Hybrid Systems Techniques for Convergence of Solutions to Switching Systems*

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***Abstract*— Invariance principles for hybrid systems are used to derive invariance principles for nonlinear switching systems with multiple Lyapunov-like functions. Dwell-time, persistent dwell-time, and weak dwell-time solutions are considered. Asymptotic stability results are deduced under observability assumptions or common bounds on the Lyapunov-like functions.**

I. Introduction

A. Background

Switching systems are dynamical systems governed by a differential equation whose right hand side is selected from a given family of functions, based on some (time or state dependent) switching rule. Stability theory of switching systems has been an active area of research over the last fifteen years. Sufficient conditions for stability were given in [15], [14], [10], [2], [9], [4], [11]. Stability under particular classes of switching signals were studied in [11], [7], [8], [1]. For much more background, see [12], [11], [7].

In this paper, we focus on invariance principles for switching systems under certain classes of signals: dwell-time, weak dwell-time, and persistent dwell-time signals. Early work on this topic includes [7], [8], [1]. Related work on invariance principles for hybrid systems — dynamical systems where solutions can evolve continuously (flow) and discontinuously (jump) — includes [13], [3], [16]. In [16] (with the results announced in [17]), invariance principles were shown for general hybrid systems in the framework of [6], which allows for nonuniqueness of solutions, multiple jumps at time instants, and Zeno behaviors, while only posing mild regularity conditions on the data.

B. Contribution

We show how some of the results of [16] can be used to obtain invariance principles for switching systems. While doing that, we recover, generalize, and/or strengthen some of the results of [7], [8], [1]. In particular:

- Corollary 5.3 strengthens [1, Theorems 1, 2] by including both forward and backward invariance conditions on the set to which solutions converge. Corollary 4.4, while giving the same invariance conditions as [1], also incorporates level sets of Lyapunov functions into the description of the invariant set.
- Corollary 4.6 is an invariance principle for nonlinear switching systems that generalizes [7, Theorem 8] stated for linear switching systems. Even in the linear setting, Corollary 4.6 yields smaller, in comparison to [7, Theorem 8], sets to which solutions converge.
- [8, Theorem 7] is derived, in Corollary 4.11, from the hybrid invariance principle in Theorem 4.1.

Invariance principles in [16], [17], are only used to prove Theorems 4.1 and 5.2. The consequences of these theorems for switching systems can be then derived in a self-contained way, by using two techniques that should prove useful for purposes other than those in this paper:

- Given a solution to a switching system, and a sequence of time intervals of length at least τ_D on which the logical mode takes on a particular value q^*, one can identify the restriction of the solution to those intervals with a function on $[0,\infty)$. The resulting object is not a solution to a switching system, as the continuous variable of the original switching system may now be only piecewise continuous. However, it is a solution to an appropriately formulated hybrid system (truly hybrid system, in which both the "continuous" variable and the logical mode jump). To this hybrid system, invariance principles of [16], [17] can be applied, with implications for the original switching system. This technique is used to obtain Corollaries 4.4, 4.6.
- In the case of multiple Lyapunov functions, i.e., when in logical mode q, a function V_q is decreasing at a rate W_q, it is often assumed that the value of V_{q^*} at the end of an interval with mode q^* is greater or equal than the value of V_{q^*} at the beginning of the next interval with mode q^*. Hence, the function $(x,q) \mapsto V_q(x)$ can not be used in the standard Lyapunov sense: it can increase during switches between different logical modes. However, it can be shown that for each bounded solution (x,q) to the switching system, the function $(x,q) \mapsto W_q(x)$ is integrable. (A similar technique was used in [8, Theorem 7].) This paves way to the application of invariance principles of [16], [17] that rely on an output function that decreases sufficiently fast to 0. Theorem 5.2 is used via this technique.

In presenting the results, we clearly separate the statements only about invariance of sets to which bounded solutions of switched systems converge (Corollaries 4.4, 4.6, and 5.3) from stronger statements about asymptotic stability that

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rely on additional information like observability or common bounds on Lyapunov functions (Corollaries 4.8, 4.11).

