On Robust Forward Invariance of Sets for Hybrid Dynamical Systems

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***Abstract*— Forward invariance with robustness to disturbances for hybrid dynamical systems modeled by hybrid inclusions with state-dependent conditions enabling flows and jumps is studied. In particular, we study forward invariance notions that apply to systems with hybrid inputs and disturbances with nonlinear dynamics, for which not every solution is unique or may exist for arbitrary long hybrid time. Given state-feedback laws, notions of robust forward invariance of sets for closed-loop system are introduced. Sufficient conditions in terms of the objects defining the system are presented. One result using Lyapunov-based conditions removes the usual Lipschitz constraint on continuous dynamics. A bouncing ball example is given to illustrate major results.**

I. Introduction

Often referred to as flow-invariance [1], positively invariance [2] or viability (or just invariance) [3], forward invariance properties of set ensure that solutions only evolve within the set when start within it. Such properties emerge in many analysis and controller design problems featuring safety and robustness design goals [4], [5], and are even more valuable under the presence of disturbances. Tools for the design of controllers for robust forward invariance of sets – a forward invariance property that is uniform over the allowed disturbances – are developed for linear discrete-time systems via convex programming in [6], for continuous-time monotone systems via stability analysis in [7] with an application in temperature regulation to achieve energy efficiency, and for nonlinear continuous-time systems via control barrier functions in [8] with an application in adaptive cruise control. Meanwhile, theoretical and computational results on forward invariance of sets for hybrid systems (without disturbances) modeled in various frameworks are available; see [9] for results for systems given by impulsive differential inclusions, [10] for hybrid automata, and [11] for hybrid inclusions. To the best of our knowledge there are no results on robust invariance for systems with hybrid behavior.

In this paper, building from previous work in [11], where forward invariance properties for hybrid dynamical systems were established, we introduce several notions of robust forward invariance and robust controlled forward invariance of sets for hybrid systems. Such notions guarantee that solutions that start within the set, stay in the set for all time and for all possible disturbances. For the purpose of verifying such properties for systems that are under the influence of state-feedback pairs, via a solution-independent approach, sufficient conditions for each notion are provided in terms of the system data. Due to the presence of disturbances in the hybrid systems under consideration, we present a result to ensure existence of nontrivial solutions, and categorize the type of solutions based on their domain and "ending behaviors." This is used to provide conditions for guaranteeing the robust forward invariance notion that requires completeness of solutions. In particular, one of the presented results requires the usual Lipschitz property on the continuous dynamics, for which a mild constraint is imposed on the disturbances; while another result removes the Lipschitz assumption and proposes a Lyapunov approach to obtain forward invariance of its sublevel sets.

The remainder of the paper is organized as follows. Section II lists basic definitions and results for hybrid systems with inputs and disturbances. Notions and sufficient conditions for robust forward invariance are given in Section III. In Section III-A, a Lyapunov-like function is used to ensure robust forward invariance of sublevel sets. Section IV introduces robust controlled forward invariance for system with inputs and disturbances. In Section V, an example is provided to illustrate major results. Proofs of the presented results will be published elsewhere.

Notation: A closed unit ball around the origin in $\mathbb{R}^n$ is denoted by $\mathbb{B}$. Given a vector x, $|x|$ denotes the 2-norm of x. Given $r \in \mathbb{R}$, the r-sublevel sets of a function $V : \mathbb{R}^n \to \mathbb{R}$ is $L_V(r) := \{x \in \mathbb{R}^n : V(x) \leq r\}$. The closure of the set K is denoted as $\overline{K}$. Given a closed set K, we denote the tangent cone of the set K at a point $x \in K$ as $T_K(x)$, the interior of K as $\operatorname{int} K$, and the boundary of K as ∂K. Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the range of M is denoted as $\operatorname{rge} M = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ s.t. } y \in M(x)\}$, and the domain of M is denoted as $\operatorname{dom} M = \{x \in \mathbb{R}^m : M(x) \neq \emptyset\}$.

II. Preliminaries

In this paper, results for the study of robust controlled forward invariance properties for hybrid system modeled using the hybrid inclusions framework in [12] are presented. We consider hybrid systems $\mathcal{H}_{u,w}$ with state x, control input $u = (u_c, u_d)$, and disturbance input $w = (w_c, w_d)$ given by

$$\mathcal{H}_{u,w} \begin{cases} \dot{x} \ \in F_{u,w}(x, u_c, w_c) & (x, u_c, w_c) \in C_{u,w} \\ x^+ \in G_{u,w}(x, u_d, w_d) & (x, u_d, w_d) \in D_{u,w}, \end{cases} \tag{1}$$

where its data is defined by the flow set $C_{u,w} \subset \mathbb{R}^n \times \mathcal{U}_c \times \mathcal{W}_c$, the flow map $F_{u,w} : \mathbb{R}^n \times \mathbb{R}^{m_c} \times \mathbb{R}^{d_c} \rightrightarrows \mathbb{R}^n$, the jump set $D_{u,w} \subset \mathbb{R}^n \times \mathcal{U}_d \times \mathcal{W}_d$, and the jump map

