epsilon\mathbb{B}) \cap D, G(\mathfrak{X}) \subset (\mathcal{A} + \epsilon\mathbb{B}) \cap D \quad (3)$$

Note that $u_D(\mathfrak{X}) \leq 0$ for all $\mathfrak{X} \subset \mathcal{A}$ and V is positive definite on D with respect to \mathcal{A} , so $G(\mathcal{A} \cap D) \subset \mathcal{A} \cap D$. Since by Assumption 2.8, G is outer semicontinuous and bounded, there exists $\gamma > 0$ so that $G(\mathcal{A} + \gamma\mathbb{B}) \subset \mathcal{A} + \epsilon\mathbb{B}$. Based on

(3), we can claim that the collection of sets

$$\mathcal{N} = \{X \subset (\mathcal{A} + \epsilon\mathbb{B}) \cap D : \sup_{x \in X} V(x) \leq r_\epsilon\}$$

is forward invariant for (1). Relying on forward invariance of \mathcal{N} , maximal solutions $\mathfrak{X} \in S(X)$, with $X \subset (\mathcal{A} + \delta\mathbb{B}) \cap D$ will be contained in $\mathcal{A} + \epsilon\mathbb{B}$, leading to the set \mathcal{A} being stable. To show attractivity, note that given $\epsilon > 0$ with $(\mathcal{A} + \epsilon\mathbb{B}) \subset \mathcal{M}$, we can find $r_\epsilon \in (0, r)$, with r as in condition ($\star\star$) so that \mathcal{N} is forward invariant and apply Theorem 4.9 from [12] ■

Example 3.5 (Illustration of Theorem 3.4): Consider the set dynamical system

$$\begin{aligned} \mathfrak{X}^+ &= G(\mathfrak{X}) \\ \mathfrak{X} &\subset D \end{aligned}$$

where $G(\mathfrak{X}) = \{g(x) : x \in \mathfrak{X}\}$, $g(x) = \begin{bmatrix} \frac{x_2}{\alpha(1+x_2^2)} \\ \frac{x_1}{\beta(1+x_2^2)} \end{bmatrix}$

and $D \subset \mathbb{R}^2$ a compact set, $\alpha, \beta > 0$, along with the sets $\mathcal{M} = \{x \in \mathbb{R}^2 : |x| \leq \bar{m}\}$, with $\bar{m} \in \mathbb{R}_{\geq 0}$ and the set $\mathcal{A} = \{(0, 0)\}$. The data of this set dynamical system satisfies Assumption 2.8 since g is continuous and D is compact. Consider the function $V(x) = x^\top x$ for each $x \in \mathbb{R}^2$ which is continuous and positive definite with respect to \mathcal{A} . Let $\epsilon \in (0, \bar{m})$ be such that $\mathfrak{X} \subset \mathcal{A} + \epsilon\mathbb{B} \subset \mathcal{M}$. Then $\sup_{x \in \mathfrak{X}} V(x) \leq \epsilon^2$. Now, with $V(\eta) = \eta_1^2 + \eta_2^2$ and

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \bigcup_{x \in \mathfrak{X}} \begin{bmatrix} \frac{x_2}{\alpha(1+x_2^2)} \\ \frac{x_1}{\beta(1+x_2^2)} \end{bmatrix}$$

we have

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) = \sup \left\{ \eta_1^2 + \eta_2^2 : (\eta_1, \eta_2) \in \bigcup_{x \in \mathfrak{X}} \begin{bmatrix} \frac{x_2}{\alpha(1+x_2^2)} \\ \frac{x_1}{\beta(1+x_2^2)} \end{bmatrix} \right\}$$

Since \mathfrak{X} is compact, we have that, for $\mathfrak{X} \subset \mathcal{A} + \epsilon\mathbb{B} \subset \mathcal{M}$,

$$\begin{aligned} \sup_{\eta \in G(\mathfrak{X})} V(\eta) &= \max \left\{ \eta_1^2 + \eta_2^2 : (\eta_1, \eta_2) \in \bigcup_{x \in \mathfrak{X}} \begin{bmatrix} \frac{x_2}{\alpha(1+x_2^2)} \\ \frac{x_1}{\beta(1+x_2^2)} \end{bmatrix} \right\} \\ &= \max_{x \in \mathfrak{X}} \left\{ \frac{x_2^2}{\alpha^2(1+x_2^2)^2} + \frac{x_1^2}{\beta^2(1+x_2^2)^2} \right\} \\ &= \max \left\{ \frac{1}{\beta^2}, \frac{1}{\alpha^2} \right\} \epsilon^2 \end{aligned}$$