II. Preliminaries

A. Switching systems

Let $O \subset \mathbb{R}^n$ be an open set, let $Q = \{1, 2, \dots, q_{max}\}$, and for each $q \in Q$, let $f_q : O \to \mathbb{R}^n$ be a continuous function. We consider switching systems given by

$$\mathcal{SW}: \qquad \dot{x} = f_q(x). \tag{1}$$

For more background on switching systems, see [11] or [7].

A complete *solution to the switching system* $\mathcal{SW}$ consists of a locally absolutely continuous function $x : [0, \infty) \to O$ and a function $q : [0, \infty) \to Q$ that is piecewise constant and has a finite number of discontinuities in each compact time interval, and $\dot{x}(t) = f_{q(t)}(x(t))$ for almost all $t \in [0, \infty)$. We will say that a complete solution (x, q) to $\mathcal{SW}$ is precompact if x is bounded with respect to O, that is, there exists a compact set $K \subset O$ such that $x(t) \in K$ for all $t \in [0, \infty)$.

Let (x, q) be a complete solution to $\mathcal{SW}$ and let $t_0 = 0$, and $t_1, t_2, \dots$ be the consecutive (positive) times at which q is discontinuous (We assume that there is indeed infinitely many such times, as otherwise, the system is not a switching system for the purposes of asymptotic analysis.) Informally, t_i is the time of the i-th switch. The solution (x, q) is a *dwell-time solution* with dwell time $\tau_D > 0$ if $t_{i+1} - t_i \geq \tau_D$ for $i = 0, 1, \dots$. (That is, jumps are separated by at least τ_D amount of time.) The solution (x, q) is a *persistent dwell-time solution* with persistent dwell time $\tau_D > 0$ and period of persistence $T > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, t_{i_2}, \dots$ of the sequence $\{t_i\}$ such that $t_{i_k+1} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \dots$ and $t_{i_{k+1}} - t_{i_k+1} \leq T$ for $k = 0, 1, \dots$. (That is, at most T amount of time passes between two consecutive intervals of length at least τ_D on which there is no jumps.) Finally, a solution (x, q) is a *weak dwell-time solution* with dwell time $\tau_D > 0$ if there exists a subsequence $0 = t_{i_0}, t_{i_1}, t_{i_2}, \dots$ of the sequence $\{t_i\}$ such that $t_{i_k+1} - t_{i_k} \geq \tau_D$ for $k = 1, 2, \dots$. (That is, there are infinitely many intervals of length τ_D with no switching.) These classes of solutions follow the definitions in [7], see also [9].

B. Hybrid systems

We consider hybrid systems of the form

$$\mathcal{H}: \quad \begin{cases} \dot{x} \in F(x) & x \in C, \\ x^+ \in G(x) & x \in D, \end{cases} \tag{2}$$

with an associated state space $\mathcal{O} \subset \mathbb{R}^m$. Above, F (respectively, G) is the possibly set-valued map describing the flow, (respectively, the jumps) while C (respectively, D) is the set on which the flow can occur (respectively, from which the jumps can occur). For more background on hybrid systems in this framework, see [5] or [6].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if it is a union of intervals $[t_j, t_{j+1}] \times \{j\}$, for some finite or infinite sequence of times $0 = t_0 \leq t_1 \leq t_2 \dots \leq t_J$, with the "last" interval possibly of the form $[t_J, T)$ with T finite or $T = \infty$. A *hybrid arc* is a function whose domain is a hybrid time domain (for a hybrid arc x, its domain will be denoted $\operatorname{dom} x$) and such that for each $j \in \mathbb{N}$, $t \to x(t, j)$ is locally absolutely continuous on $\operatorname{dom} x \cap ([0, \infty) \times \{j\})$.