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$G_{u,w} : \mathbb{R}^n \times \mathbb{R}^{m_d} \times \mathbb{R}^{d_d} \rightrightarrows \mathbb{R}^n$. The space for the state x is $\mathbb{R}^n$, the space for the input $u = (u_c, u_d)$ is $\mathcal{U} = \mathcal{U}_c \times \mathcal{U}_d \subset \mathbb{R}^{m_c} \times \mathbb{R}^{m_d}$, and the space for the disturbance $w = (w_c, w_d)$ is $\mathcal{W} = \mathcal{W}_c \times \mathcal{W}_d \subset \mathbb{R}^{d_c} \times \mathbb{R}^{d_d}$. The sets $C_{u,w}$ and $D_{u,w}$ define conditions that x, u, and w should satisfy for flows or jumps to occur, respectively. We assume that $C_{u,w}$ and $D_{u,w}$ define conditions on u that only depend on x and conditions on w that only depend on x, where $w = 0$, meaning there is no disturbance in system, always qualifies as a value for disturbance.

We are interested in the forward invariance properties of a set that are uniform in the disturbances w for the closed-loop system resulting from $\mathcal{H}_{u,w}$ under effect of a $\mathcal{H}_{u,w}$-admissible state-feedback law (κ_c, κ_d).[1] Such closed-loop system is given by

$$\mathcal{H} \begin{cases} \dot{x} \in F(x, w_c) := F_{u,w}(x, \kappa_c(x), w_c) & (x, w_c) \in C \\ x^+ \in G(x, w_d) := G_{u,w}(x, \kappa_d(x), w_d) & (x, w_d) \in D, \end{cases} \quad (2)$$

with $C := \{(x, w_c) \in \mathbb{R}^n \times \mathcal{W}_c : (x, \kappa_c(x), w_c) \in C_{u,w}\}$ and $D := \{(x, w_d) \in \mathbb{R}^n \times \mathcal{W}_d : (x, \kappa_d(x), w_d) \in D_{u,w}\}$.

Next, we define the class of hybrid disturbances we allow in this paper.

Definition 2.1 (hybrid disturbance): Hybrid disturbances w are functions of hybrid time that are generated by some hybrid exosystem $\mathcal{H}_e$ of the form

$$\mathcal{H}_e \begin{cases} \dot{w} \in F_e(w) & w \in C_e \\ w^+ \in G_e(w) & w \in D_e, \end{cases}$$

with state (and output) $w = (w_c, w_d) \in \mathcal{W}$.[2] A hybrid disturbance w that satisfies the dynamics of the hybrid system $\mathcal{H}_{u,w}$ is said to be $\mathcal{H}_{u,w}$-admissible. □

Solutions to a hybrid system $\mathcal{H}$ as in (2) are parameterized by hybrid time domains E, which are subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T, J) \in E, E \cap ([0, T] \times \{0, 1, ..., J\})$ can be written as $\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. Moreover, following [14, Definition 2.4], a hybrid arc ϕ is a function on a hybrid time domain that, for each $j \in \mathbb{N}, t \mapsto \phi(t, j)$ is absolutely continuous on the interval $I^j := \{t : (t, j) \in \operatorname{dom} \phi\}$, where $\operatorname{dom} \phi$ denotes the hybrid time domain of ϕ. As an extension to [14, Definition 2.6] a solution to $\mathcal{H}$ is defined as follows.

Definition 2.2: (solutions to $\mathcal{H}$) A pair (ϕ, w) consisting of a hybrid arc ϕ and a hybrid disturbance $w = (w_c, w_d)$, with $\operatorname{dom} \phi = \operatorname{dom} w$ $(= \operatorname{dom}(\phi, w))$, is a solution pair to the hybrid system $\mathcal{H}$ in (2) if $(\phi(0, 0), w_c(0, 0)) \in \overline{C}$ or $(\phi(0, 0), w_d(0, 0)) \in D$, and

(S1) for all $j \in \mathbb{N}$ such that I^j has nonempty interior

$$(\phi(t, j), w_c(t, j)) \in C \qquad \text{for all } t \in \operatorname{int} I^j,$$
$$\frac{d\phi}{dt}(t, j) \in F(\phi(t, j), w_c(t, j)) \quad \text{for almost all } t \in I^j,$$

(S2) for all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$,

$$(\phi(t, j), w_d(t, j)) \in D$$
$$\phi(t, j+1) \in G(\phi(t, j), w_d(t, j)).$$

In addition, a solution pair (ϕ, w) to $\mathcal{H}$ is said to be

1) complete if $\operatorname{dom}(\phi, w)$ is unbounded;
2) maximal if there does not exist another $(\phi, w)'$ such that (ϕ, w) is a truncation of $(\phi, w)'$ to some proper subset of $\operatorname{dom}(\phi, w)'$. □

Given $K \subset \mathbb{R}^n$, we use $\mathcal{S}_{\mathcal{H}}(K)$ to denote a set that includes all maximal solutions (ϕ, w) to the hybrid system $\mathcal{H}$ with initial condition $\phi(0, 0)$ in K. In addition, for ease of presentation, we define the following (note that $\star \in \{c, d\}$ in the following expressions):