Recalling that

$$u_D(\mathfrak{X}) = \begin{cases} \sup_{\eta \in G(\mathfrak{X})} V(\eta) - \sup_{x \in \mathfrak{X}} V(x) & \text{if } \mathfrak{X} \subset D \\ -\infty & \text{otherwise} \end{cases}$$

since we have that $\sup_{\eta \in G(\mathfrak{X})} V(\eta) \leq \max \left\{ \frac{1}{\beta^2}, \frac{1}{\alpha^2} \right\} \epsilon^2$ and

$\sup_{x \in \mathfrak{X}} V(x) \leq \epsilon^2$ it follows that for $\alpha, \beta \geq 1$ we have that $u_D(\mathfrak{X}) \leq 0$. With $\sup_{x \in \mathfrak{X}} V(x) \leq \epsilon^2$, we have $\mathfrak{X} \subset \mathcal{A} + \epsilon\mathbb{B}$ implies $G(\mathfrak{X}) \subset \mathcal{A} + \epsilon\mathbb{B}$. Then we can find r^* as in condition ($\star\star$) in Theorem 3.4 and the sublevel set $L_V(r)$ defines a forward invariant set for $\mathfrak{X} \subset \mathcal{A} + \epsilon\mathbb{B}$. Now, since V is continuous, we can find $\delta > 0$ such that solutions starting

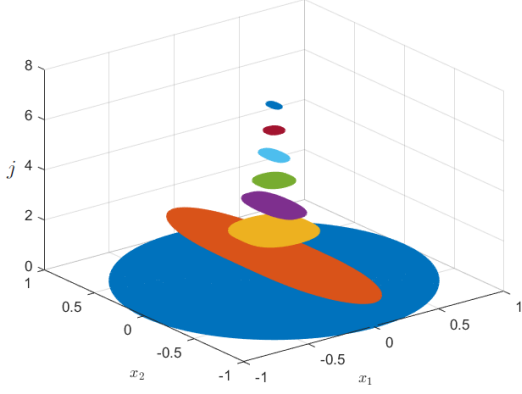


Fig. 1. Solution to the set dynamical system in Example 3.5, with $\alpha = 2$ and $\beta = 1$ and initial condition: $\mathfrak{X}_0 = \{(x_1, x_2) \in \mathbb{R}^2 : |x| \leq 1\}$

inside of $\delta\mathbb{B}$ are within the sublevel set, and since $L_V(r)$ is invariant, solutions will remain there. In particular, we can select $\delta = \epsilon$, and solutions that start inside of the set defined by $\delta\mathbb{B}$ will remain in $\mathcal{A} + \epsilon\mathbb{B}$, thus leading to $\mathcal{A} = \{(0, 0)\}$ being stable. An illustration for the case where $\alpha = 2$ and $\beta = 1$, where the initial set starts in $\epsilon\mathbb{B}$ is shown in Figure 1.

Theorem 3.6: (Lyapunov-type sufficient conditions for set dynamical systems) Given a set dynamical system as defined in (1) with data satisfying Assumption 2.8, a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a set $\mathcal{M} \subset \mathbb{R}^n$ that contains a neighborhood of \mathcal{A} , suppose that (\star) of Theorem 3.4 holds and that, furthermore, $u_D(\mathfrak{X}) < 0$ for all $\mathfrak{X} \subset \mathcal{M} \setminus \mathcal{A}$. Then, \mathcal{A} is attractive and, hence, locally asymptotically stable.

Example 3.7 (Illustration of Theorem 3.6): Consider the system from Example 3.5, now with both $\alpha, \beta > 1$. Its data satisfies Assumption 2.8. Consider the function V as defined in Example 3.5. Now, since $\alpha, \beta > 1$ we have that for $\mathfrak{X} \subset \mathcal{A} + \epsilon\mathbb{B} \subset \mathcal{M}$, $\sup_{x \in \mathfrak{X}} V(x) \leq \epsilon^2$ and

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) \leq \max \left\{ \frac{1}{\beta^2}, \frac{1}{\alpha^2} \right\} \epsilon^2$$

so $u_D(\mathfrak{X}) < 0$ for all $\mathfrak{X} \subset D \setminus \mathcal{A}$. Because

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) < \sup_{x \in \mathfrak{X}} V(x)$$

the sequence given by $\{\sup_{x \in \mathfrak{X}_j} V(x)\}_{j=0}^{\infty}$ converges to 0. Then, $r = 0$ and the sublevel set $L_V(0)$ is forward invariant. An illustration for $\alpha = \beta = 2$ is shown in Figure 2.