A hybrid arc x is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$, $x(t, j) \in \mathcal{O}$ for all $(t, j) \in \operatorname{dom} x$, and

(S1) for all $j \in \mathbb{N}$ such that $I_j \times \{j\} := \operatorname{dom} x \cap ([0, \infty) \times \{j\})$ is nonempty, $x(\cdot, j)$ is locally absolutely continuous in t on I_j and, for almost all $t \in I_j$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j));$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)).$$

Results on structural properties of solutions to $\mathcal{H}$, like (appropriately understood) sequential compactness of the space of solutions and outer/upper semicontinuous dependence of solutions on initial conditions, were obtained in [6]. These results made possible the general invariance principles of [16], [17]. The assumptions on the data $(\mathcal{O}, F, G, C, D)$ of $\mathcal{H}$ that enabled the results of [6], [16], [17] are as follows:

(A0) $\mathcal{O}$ is open;

(A1) C and D are relatively closed subsets of $\mathcal{O}$;

(A2) $F : \mathcal{O} \rightrightarrows \mathbb{R}^m$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$;

(A3) $G : \mathcal{O} \rightrightarrows \mathbb{R}^m$ is outer semicontinuous, and $G(x)$ is nonempty and such that $G(x) \subset \mathcal{O}$ for all $x \in D$.

(The set-valued map $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for every convergent sequence of x_i's with $\lim x_i \in \mathcal{O}$, and every convergent sequence of $y_i \in F(x_i)$, $\lim y_i \in F(\lim x_i)$. F is *locally bounded* if for every compact $K \subset \mathcal{O}$ there exists a compact $K' \subset \mathbb{R}^n$ such that $F(K) \subset K'$. Similarly for G.) Let us say that all hybrid systems we write down in this paper do satisfy the assumptions just stated.

III. Switching systems as hybrid systems

Given a switching system $\mathcal{SW}$ as presented in Section II-A, consider the hybrid system

$$\mathcal{H}_{\mathcal{SW}}: \quad \begin{cases} \dot{x} = f_q(x) & x \in O, q \in Q \\ q^+ \in Q & x \in O, q \in Q \end{cases} \tag{3}$$

with the variable $(x, q) \in \mathbb{R}^{n+1}$. (Not mentioning $\dot{q}$ in the description of flow or x^+ in the description of jumps means that q remains constant during flow while x does not change during jumps.) To view the system (3) as a special case of (2), one can take the state space to be $\mathcal{O} = O \times \mathbb{R}$, the flow set $C = O \times Q$, the flow map $F(x, q) = (f_q(x), 0)$ if $(x, q) \in C$ and $F(x, q) = \emptyset$ otherwise; the jump set $D = O \times Q$; and the (set-valued!) jump map $G(x, q) = (x, Q)$ for $(x, q) \in D$ and $G(x, q) = \emptyset$ otherwise. With such data, the conditions (A0)-(A3) are satisfied.

To every solution to $\mathcal{SW}$ there corresponds a solution to the hybrid system. Indeed, if $t_0 = 0$ and $t_1, t_2, \dots$ are the times at which q is discontinuous, one can easily build a solution to $\mathcal{H}_{\mathcal{SW}}$ on a hybrid time domain $E = \bigcup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\})$ that corresponds to (x, q). Of course, there are solutions to $\mathcal{H}_{\mathcal{SW}}$ that do not correspond to any solution to $\mathcal{SW}$, for example $\mathcal{H}_{\mathcal{SW}}$ has solutions that only jump (instantaneous Zeno solutions). While $\mathcal{H}_{\mathcal{SW}}$ satisfies (A0)-(A3), using invariance principles applied to $\mathcal{H}_{\mathcal{SW}}$ to deduce convergence of, say, dwell-time solutions to it (and behavior of these reflects the behavior of dwell time solutions to $\mathcal{SW}$) may lead to invariant sets whose invariance is verified by the said Zeno solutions. This does not lead to useful conclusions for the underlying switching system. Thus, better hybrid representations of $\mathcal{SW}$ under dwell time and other classes of switching signals are needed.