- Given a set $K \subset \mathbb{R}^n$ and a set $S \subset \mathbb{R}^n \times \mathbb{R}^{d_\star}$, we define $\Upsilon(K, S) := \{(x, w) \in S : x \in K\}$, and $\Upsilon_c(K) := \Upsilon(K, C), \Upsilon_d(K) := \Upsilon(K, D)$.
- Given a set $K \subset \mathbb{R}^n \times Y$, where $Y \in \{\mathbb{R}^{m_c} \times \mathbb{R}^{d_c}, \mathbb{R}^{d_c}\}$, we define its projection onto $\mathbb{R}^n$ as $\Pi(K) := \{x \in \mathbb{R}^n : \exists y \in Y \text{ s.t. } (x, y) \in K\}$.
- Given a set $K \subset \mathbb{R}^n \times \mathbb{R}^{m_\star} \times \mathbb{R}^{d_\star}$, we define $\Psi^w(x, K) := \{w : \exists u \text{ s.t. } (x, u, w) \in K\}$. For each $x \in \mathbb{R}^n$, we define the set-valued maps $\Psi^w_c(x) := \Psi^w(x, C_{u,w})$, and $\Psi^w_d(x) := \Psi^w(x, D_{u,w})$.[3]

III. Robust Forward Invariance Properties for Hybrid Systems

Following notions in [11], in this work, we introduce notions for robust forward invariance properties of sets for the closed-loop system $\mathcal{H}$ given as in (2).

A. Robust Forward pre-Invariance of Sets

In this section, weak forward pre-invariance and forward pre-invariance of given set $K \subset \mathbb{R}^n$, uniform in the disturbances, for $\mathcal{H}$ is considered. In particular, these notions require solutions to stay in K when they start in K.

Definition 3.1: (robust weak forward pre-invariance of a set) The set $K \subset \mathbb{R}^n$ is said to be robustly weakly forward pre-invariant for $\mathcal{H}$ if for every $x \in K$, there exists one solution $(\phi, w) \in \mathcal{S}_{\mathcal{H}}(x)$ is such that $\operatorname{rge} \phi \subset K$. □

Definition 3.2: (robust forward pre-invariance of a set) The set $K \subset \mathbb{R}^n$ is said to be robustly forward pre-invariant for $\mathcal{H}$ as in (2) if every $(\phi, w) \in \mathcal{S}_{\mathcal{H}}(K)$ such that $\operatorname{rge} \phi \subset K$. □

To avoid solving solutions explicitly for verifying notions in Definition 3.1 and Definition 3.2, we present, when possible, solution independent conditions to check for these

[1] A state-feedback pair (κ_c, κ_d), where $\kappa_c : \mathbb{R}^n \to \mathbb{R}^{m_c}$ and $\kappa_d : \mathbb{R}^n \to \mathbb{R}^{m_d}$ is said to be $\mathcal{H}_{u,w}$-admissible if the pair satisfies the dynamics of $\mathcal{H}_{u,w}$.

[2] The disturbances w are not necessarily differentiable but must be continuous over each interval of flow. For example of hybrid exosystems of the form $\mathcal{H}_e$, see [13].

[3] Note that maps $\Psi^w_\star$ are defined for $C_{u,w}$ and $D_{u,w}$. In the sections to follow, we also use them for the flow and jump sets of the closed-loop system $\mathcal{H}$.

properties. To this end, we impose the following mild assumptions on system $\mathcal{H}$ and set K.

Assumption 3.3: The sets K, C, and D are such that $K \subset \overline{\Pi(C)} \cup \Pi(D)$ and that $K \cap \Pi(C)$ is closed. The map $F : \mathbb{R}^n \times \mathbb{R}^{d_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded on C, and $F(x, w_c)$ is convex for every $(x, w_c) \in C$.

In addition, we use the concept of tangent cone to a set K in these conditions.

Definition 3.4: (tangent cone [15, Definition 6.10]) The tangent cone to a closed set $K \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted as $T_K(x)$, is given as

$$T_K(x) = \left\{ \omega \in \mathbb{R}^n : \liminf_{\tau \searrow 0} \frac{|x + \tau\omega|_K}{\tau} = 0 \right\}. \quad (3)$$

Next, we propose sufficient conditions to ensure robust weak forward pre-invariance of a set for $\mathcal{H}$.

Proposition 3.5: (conditions for robust weak forward pre-invariance of a set) Let the set $K \subset \mathbb{R}^n$ be closed and C, F, D and K satisfy Assumption 3.3. The set $K \subset \mathbb{R}^n$ is robust weakly forward pre-invariant for hybrid system $\mathcal{H} = (C, F, D, G)$ given as in (2) *if the following conditions hold:*

1.1) For every $(x, w_d) \in \Upsilon_d(K \cap \Pi(D)), G(x, w_d) \cap K \neq \emptyset$;
1.2) For every $(x, w_c) \in \Upsilon_c(K \backslash \Pi(D)), F(x, w_c) \cap T_K(x) \neq \emptyset$.