IV. CONTROL OF SET DYNAMICAL SYSTEMS

We now consider set dynamical systems with a set-valued external control input. More precisely, we consider a controlled system consisting of a physical process with dynamics described by

$$\begin{aligned} \mathfrak{X}_p^+ &= G_p(\mathfrak{X}_p, \mathcal{U}_p) \\ \mathcal{Y}_p &= H_p(\mathfrak{X}_p, \mathcal{U}_p) \\ (\mathfrak{X}_p, \mathcal{U}_p) &\subset D_p, \mathfrak{X}_p \subset \mathbb{R}^{n_p}, \mathcal{U}_p \subset \mathbb{R}^{m_p} \end{aligned} \quad (4)$$

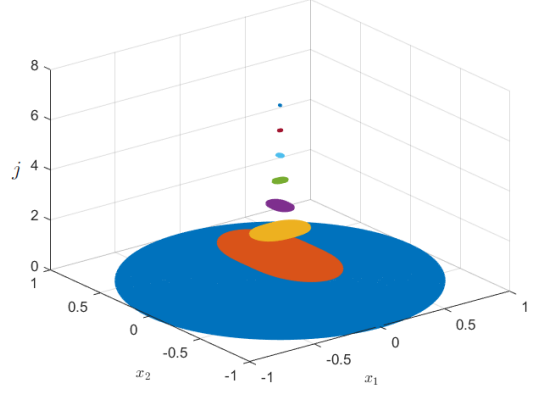


Fig. 2. Solution to the set dynamical system in Example 3.7, with $\alpha = \beta = 2$ and initial condition: $\mathfrak{X}_0 = \{(x_1, x_2) \in \mathbb{R}^2 : |x| \leq 1\}$

and a set-valued controller described as

$$\begin{aligned} \mathfrak{X}_c^+ &= G_c(\mathfrak{X}_c, \mathcal{U}_c) \\ \mathcal{Y}_c &= H_c(\mathfrak{X}_c) \\ \mathfrak{X}_c &\subset \mathbb{R}^{n_c}, \mathcal{U}_c \subset \mathbb{R}^{m_c} \end{aligned} \quad (5)$$

The closed-loop system resulting from controlling the set dynamical system (4) with the set-valued controller in (5) obtained through the assignment $\mathcal{U}_p = \mathcal{Y}_c$ and $\mathcal{U}_c = \mathcal{Y}_p$ can be represented as a set dynamical system in (1) with

$$\begin{aligned} G(\mathfrak{X}) &= \left[\begin{array}{c} G_p(\mathfrak{X}_p, H_c(\mathfrak{X}_c)) \\ G_c(\mathfrak{X}_c, H_p(\mathfrak{X}_p, H_c(\mathfrak{X}_c))) \end{array} \right] \\ D &:= \{(\mathfrak{X}_p, \mathfrak{X}_c) \subset \mathbb{R}^n : (\mathfrak{X}_p, H_c(\mathfrak{X}_c)) \subset D_p\} \end{aligned} \quad (6)$$

and an arbitrary function H , where $\mathfrak{X} = (\mathfrak{X}_p, \mathfrak{X}_c) \subset \mathbb{R}^n$ is now the set-valued state and $n = n_p + n_c$.

Given (D_p, G_p, H_p) the results presented in Theorems 3.4 and 3.6 can be used to design (5), by specifying conditions on the data (G_c, H_c) defining the controller such that the resulting closed-loop system has a stability property. The following result provides conditions for stability of the closed-loop set dynamical system with data in (6).

Theorem 4.1: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, a set $\mathcal{M} \subset \mathbb{R}^n$ that contains a neighborhood of \mathcal{A} , a plant represented by the set dynamical system in (4), and a controller defined in terms of (5), suppose