To each dwell-time solution (x, q), with dwell time $\tau_D > 0$, to $\mathcal{SW}$ there corresponds a solution (x, q, τ) to the following hybrid system:

$$\mathcal{H}_{\tau_D} \begin{cases} \dot{x} = f_q(x), \ \dot{\tau} \in \kappa_{\tau_D}(\tau) & \tau \in [0, \tau_D] \\ q^+ \in Q, \ \tau^+ = 0 & \tau = \tau_D \ . \end{cases} \tag{4}$$

Above, $\kappa_{\tau_D} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the (set-valued) map given by

$$\kappa_{\tau_D}(\tau) = \begin{cases} 1 & \text{if} \quad \tau < \tau_D \\ [0, 1] & \text{if} \quad \tau = \tau_D \\ 0 & \text{if} \quad \tau > \tau_D \end{cases} .$$

Solutions to $\dot{\tau} \in \kappa_{\tau_D}(\tau)$ increase at the rate 1 when $\tau < \tau_D$ and remain constant otherwise. The map κ_{τ_D} is such that the variable τ remains bounded (by τ_D) regardless of the length of the flow intervals.

In the opposite direction, some solutions to (4) may flow before the first jump for less than τ_D amount of time, but those that have $\tau(0, 0) = 0$ do correspond directly to dwell time solutions, with dwell time τ_D, to $\mathcal{SW}$.

Let $F : O \rightrightarrows \mathbb{R}^n$ be the set valued map defined by

$$F(x) = \overline{\text{con}} \bigcup_{q \in Q} f_q(x) \ , \tag{5}$$

where $\overline{\text{con}} S$ stands for the closed convex hull of the set S. To each persistent dwell-time solution (x, q) to $\mathcal{SW}$, with dwell time $\tau_D > 0$ and period of persistence $T > 0$, there corresponds a solution (x, q, τ_1, τ_2) to the following system:

$$\mathcal{H}_{\tau_D, T} \begin{cases} \dot{x} = f_q(x), \dot{\tau}_1 \in \kappa_{\tau_D}(\tau_1) & q \in Q, \tau_1 \in [0, \tau_D] \\ \dot{x} \in F(x), \dot{\tau}_2 = 1 & q = 0, \tau_2 \in [0, T] \\ \left.\begin{array}{c} q^+ \in Q \cup \{0\}, \\ \tau_1^+ = 0, \tau_2^+ = 0 \end{array}\right\} & q \in Q, \tau_1 = \tau_D \\ \left.\begin{array}{c} q^+ \in Q, \\ \tau_1^+ = 0, \tau_2^+ = 0 \end{array}\right\} & q = 0, \tau_2 \in [0, T] \ . \end{cases}$$

In other words, solutions x to $\dot{x} = f_q(x)$ under arbitrary switching signals q are solutions to the inclusion $\dot{x} \in F(x)$. (In fact, x is a solution to the inclusion, on some bounded time interval, if and only if it is a uniform limit of some sequence of solutions generated via switching.)

IV. Hybrid invariance principle using a nonincreasing function, and consequences

In this section, we present invariance principles to establish convergence of dwell-time, persistent dwell-time, and weak dwell-time solutions to switching systems. The foundation to those will be an invariance principle for hybrid systems, which comes out of [17], and is based on a nonincreasing Lyapunov function.

A. A hybrid invariance principle using a nonincreasing function

The following result follows from [17, Corollary 4.3], specialized to dwell time solutions along the lines of [17, Corollary 4.2], or more directly from [16, Corollary 4.4].