The next set of sufficient conditions guarantees that every solution to $\mathcal{H}$ stays within the set K when starting within, which require the disturbances w and the set K to satisfy the following assumption.

($\star$) for every $\xi \in (\partial K) \cap \Pi(C)$, there exists a neighborhood U of ξ such that $\Psi^w_c(x) \subset \Psi^w_c(\xi)$ for every $x \in U$.

In addition, we recall the Lipschitz properties of set-valued maps from [9].

Definition 3.6: (locally Lipschitz set-valued maps) A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally Lipschitz on a set $K \subset \mathbb{R}^n$ if for every $x \in K$, there exist a neighborhood U of x and a constant $\lambda \geq 0$ such that

$$F(x) \subset F(\xi) + \lambda|x - \xi|\mathbb{B} \qquad \forall \xi \in U \cap \operatorname{dom} F.$$

Furthermore, F is locally Lipschitz when it is locally Lipschitz on $\operatorname{dom} F$ (see [16, Chapter 1, Definition 4]). □

Then, we propose, in the following result, conditions for robust forward pre-invariance of a set.

Proposition 3.7: (conditions for robust forward pre-invariance of a set) Let the set $K \subset \mathbb{R}^n$ be closed and C, F, D and K satisfy Assumption 3.3. Suppose F is locally Lipschitz on C. The set $K \subset \mathbb{R}^n$ is robustly forward pre-invariant for hybrid system $\mathcal{H} = (C, F, D, G)$ given as in (2) *if* ($\star$) *holds and the following conditions hold:*

2.1) For every $(x, w_d) \in \Upsilon_d(K \cap \Pi(D)), G(x, w_d) \subset K$;
2.2) For every $(x, w_c) \in \Upsilon_c(K \cap \Pi(C)), F(x, w_c) \subset T_K(x)$.

The Lipschitzness of the set-valued map F in Proposition 3.7 and property ($\star$) are crucial to ensure that every solution stays in the designated set during flows. Note that condition ($\star$) guarantees such property uniformly in w_c. An example is given below Theorem 3.1 in [2] to show that solutions can leave a set due to the absence of locally Lipschitz right-hand side of a continuous-time system.

Inspired by the Lyapunov conditions for stability of hybrid systems in [14, Theorem 3.18], the sufficient conditions in Proposition 3.7 are exploited to guarantee robust forward pre-invariance for $\mathcal{H}$. More precisely, given a qualifying Lyapunov-like function $W : \mathbb{R}^n \to \mathbb{R}$ for the closed-loop $\mathcal{H}$ given as in (2), conditions on the system data (C, F, D, G) are proposed to guarantee robust forward pre-invariance of a subset of the $r-$sublevel set of W given by

$$\mathcal{M}_r = L_W(r) \cap (\Pi(C) \cup \Pi(D)). \quad (4)$$

Proposition 3.8: (robust forward pre-invariance of $\mathcal{M}_r$) Given a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (2)*, suppose there exist a constant $r^* \in \mathbb{R}$ and a function $W : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $L_W(r^*) \cap \Pi(\overline{C})$ such that*

$$\langle \nabla W(x), \eta \rangle \leq 0 \qquad \forall (x, w_c) \in \Upsilon_c(L_W(r^*) \cap \Pi(C)), \quad \eta \in F(x, w_c), \quad (5)$$

$$W(\eta) - W(x) \leq 0 \quad \forall (x, w_d) \in \Upsilon_d(L_W(r^*) \cap \Pi(D)), \quad \eta \in G(x, w_d). \quad (6)$$

Then, for each $r \in (-\infty, r^*)$ such that $\mathcal{M}_r$ is closed, $\mathcal{M}_r$ has the following properties:

L1) $\mathcal{M}_r$ is robust weakly forward pre-invariant for $\mathcal{H}$ if

$$G(\Upsilon_d(\mathcal{M}_r \cap \Pi(D))) \cap (\Pi(C) \cup \Pi(D)) \neq \emptyset; \quad (7)$$

L2) $\mathcal{M}_r$ is robustly forward pre-invariant for $\mathcal{H}$ if

$$G(\Upsilon_d(\mathcal{M}_r \cap \Pi(D))) \subset \Pi(C) \cup \Pi(D). \quad (8)$$

For the given Lyapunov-like function W, (5) in Proposition 3.8 requires all solutions to have "nonincreasing values" of W within $L_W(r^*)$. This is to ensure solutions stay within $L_W(r)$ for any given $r < r^*$. Note that, when provided a fixed $r \in \mathbb{R}^n$, this flow condition can be replaced by one that only seeks for "nonincreasing" W in a neighborhood of $\partial L_W(r)$ on $\mathbb{R}^n$, namely, if there exist U of every $(\xi, v_c) \in \Upsilon_c(L_W(r^*) \cap \Pi(C))$ such that for every $(x, w_c) \in U$, $\langle \nabla W(x), \eta \rangle \leq 0$ for every $\eta \in F(x, w_c)$. Other alternative conditions may involve a locally Lipschitz flow map F and a similar assumption to ($\star$) used in Proposition 3.7.