- 1) *There exist functions G_c and H_c such that the resulting closed-loop system (6) satisfies (A0) and (A1) in Assumption 2.8; and*
- 2) *There exists a function $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$, continuous on \mathcal{M} and positive definite on $\mathcal{M} \cap D$ with respect to \mathcal{A} , such that the map $u_D : \mathcal{M} \rightarrow [-\infty, +\infty]$ defined as*

$$u_D(\mathfrak{X}) = \begin{cases} \sup_{\eta \in G(\mathfrak{X})} V(\eta) - \sup_{x \in \mathfrak{X}} V(x) & \text{if } \mathfrak{X} \subset D \\ -\infty & \text{otherwise} \end{cases}$$

where

$\eta = (\eta_p, \eta_c)$, $G(\mathfrak{X}) = (G_p(\mathfrak{X}_p, \mathcal{U}_p), G_c(\mathfrak{X}_c, \mathcal{U}_c))$, and $\mathfrak{X} = (\mathfrak{X}_p, \mathfrak{X}_c)$ satisfies $u_D(\mathfrak{X}) \leq 0$ for all $\mathfrak{X} \subset \mathcal{M}$ and $u_D(\mathfrak{X}) < 0$ for all $\mathfrak{X} \subset \mathcal{M} \setminus \mathcal{A}$.

Then, the set \mathcal{A} is asymptotically stable for the closed-loop system in (6).

Using Theorem 4.1, controller design can be performed by either selecting G_c and H_c , and then finding a Lyapunov-like function V or, by selecting a candidate function V and then defining a control law that satisfies the previously stated stability conditions. This methodology is illustrated in examples below.

Example 4.2 (Control design with unmodeled dynamics):

Consider the special case given by the dynamical system described by $x^+ = g(x, u)$ with state dependent disturbances d_i , with $i \in \{1, 2, 3\}$, associated to state, input and unmodeled dynamics respectively. Effects of these disturbances can be represented by the set dynamical system

$$G_p(\mathfrak{X}_p, \mathcal{U}_p) = \bigcup_{x \in \mathfrak{X}_p} \bigcup_{u \in \mathcal{U}_p} (g(x + d_1(x)\mathbb{B}, u + d_2(x)\mathbb{B}) + d_3(x)\mathbb{B})$$

Let us consider $\mathcal{Y}_p = \mathfrak{X}_p$. A feedback control strategy can be designed in terms of (6) by defining a controller with output $\mathcal{Y}_c = H_c(\mathfrak{X}_c)$ and dynamics $\mathfrak{X}_c^+ = G_c(\mathfrak{X}_c, \mathfrak{X}_p)$ where G_c and H_c are selected such that the resulting feedback system has a desired set asymptotically stable. Such a property would be robust to the effects of state, input and unmodeled dynamics disturbances.

Now, consider the particular choice $x^+ = Ax + Bu$ with bounded unmodeled dynamics disturbances d_3 associated to parameter variation, with $|x| \leq \gamma$, for some $\gamma \geq 0$. This system can be represented in terms of (6) with

$$G_p(\mathfrak{X}_p, \mathcal{U}_p) = \bigcup_{x \in \mathfrak{X}_p} \bigcup_{u \in \mathcal{U}_p} ((Ax + Bu) + d_3(x))$$

and $D_p = \{x \in \mathbb{R}^{n_p} : |x| \leq \rho\}$, $\rho \in \mathbb{R}_{\geq 0}$. Let $\mathcal{Y}_p = \mathfrak{X}_p$ and \mathcal{U}_p be the controlled plant input. The function d_3 represents parameter variation and is taken to be a bounded function with respect to the state such that $d_3(x) \subset \Delta\mathbb{B}$ for some $\Delta \geq 0$. Consider the case where $\mathfrak{X}_p \subset \mathbb{R}^{n_p}$ and the problem of designing the controller data (G_c, H_c), such that the set \mathcal{A} is stable. Let $\mathcal{U}_c = \mathfrak{X}_p$, and $G_c(\mathfrak{X}_c, \mathcal{U}_c) = G_c(\mathfrak{X}_p)$. The feedback system can be represented as in (6) with

$$G(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}_p} \begin{bmatrix} (Ax + BG_c(x) + d_3(x)) \\ G_c(x) \end{bmatrix}$$

Consider in particular the case where $d_3(x) = \Delta_A x$, $G_c(\mathfrak{X}_c, \mathcal{U}_c) = K\mathfrak{X}_p$, with K a matrix of appropriate dimension, and the problem of designing the controller data such that the set $\mathcal{A} = \{0\}$ is stable. Since D is compact and by construction both G_p, G_c are linear functions of closed sets, the closed-loop system data satisfies Assumption 2.8. Consider the function $V(x) = x^\top Px$, with $P = P^\top > 0$. In order to achieve asymptotic stability of \mathcal{A} , we want $u_D(\mathfrak{X}) < 0$, namely

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) < \sup_{x \in \mathfrak{X}} V(x)$$

for all $\mathfrak{X} \subset D \setminus \mathcal{A}$. Here $\sup_{x \in \mathfrak{X}} V(x) = \sup_{x \in \mathfrak{X}} \{x^\top Px\}$