Theorem 4.1: *Let $O \subset \mathbb{R}^n$ be open, $f : O \to \mathbb{R}^n$ be continuous, $K \subset O$ be nonempty and compact, $V : O \to \mathbb{R}$ be continuously differentiable, $W : O \to \mathbb{R}_{\geq 0}$ be continuous and such that $\nabla V(x) \cdot f(x) \leq -W(x)$ for all $x \in O$. Consider a hybrid system*

$$\mathcal{H}_1 : \quad \begin{cases} \dot{x} = f(x), \dot{\tau} \in \kappa_{\tau_D}(\tau) & \tau \in [0, \tau_D], \\ x^+ \in K, \tau^+ = 0 & \tau = \tau_D, \end{cases} \tag{6}$$

on the state space $O \times \mathbb{R}$. Let $(x, \tau) : \text{dom}(x, \tau) \to O \times \mathbb{R}_{\geq 0}$ be a complete solution to $\mathcal{H}_1$ such that $x(t, j) \in K$ for all $(t, j) \in \text{dom}\, x$ and such that $V(x(t, j+1)) \leq V(x(t, j))$ for all $(t, j) \in \text{dom}\, x$ such that $(t, j+1) \in \text{dom}\, x$. Then, for some constant $r \in \mathbb{R}$, x approaches the largest subset M of

$$V^{-1}(r) \cap K \cap W^{-1}(0)$$

that is invariant in the following sense: for each $x_0 \in M$ there exists a solution ξ to $\dot{x} = f(x)$ on $[0, \tau_D/2]$ such that $\xi(t) \in M$ for all $t \in [0, \tau_D/2]$ and either $\xi(0) = x_0$ or $\xi(\tau_D/2) = x_0$.

B. Invariance principles for switching systems

We now apply Theorem 4.1 to switching systems. The results are shown for the case of multiple Lyapunov functions, under the following assumptions.

Assumption 4.2: *$O \subset \mathbb{R}^n$ is an open set, $Q = \{1, 2, \ldots, q_{max}\}$, and for each $q \in Q$, $f_q : O \to \mathbb{R}^n$ is a continuous function, $V_q : O \to \mathbb{R}$ is a continuously differentiable function, $W_q : O \to \mathbb{R}_{\geq 0}$ is a continuous function, and $\nabla V_q(x) \cdot f_q(x) \leq -W_q(x)$ for all $x \in O$.*

Assumption 4.3: *The solution (x, q) to $\mathcal{SW}$ is such that, for each $q^* \in Q$, for any two consecutive intervals (t_j, t_{j+1}), (t_k, t_{k+1}) such that $q(t) = q^*$ for all $t \in (t_j, t_{j+1})$ and all $t \in (t_k, t_{k+1})$, we have $V_{q^*}(x(t_{j+1})) \geq V_{q^*}(x(t_k))$.*

In short, the value of V_{q^*} at the end of an interval on which $q = q^*$ is greater or equal to the value of V_{q^*} at the beginning of the next interval on which $q = q^*$. This assumption is usually needed when establishing convergence and stability results for switching systems, see e.g. [7],[1].

1) Invariance principle for dwell-time solutions to $\mathcal{SW}$: We begin with an application of Theorem 4.1 to dwell-time solutions of $\mathcal{SW}$.

Corollary 4.4: *Let Assumption 4.2 hold, and let (x, q) be a precompact dwell-time solution, with dwell time $\tau_D > 0$, to the switching system $\mathcal{SW}$ satisfying Assumption 4.3. Then there exist $r_1, \ldots, r_{q_{max}} \in \mathbb{R}$ such that x approaches*

$$M = \bigcup_{q \in Q} M_q(r_q, \tau_D), \tag{7}$$

where $M_q(r_q, \tau_D)$ is the largest subset of $V_q^{-1}(r_q) \cap W_q^{-1}(0)$ that is invariant in the following sense: for each $x_0 \in M_q(r_q, \tau_D)$ there exists a solution ξ to $\dot{x} = f_q(x)$ on $[0, \tau_D/2]$ such that $\xi(t) \in M_q(r_q, \tau_D)$ for all $t \in [0, \tau_D/2]$ and either $\xi(0) = x_0$ or $\xi(\tau_D/2) = x_0$.

If, given a continuously differentiable $V : O \to \mathbb{R}^n$, and a continuous $W : O \to \mathbb{R}_{\geq 0}$, we have that $V_q = V$, $W_q = W$ for all $q \in Q$, the conclusion of Corollary 4.4 is stronger than that of Theorem 1 in [1]. One of the reasons is due to [1] not taking advantage of the invariant set to which solutions converge being a subset of some level set (and not just a sublevel set) of V. Further strengthening of this result will be carried out in Theorem 5.2 and Corollary 5.3.