B. Robust Forward Invariance of Sets

This section pertains to "stronger" robust forward invariance properties of sets, similar to the ones in [11, Definition 2.4 and Definition 2.6], for $\mathcal{H}$ given as in (2). These notions require existence of nontrivial solutions from every point in the set of interests and completeness of maximal solutions.

Definition 3.9: (robust weak forward invariance of a set) The set $K \subset \mathbb{R}^n$ is said to be robustly weakly forward invariant for $\mathcal{H}$ if for every $x \in K$, there exists a complete $(\phi, w) \in \mathcal{S}_\mathcal{H}(x)$ such that $\operatorname{rge} \phi \subset K$. □

Definition 3.10: (robust controlled forward invariance of a set) The set $K \subset \mathbb{R}^n$ is said to be robustly forward invariant for $\mathcal{H}$ if for every $x \in K$ there exists a solution to $\mathcal{H}$ and every $(\phi, w) \in \mathcal{S}_\mathcal{H}(K)$ is complete and such that $\operatorname{rge} \phi \subset K$. □

Next, we present a variation of [14, Proposition 2.10] for hybrid systems with disturbances given as in (2).

Proposition 3.11: (basic existence) Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ *as in* (2)*. Let* $\xi \in \overline{\Pi(C)} \cup \Pi(D)$*. If* $\xi \in \Pi(D)$*, or*

(VC$_w$) there exist $\varepsilon > 0$*, an absolutely continuous function* $\widetilde{z} : [0, \varepsilon] \to \mathbb{R}^n$ *such that* $\widetilde{z}(0) = \xi$, $(\widetilde{z}(t), w_c(t, 0)) \in C$ *for all* $t \in (0, \varepsilon)$ *and* $\dot{\widetilde{z}}(t) \in F(\widetilde{z}(t), w_c(t, 0))$*, for almost all* $t \in [0, \varepsilon]$*, where* $w_c(t, 0) \in \Psi_c^w(\widetilde{z}(t))$ *for every* $t \in [0, \varepsilon]$*,*[4]

then, there exists a nontrivial solution pair (ϕ, w) *with* $\mathcal{H}_{u,w}$*-admissible* $w = (w_c, w_d)$ *to* $\mathcal{H}$ *from the initial state* $\phi(0, 0) = \xi$*. If* $\xi \in \Pi(D)$ *and (VC$_w$) holds for every* $\xi \in \overline{\Pi(C)} \setminus \Pi(D)$*, then there exists a nontrivial solution pair to* $\mathcal{H}$ *from every initial state* $\xi \in \overline{\Pi(C)} \cup \Pi(D)$*, and every solution pair* $(\phi, w) \in \mathcal{S}_\mathcal{H}$ *from such points satisfies exactly one of the following:*

(a) the solution pair (ϕ, w) *is complete;*

(b) (ϕ, w) *is not complete and "ends with flow": with* $(T, J) = \sup \operatorname{dom}(\phi, w)$*, the interval* I^J *has nonempty interior, and either*

- *b.1)* I^J *is closed, in which case either*
  - *b.1.1)* $\phi(T, J) \in \overline{\Pi(C)} \setminus (\Pi(C) \cup \Pi(D))$*, or*
  - *b.1.2) from* $\phi(T, J)$ *flow within* $\Pi(C)$ *is not possible, meaning that there is no* $\varepsilon > 0$*, absolutely continuous function* $\widetilde{z} : [0, \varepsilon] \to \mathbb{R}^n$ *such that* $\widetilde{z}(0) = \phi(T, J)$*,* $(\widetilde{z}(t), w_c(t, J)) \in C$ *for all* $t \in (0, \varepsilon)$*, and* $\dot{\widetilde{z}}(t) \in F(z(t), w_c(t, J))$ *for almost all* $t \in [0, \varepsilon]$*, where* $w_c(t, J) \in \Psi_c^w(\widetilde{z}(t))$ *for every* $t \in [0, \varepsilon]$*, or*
- *b.2)* I^J *is open to the right, in which case* $(T, J) \notin \operatorname{dom}(\phi, w)$ *due to the lack of existence of an absolutely continuous function* $\widetilde{z} : \overline{I^J} \to \mathbb{R}^n$ *satisfying* $(\widetilde{z}(t), w_c(t, J)) \in C$ *for all* $t \in \operatorname{int} I^J$*,* $\dot{\widetilde{z}}(t) \in F(\widetilde{z}(t), w_c(t, J))$ *for almost all* $t \in I^J$*, and such that* $\widetilde{z}(t) = \phi(t, J)$ *for all* $t \in I^J$*, where* $w_c(t, J) \in \Psi_c^w(\widetilde{z}(t))$ *for every* $t \in [0, \varepsilon]$*;*

(c) (ϕ, w) *is not complete and "ends with jump": with* $(T, J) = \sup \operatorname{dom}(\phi, w) \in \operatorname{dom}(\phi, w)$*,* $(T, J-1) \in \operatorname{dom}(\phi, w)$*,* $\phi(T, J) \notin \Pi(D)$*, and either*