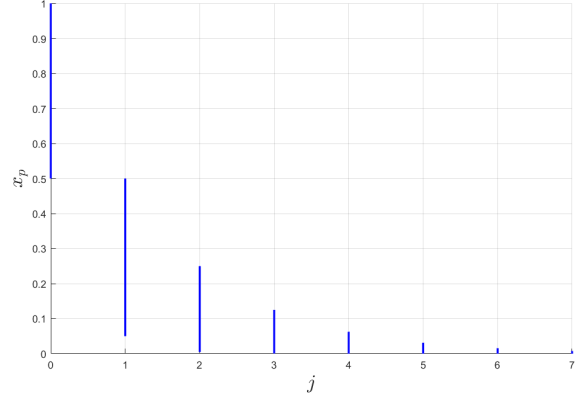


Fig. 3. Solution to the set dynamical system in Example 4.2, with initial condition $\mathfrak{X}_0 = \{x \in \mathbb{R} : 0.5 \leq x \leq 1\}$, parameters $A = 1$, $B = 1$, $\Delta_A = 0.2$ and designed controller gain $K = -0.7$

and

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) = \sup \left\{ \eta^\top P \eta : \eta \in \bigcup_{x \in \mathfrak{X}} (A + \Delta_A \mathbb{B} + BK)x \right\}$$

Since \mathfrak{X} is compact, we have that

$$\sup_{x \in \mathfrak{X}} \{x^\top Px\} = \max_{x \in \mathfrak{X}} \{x^\top Px\}$$

and

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) = \max \left\{ \eta^\top P \eta : \eta \in \bigcup_{x \in \mathfrak{X}} (A + \Delta_A \mathbb{B} + BK)x \right\}$$

Consider in particular the case where $P = I$, $A = 1$, $B = 1$ and the parameter variation has a maximum of $\Delta_A = 0.2$. We can select values in K such that $u_D(\mathfrak{X}) < 0$ for all $\mathfrak{X} \subset D \setminus \mathcal{A}$ to achieve asymptotic stability for the closed-loop system. This condition can be satisfied by selecting K such that

$$A + \Delta_A \mathbb{B} + BK < 1$$

is satisfied within all the range of parameter variation. In particular, selecting $K = -0.7$ will render the set $\mathcal{A} = \{0\}$ asymptotically stable. Figure 3 presents a solution for parameters $A = 1$, $B = 1$, $\Delta_A = 0.2$, and designed state-feedback controller gain $K = -0.7$ from the initial set $\mathfrak{X}_0 = \{x \in \mathbb{R} : 0.5 \leq x \leq 1\}$.

Example 4.3 (Control design with data fusion):

Consider the system

$$\begin{aligned} x^+ &= Ax + u \\ y &= H(x) \end{aligned}$$

with $x \in D$, $u \in U$ with $D, U \subset \mathbb{R}$, where the system's output y is a state estimation, obtained from multiple noisy sensors. Here the output dynamics correspond to a continuous function representing the effect of a sensor fusion filter. The behavior of each sensor can be represented as

$$S_j^i(x) = x_j + \delta_i(x)\mathbb{B}$$

where $i = 1, 2, \dots, m$ corresponds to the sensor number, δ_i

corresponds to state dependent additive noise, characteristic of the sensor i , and x_j corresponds to the plant's state at discrete time j . A simple data fusion strategy can be defined by generating an estimate of x combining the sensors measures by weighting their relevance based on their deviation from the actual state:

$$\mathcal{Y} = H(\mathfrak{X}_p) = \bigcup_{x \in \mathfrak{X}_p} \bigcup_{i \in \{1, 2, \dots, m\}} \omega_i S_i(x)$$

where ω_i is computed based on the maximum value of δ_i . Consider the problem of stabilizing the set $\mathcal{A} = \{0\}$ with a state-feedback controller, implemented using the output measured from the data fusion system, for the case where $i = 2$ and $D = \{x \in \mathbb{R} : |x| \leq \rho\}$, with $\rho \in \mathbb{R}_{\geq 0}$. This system can be described in terms of (6) by

$$G_p(\mathfrak{X}_p, \mathcal{U}_p) = \bigcup_{x \in \mathfrak{X}_p} \bigcup_{u \in \mathcal{U}_p} (Ax + u)$$

$$H_p(\mathfrak{X}_p, \mathcal{U}_p) = \bigcup_{x \in \mathfrak{X}_p} H(x)$$

and a controller with $\mathfrak{X}_c^+ = G_c(\mathfrak{X}_c, \mathcal{U}_c) = KH_p(\mathfrak{X}_p)$ and $\mathcal{Y}_c = \mathcal{U}_p$. Since D is compact and both G_p and G_c are continuous, Assumption 2.8 in Theorem 4.1 is satisfied. Consider $V(x) = |x|^2$, continuous and positive definite for all $x \in D \setminus \mathcal{A}$. In order to achieve asymptotic stability, $u_D(\mathfrak{X}) < 0$,