Example 4.5: Consider the switching system in [1, Example 5] given by

$$f_1(x) = \begin{bmatrix} -x_1 - x_2 \\ x_1 \end{bmatrix},$$
$$f_2(x) = \begin{cases} \begin{bmatrix} -x_1 - x_2 \\ x_1 \end{bmatrix} & \text{if } x_1 < 0 \\ \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} & \text{if } x_1 \geq 0 \end{cases}$$

where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$. Let $Q = \{1, 2\}$. With the quadratic function $V(x) = x_1^2 + x_2^2$, we get $W_1(x) = -2x_1^2$ and $W_2(x) = -2x_1^2$ if $x_1 < 0$ and $W_2(x) = 0$ if $x_1 \geq 0$. [1, Theorem 1] establishes that bounded solutions to the switching system starting from $x^0 \in \mathbb{R}^2$ converge to $S := \{x \in \mathbb{R}^2 \mid x_1 \geq 0\} \cap \{x \in \mathbb{R}^2 \mid V(x) \leq V(x^0)\}$ since

$$W_1^{-1}(0) = \{x \in \mathbb{R}^2 \mid x_1 = 0\},$$
$$W_2^{-1}(0) = \{x \in \mathbb{R}^2 \mid x_1 \geq 0\},$$

and the largest invariant set in $\bigcup_{q \in Q} W_q^{-1}(0) \cap \{x \in \mathbb{R}^2 \mid V(x) \leq V(x^0)\} = S$ is the set S itself.

For each $q \in \{1, 2\}$, the only invariant set in $V^{-1}(r) \cap W_q^{-1}(0)$ (in the sense of Corollary 4.4) is for $r = 0$. Hence $M_q = 0$ for $q \in \{1, 2\}$ and Corollary 4.4 implies that every precompact dwell-time solution to the switching system is such that the x component converges to the origin. △

In addition to the improvement due to using a level set of V in Corollary 4.4, the invariance properties requested in Corollary 4.4 are stronger than those in [1, Theorem 1].

2) Invariance principle for persistent dwell-time solutions to $\mathcal{SW}$: Given $f_1, \ldots, f_{q_{max}}$ as in Assumption 4.2, let $F : O \rightrightarrows \mathbb{R}^n$ be the set-valued map given by (5). Given sets $S_1, S_2 \subset \mathbb{R}^n$, let $\mathcal{F}_T(S_1, S_2)$ be the set of all points that can be expressed as $\xi(t)$ where $\xi : [0, T'] \to O$, with some $T' \in [0, T]$, is a solution to $\dot{\xi} \in F(\xi)$ such that $\xi(0) \in S_1$ and $\xi(T') \in S_2$. Note that considering $T' = 0$ suggests that $S \subset \mathcal{F}_T(S, S)$ for any set $S \subset \mathbb{R}^n$.

Corollary 4.6: *Under Assumption 4.2, let (x, q) be a precompact persistent dwell-time solution to $\mathcal{SW}$, with dwell time $\tau_D > 0$ and period of persistency $T > 0$, satisfying Assumption 4.3. Then, there exist $r_1, \ldots, r_{q_{max}} \in \mathbb{R}$ such that x approaches $\mathcal{F}_T(M, M)$, with M as in Corollary 4.4.*

Consider the case of linear vector fields $f_q(x) = A_q x$, quadratic $V_q(x) = x^T P_q x$, $W_q(x) = x^T C_q^T C_q x$. A very similar case was treated by [7, Theorem 8]. [7, Theorem 8] concludes that every precompact persistent dwell-time solution (x, q) to $\mathcal{SW}$ is such that x converges to L, the smallest subspace that is A_q-invariant for each $q \in Q$ and contains the unobservable subspaces of all the pairs (A_q, C_q). Corollary 4.6 gives a more precise statement, taking into account the period of persistency T. The set M of Corollary 4.4 is the union of unobservable subspaces of all the pairs (A_q, C_q). While $\mathcal{F}_T(M, M) \subset L$, the set $\mathcal{F}_T(M, M)$ is a strict subset of L (and not a subspace) for each T. In fact, $\mathcal{F}_T(M, M)$ is a subset of a neighborhood of M, the radius of which depends on T and on the matrices A_q.