- *c.1)* $\phi(T, J) \notin \overline{\Pi(C)}$*, or*
- *c.2)* $\phi(T, J) \in \overline{\Pi(C)}$*,*[5] *and from* $\phi(T, J)$ *flow within* $\Pi(C)$ *as defined in b.1.2) is not possible.*

Proposition 3.11 presents conditions guaranteeing existence of nontrivial solutions to $\mathcal{H}$ from every initial state $\xi \in \overline{\Pi(C)} \cup \Pi(D)$, as well as characterizes all possibilities for maximal solution pairs. In particular, maximal solution pairs that are not complete can either "end with flow" or "end with jump." In short, the former means that I^J has a nonempty interior over which $(\phi(t, J), w_c(t, J)) \in C$ for all $t \in \operatorname{int} I^J$ and $\frac{d\phi}{dt}(t, J) \in F(\phi(t, J), w_c(t, J))$ for almost all $t \in \operatorname{int} I^J$, where $(T, J) = \sup \operatorname{dom}(\phi, w)$. In particular, case *b.1.1)* corresponds to a solution pair ending at the boundary of $\overline{C}$, case *b.1.2)* to the case of a solution pair ending after flowing and at a point where flow is not possible, while case *b.2)* covers the case of a solution pair escaping to infinity in finite time. The case "end with jump" means that $(T, J), (T, J-1) \in \operatorname{dom}(\phi, w), (\phi(T, J-1), w_d(T, J-1)) \in D$, and the solution pair ends either with $\phi(T, J) \in \Pi(C) \setminus \Pi(D)$ due to flow not being possible or with $\phi(T, J) \notin \Pi(C) \cup \Pi(D)$, where $(T, J) = \sup \operatorname{dom}(\phi, w)$.

Remark 3.12: Case *c.1)* in Proposition 3.11 does not hold when $G(D) \subset \overline{\Pi(C)} \cup \Pi(D)$.[6] Moreover, when C is closed and $F : \mathbb{R}^n \times \mathbb{R}^{d_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous[7] relative to C and locally bounded, and for all $(x, w_c) \in C$, $F(x, w_c)$ is nonempty and convex, *(VC$_w$)* can be guaranteed by the existence of a neighborhood U of ξ such that for every $x \in U \cap \Pi(C)$, $F(x, w_c) \cap T_{\Pi(C)}(x) \neq \emptyset$.

Then, based on Proposition 3.11, we propose sufficient conditions for robust weak forward invariance of a set for $\mathcal{H}$ as follows.

Theorem 3.13: (conditions for robust weak forward invariance of a set) Let the set the $K \subset \mathbb{R}^n$ *be closed and* C, F, D *and* K *satisfy Assumption 3.3. The set* $K \subset \mathbb{R}^n$ *is robustly weakly forward invariant for hybrid system* $\mathcal{H} = (C, F, D, G)$ *given as in* (2) *if 1.1) in Proposition 3.5 and the following conditions hold:*

3.1) For every $(x, w_c) \in \Upsilon_c(K \setminus \Pi(D)), F(x, w_c) \cap T_{K \cap \Pi(C)}(x) \neq \emptyset$*;*

($\star\star$) For every $(\phi, w) \in \mathcal{S}_\mathcal{H}(K \cap \Pi(C))$ *with* $\operatorname{rge} \phi \subset K$*, case b.2) in Proposition 3.11 does not hold.*

Then, imposing Lipschitzness of F and assumption ($\star$), conditions to guarantee robust forward invariance of a set are given as below.

Proposition 3.14: (conditions for robust forward invariance of a set) Let the set $K \subset \mathbb{R}^n$ *be closed and* C, F, D *and* K *satisfy Assumption 3.3. Suppose* F *is locally Lipschitz on* C*. The set* $K \subset \mathbb{R}^n$ *is robustly forward invariant for hybrid system* $\mathcal{H} = (C, F, D, G)$ *given as in* (2) *if* ($\star$), ($\star\star$), *2.1) in Proposition 3.5 and the following condition hold:*

4.1) For every $(x, w_c) \in \Upsilon_c(K \cap \Pi(C))$*,* $F(x, w_c) \subset T_{K \cap \Pi(C)}(x)$*.*

Remark 3.15: Although, in principle, condition ($\star\star$) is a solution-dependent property, it is worth noting that ($\star\star$) can be guaranteed by either of the system data based conditions as follows:

1) $K \cap \Pi(C)$ is compact;
2) F is bounded on $\Upsilon_c(K \cap \Pi(C))$;
3) F is Marchaud[8] on $\Upsilon_c(K \cap \Pi(C))$.

[4]Note that every $\Psi_\star^w(x)$ is nonempty since $0 \in \Psi_\star^w(x)$.

[5]As a consequence of $\phi(T, J) \notin \Pi(D)$, $\phi(T, J) \in \overline{\Pi(C)} \setminus \Pi(D)$ under the condition in case *c.2)*.

[6]$G(D) := \{x \in G(x, w_d) : \exists (x, w_d) \in D\}$

[7]A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous iff it has closed graph (see [14, Definition 5.9]).