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) < \sup_{x \in \mathfrak{X}} V(x)$$

for all $\mathfrak{X} \subset D \setminus \mathcal{A}$. Here, $\sup_{x \in \mathfrak{X}} V(x) = \sup_{x \in \mathfrak{X}} \{x^2\}$ and

$$\mathcal{U}_p = KH_p(\mathfrak{X}_p)$$

$$\mathcal{U}_p = \bigcup_{x \in \mathfrak{X}_p} (K(\omega_1(x + d_1(x)\mathbb{B}) + \omega_2(x + d_2(x)\mathbb{B})))$$

For the case where $\delta_1(x) = D_1x$ and $\delta_2(x) = D_2x$, with $D_1, D_2 \in \mathbb{R}$, we can define $\omega_1 = \frac{D_2}{D_1 + D_2}$ and $\omega_2 = \frac{D_1}{D_1 + D_2}$. Then,

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) = \sup\{\eta^2 : \eta \in \bigcup_{x \in \mathfrak{X}} (A + K(1 + 2\omega_2 D_2 \mathbb{B}))x\}$$

Since \mathfrak{X} is compact, we have that

$$\sup_{\eta \in G(\mathfrak{X})} V(\eta) = \max\{\eta^2 : \eta \in \bigcup_{x \in \mathfrak{X}} (A + K(1 + 2\omega_2 D_2 \mathbb{B}))x\}$$

Consider in particular the case where $A = 1$ and the sensor noise parameters are $D_1 = 0.1$ and $D_2 = 0.05$. We can select a value for K such that $u_D(\mathfrak{X}) < 0$ for all $\mathfrak{X} \subset D \setminus \mathcal{A}$ to achieve stability for the closed-loop system. In particular, for $K = -0.6$ condition is satisfied for the closed-loop system. Figure 4 presents a solution with initial condition $\mathfrak{X}_0 = \{x \in \mathbb{R} : 0.7 \leq x \leq 1.1\}$ and the designed state-feedback controller with gain $K = -0.6$.

V. CONCLUSION

Convergence and stability properties for set dynamical systems with inputs were studied in this paper. Krasovskii and Lyapunov results on stability were presented for systems with set-valued states, under the assumption of outer semi-continuity of the set-valued maps that define the system's

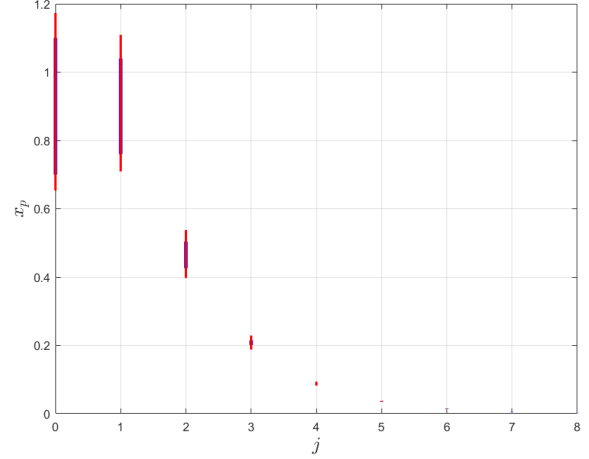


Fig. 4. Solution to the set dynamical system in Example 4.3, with $\mathfrak{X}_0 = \{x \in \mathbb{R} : 0.7 \leq x \leq 1.1\}$, $A = 1$, sensor noise parameters $D_1 = 0.1$, $D_2 = 0.05$, and designed value for the controller $K = -0.6$. Red lines represent state estimated using the data fusion strategy, and blue lines represent the evolution of set-valued state.

dynamics. The mathematical framework in [11] and results in [12] were extended to a formulation for closed-loop control systems with output feedback within the framework of set dynamical systems. The set dynamical systems framework for controller design can be also used for designing other types of controllers, such as fuzzy and predictive approaches, where the set formulation can help to represent variability in current or future states.

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