Further improvement in Corollary 4.6 can be made by noting that one can replace M in that corollary by M', with M' being the union of only those sets $M_{q^*}(r_{q^*}, \tau_D)$ from Corollary 4.4 for which q^* is attained by the variable q for at least τ_D units of time, infinitely many times.

3) Observability and stability: We will say that a pair of functions (f, W) is observable if, for each $a < b$, the only solution $x : [a, b] \to \mathbb{R}^n$ to $\dot{x} = f(x)$ with $W(x(t)) = 0$ for all $t \in [a, b]$ is $x(t) = 0$ for all $t \in [a, b]$.

Assumption 4.7: *For each $q \in Q$, (f_q, W_q) is observable.*

This assumption implies, in particular, that the sets $M_q(r, \tau_D)$ in Corollary 4.4 all equal $\{0\}$.

Corollary 4.8: *Let Assumptions 4.2, 4.7 hold. Then, every precompact dwell-time solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 is such that x converges to the origin. If furthermore, for each $q \in Q$, f_q is locally Lipschitz continuous and $f_q(0) = 0$, then every precompact persistent dwell-time solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 is such that x converges to the origin.*

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class-$\mathcal{K}_\infty$ if $\gamma(0) = 0$ and γ is continuous, strictly increasing, and unbounded.

Assumption 4.9: *There exist class-$\mathcal{K}_\infty$ functions $\alpha, \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\alpha(|x|) \leq V_q(x) \leq \beta(|x|)$ for each $q \in Q$, all $x \in O$.*

The following result is immediate; see [2, Theorem 2.3].

Lemma 4.10: *Under Assumptions 4.2, 4.9 there exists a class-$\mathcal{K}_\infty$ function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for any solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3, $|x(t)| \leq \gamma(|x(0)|)$. In particular, for any $\varepsilon > 0$ there exists $\delta > 0$ such that every solution (x, q) to $\mathcal{SW}$ satisfying Assumption 4.3 with $|x(0)| \leq \delta$ satisfies $|x(t)| \leq \varepsilon$ for all $t \in \mathbb{R}_{\geq 0}$.*

In particular, Assumption 4.9 implies that all solutions to $\mathcal{SW}$ are bounded. Furthermore, it guarantees stability of 0, and hence quite weak conditions are sufficient for solutions (x, q) to $\mathcal{SW}$ to be such that $x \to 0$. In particular, we have the following result, that parallels [8, Theorem 7], and also [7, Theorem 4] that was given for the case of linear systems and quadratic Lyapunov functions.

Corollary 4.11: *Let Assumptions 4.2, 4.7, and 4.9 hold. Then any weak dwell time solution (x, q) to $\mathcal{SW}$ satisfying*

Assumption 4.3 is bounded and any such complete solution is such that x converges to the origin.

V. Hybrid invariance principle using a meagre function, and consequences

We now improve one of our results, Corollary 4.4, by relying on an invariance principle for hybrid systems from [16], [17] that does not involve a nondecreasing Lyapunov function, but rather, an appropriately fast vanishing output. We will rely on the following version of Assumption 4.3 which is appropriate for solutions to hybrid systems.