[8]A map F is said to be Marchaud if its graph and its domain are nonempty and closed; $F(x)$ is convex, compact, and nonempty for each $x \in \operatorname{dom} F$; and F has linear growth; see [17, Definition 10.3.2].

It is not trivial to derive similar result to Proposition 3.8 for robust weak forward invariance and robust forward invariance of Lyapunov sublevel sets. This is due to the fact that intersection of $T_{\Pi(C)}(x)$ and $T_K(x)$ does not match $T_{K\cap\Pi(C)}(x)$ in general without further assumptions on system data, such as set separations or nonempty interior of tangent cones [15], [18]. The needed assumptions and detail discussions will be published elsewhere.

IV. Robust Controlled Forward Invariance of Sets for Hybrid Systems

In this section, we introduce notions for robust controlled forward invariance of a set for hybrid systems $\mathcal{H}_{u,w}$ given as in (1). In particular, a set enjoys robust controlled forward invariance when an $\mathcal{H}_{u,w}$-admissible state-feedback pair (κ_c, κ_d) ensures that the state evolution stays within the set regardless of the value of the disturbance w. Thus, based on the notions in Section III, we define notions for robust controlled forward invariance of sets for $\mathcal{H}_{u,w}$.

Definition 4.1: (robust controlled forward pre-invariance of a set) The set $K \subset \mathbb{R}^n$ is said to be robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via a state-feedback pair (κ_c, κ_d) as in (1) if the set K is robustly forward pre-invariant for the resulting closed-loop system $\mathcal{H}$ given as in (2). □

Definition 4.2: (robust controlled forward invariance of a set) The set $K \subset \mathbb{R}^n$ is said to be robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via a state-feedback pair (κ_c, κ_d) as in (1) if the set K is robustly forward invariant for the resulting closed-loop system $\mathcal{H}$ given as in (2). □

To obtain robust controlled forward invariance properties via feedback pair as introduced above, (κ_c, κ_d) can be designed for given $\mathcal{H}_{u,w}$ using results from Section III. In particular, with the availability of the Lyapunov-like function as in Proposition 3.8, one can construct feedback pairs to render the sublevel set robustly controlled forward pre-invariant using selection theorems.

V. A Bouncing Ball Example

In this section, we use a bouncing ball example to illustrate the major results in this section. Consider a hybrid system $\mathcal{H}_{u,w} = (C, f, D_{u,w}, g_{u,w})$ on $\mathbb{R}^2$ modeling a bouncing ball moving vertically and controlled at impacts at zero height. This system is given by

$$\mathcal{H}_{u,w} \begin{cases} \dot{x} = f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} & x \in C \\ x^+ = g_{u,w}(x, u_d, w_d) & (x, u_d, w_d) \in D_{u,w}, \end{cases} \tag{9}$$

where the flow set is

$$C := \{x \in \mathbb{R}^2 : x_1 \geq 0\},$$

the jump map is

$$g_{u,w}(x, u_d, w_d) := \begin{bmatrix} 0 \\ -w_d x_2 + u_d \end{bmatrix}$$

and the jump set is

$$\begin{aligned} D_{u,w} := \{&(x, u_d, w_d) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} : \\ &x_1 = 0, x_2 \leq 0, u_d \in [b_1, b_2], w_d \in [e_1, e_2]\}. \end{aligned}$$

In this model, the disturbance w_d represents the uncertain coefficient of restitution which take values from a known range $[e_1, e_2]$, where $0 < e_1 < e_2 < 1$, $\gamma > 0$ is the gravity constant, and $0 < b_1 < b_2$ are the lower and upper bounds on the input u_d, respectively. The state variable x_1 models the height of the ball and x_2 represents its velocity. In addition, the jump map $g_{u,w}$ models impacts between the ball and a controlled surface at $x_1 = 0$: before every impact, x_2 is nonpositive and after each impact, the ball velocity reverses its sign and updates according to $g_{u,w}$.

We have the following control design goal: control the ball, in the presence of disturbances, such that the peak height after each bounce is at least h, when the ball is dropped from higher than h. This problem can be solved by rendering the set

$$K = L_W(\gamma h) = \{x \in \mathbb{R}^2 : W(x) \in [-\infty, -\gamma h], x_1 \geq 0\}$$

robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$, where $W : \mathbb{R}^2 \to \mathbb{R}$ describes (minus) the total energy of the ball and is given by

$$W(x) = -\left(\frac{x_2^2}{2} + \gamma x_1\right). \tag{10}$$

Then, given $b_1 = \sqrt{2\gamma h}(1 - e_1)$ and $b_2 = \infty$, applying Proposition 3.8, we show the set K is robustly forward pre-invariant for $\mathcal{H}$ via a feedback law that is given by

$$\kappa_d(x) = (e_1 - 1)x_2 \quad \forall x \in \Pi(D_{u,w}).$$

Firstly, with $r^* \in [0, \infty)$ and the given feedback law κ_d, we verify that W function in (10) satisfies the conditions in Proposition 3.8. Note that during flows, W remains constant due to conservation of energy, i.e., the total system energy level stays at the same when the ball is in the air (not touching the ground) and we have

$$\langle \nabla W(x), f(x) \rangle = -\left(\frac{2x_2\dot{x}_2}{2} + \gamma \dot{x}_1\right) = 0. \tag{11}$$

Thus, relationship (5) holds. Next, for every $(x, w_d) \in \Upsilon_d(K \cap \Pi(D_{u,w}))$, we compute $W(g_{u,w}(x, u_d, w_d)) - W(x)$. Since $x_1 = 0$ for every $x \in K \cap \Pi(D_{u,w})$, we have

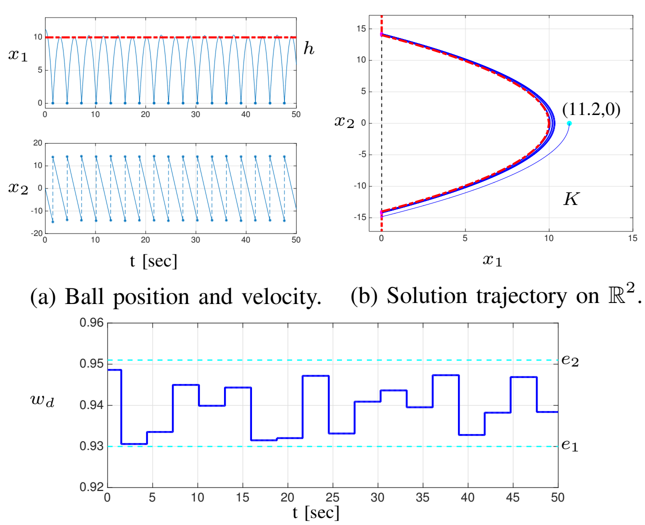
$$\begin{aligned} &W(g_{u,w}(x, u_d, w_d)) - W(x) \qquad (12) \\ &= -\frac{(-w_d x_2 + \kappa_d(x))^2}{2} - (-\frac{x_2^2}{2}) \\ &= \frac{x_2^2}{2} - \frac{(-w_d x_2 + (e_1 - 1)x_2)^2}{2} \\ &= \frac{x_2^2(1 - (e_1 - 1 - w_d)^2)}{2}, \end{aligned}$$

which is less than or equal to zero for every $w_d \in [e_1, e_2]$, since $0 < e_1 < e_2 < 1$ and $(e_1 - 1 - w_d) < -1$. Thus, (6) holds.

Then, by definition of K, the condition on jump dynamics in item *L2)* of Proposition 3.8 holds because $\Pi(D_{u,w}) \cup C = \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $\kappa_d(x) = (e_1 - 1)x_2 \geq -w_d x_2$ for every $x \in K \cap \Pi(D_{u,w})$. Hence, the feedback κ_d renders the set K robustly forward pre-invariant for the closed-loop system, and therefore, renders K robustly forward pre-invariant for $\mathcal{H}_{u,w}$ by Definition 4.1.

Furthermore, the fact that expression (12) is always non-positive implies that condition *2.1)* in Proposition 3.7 holds. Then, (11) implies condition *4.1)* in Proposition 3.14. Moreover, Assumption 3.3 and $(\star)$ hold trivially. Condition $(\star\star)$ holds since f is a continuous linear function, which is also locally Lipschitz. Hence, by application of Proposition 3.14, the feedback κ_d renders the set K robustly forward invariant for the closed-loop system, and therefore, renders K robustly forward invariant for $\mathcal{H}_{u,w}$ by Definition 4.2.

A simulation is performed to show robust controlled forward invariance of K for $\mathcal{H}_{u,w}$. We choose the least height value to be $h = 10$, set γ to 9.81, while the disturbance w_d is randomly generated within interval $[e_1, e_2]$ with $e_1 = 0.93$ and $e_2 = 0.951$ for each impact. One solution that started from initial value of $(11.2, 0)$ is shown in Figure 1.



(a) Ball position and velocity. (b) Solution trajectory on $\mathbb{R}^2$.

(c) Varying coefficient of restitution w_d over simulation time.

Fig. 1: Simulation of closed-loop system.

As shown above, with a randomly assigned w_d signal for $\mathcal{H}_{u,w}$ in Figure 1c, the resulting height peaks in between impacts stays above $h = 10$ in Figure 1a, while Figure 1b shows that, on the $\mathbb{R}^2$ plane, the solution stays within the set K (the region bounded by red dashed line).

VI. Conclusion

Forward invariance properties of sets that are uniform over the disturbances for hybrid systems with inputs and disturbances are studied in this paper. Notions and sufficient conditions of robust forward invariance properties for hybrid systems in the hybrid inclusions framework are presented. Some conditions are developed employing a Lyapunov-like function to derive forward invariance for the closed-loop system $\mathcal{H}$, in which the usual Lipschitz constraint on continuous dynamics is relaxed. Future and ongoing research include results on existence of invariance inducing state-feedback laws using robust control Lyapunov functions for forward invariance, constructing state-feedback laws using a pointwise minimum norm selection scheme, and optimality properties of the chosen selections via inverse optimality.

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