Assumption 5.1: *The hybrid arc (x, q), with $\mathrm{dom}(x, q) = \bigcup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\}$ where $J \in \mathbb{N} \cup \{\infty\}$, is such that, for each $q^* \in Q$, for any two consecutive numbers $j_* < j^*$ such that $q(t, j_*) = q^*$ for all $t \in [t_{j_*}, t_{j_*+1}]$ and $q(t, j^*) = q^*$ for all $t \in [t_{j^*}, t_{j^*+1}]$, one has $V_{q^*}(x(t_{j_*+1})) \geq V_{q^*}(x(t_{j^*}))$.*

Now, [16, Corollaries 5.4, 5.6] yield the following result:

Theorem 5.2: *Let Assumption 4.2 hold. Let (x, q, τ) be a precompact solution to $\mathcal{H}_{\tau_D}$ in (4) such that (x, q) satisfies Assumption 5.1. Then x approaches the largest subset N of*

$$\bigcup_{p \in Q} W_p^{-1}(0)$$

that is invariant in the following sense: for each $x_0 \in N$ there exist $p_1, p_2 \in Q$ such that $x_0 \in W_{p_1}^{-1}(0) \cup W_{p_2}^{-1}(0)$, $t_1, t_2 > 0$ with $t_1 + t_2 \geq \tau_D$, a solution $\xi_1 : [-t_1, 0] \to W_{p_1}^{-1}(0) \cap N$ to $\dot{\xi}_1 = f_{p_1}(\xi_1)$ such that $\xi_1(0) = x_0$, and a solution $\xi_2 : [0, t_2] \to W_{p_2}^{-1}(0) \cap N$ to $\dot{\xi}_2 = f_{p_2}(\xi_2)$ such that $\xi_2(0) = x_0$.

Corollary 5.3: *Let Assumption 4.2 hold. Let (x, q) be a precompact dwell-time solution to $\mathcal{SW}$ that satisfies Assumption 4.3. Then the conclusions of Theorem 5.2 hold.*

When compared to [1, Theorem 2], Corollary 5.3 gives stronger invariance conditions on the set to which x must converge. In [1], it is only required that there exist either a forward or a backward solution (i.e., either ξ_1 or ξ_2) while here, Theorem 5.2 calls for the existence of both a forward and a backward solution.

Example 5.4: Consider the switching system in [1, Example 4] given by

$$f_1(x) = \begin{bmatrix} -x_1 - x_2 \\ x_1 \end{bmatrix}, \; f_2(x) = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

where $x = [x_1 \; x_2]^T \in \mathbb{R}^2$. Let $Q := \{1, 2\}$. Following [1, Example 4], with $V(x) = x_1^2 + x_2^2$ we get $W_1(x) = -2x_1^2$ and $W_2(x) = -V(x)$. Then

$$W_1^{-1}(0) = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 = 0 \right\}, \quad W_2^{-1}(0) = \{0\},$$

and the largest invariant set in $\bigcup_{q \in Q} W_q^{-1}(0)$ is equal to $\left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 = 0 \right\}$. Then, via [1, Theorem 1], every solution starting from x^0 converges to $\left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 = 0 \right\} \cap \left\{ x \in \mathbb{R}^2 \;\middle|\; V(x) \leq V(x^0) \right\}$, which corresponds to a segment on the x_2-axis centered at the origin.

Convergence to the origin can be shown using Corollary 4.4. Let us apply Corollary 5.3 instead. It is more similar to [1, Theorem 1], as it does not use a level set of V_q in the characterization of the invariant set. The basic difference between [1, Theorem 1] and Corollary 5.3 is the notion of invariance.

We have $\bigcup_{p \in Q} W_p^{-1}(0) = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 = 0 \right\}$. Any point $x_0 \neq 0$ in this set is in $W_1^{-1}(0)$ but not in $W_2^{-1}(0)$. Now, the fact that no subset of $W_1^{-1}(0) = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 = 0 \right\}$ except $\{0\}$ is invariant under f_1 implies that the subset N of $\bigcup_{p \in Q} W_p^{-1}(0)$, invariant in the sense of Theorem 5.2, is exactly $\{0\}$. Hence, all solutions of the system under discussion have x converging to $\{0\}$. △

The example above shows that one way to obtain stronger results from invariance principles is by considering invariance notions that involve both forward and backward invariance. This is, of course, the case in simpler settings.

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