On Asymptotic Synchronization of Interconnected Hybrid Systems with Applications

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Abstract— **In this paper, we consider the synchronization of the states of a multiagent network system, where each agent exhibits hybrid behavior. Specifically, the state of each agent may evolve continuously according to a differential equation, and, at times, jump discretely according to a difference equation. We develop a notion of asymptotic synchronization for part of the state of the system. Our definition of asymptotic synchronization imposes both Lyapunov stability and attractivity on the difference between the agents' states. We recast synchronization as a set stabilization problem, for which tools for the study of asymptotic stability of sets for hybrid systems are suitable. As applications, we introduce two synchronization problems for hybrid systems: the interconnection of continuous-time systems connected over an intermittent communication network and the synchronization of interconnected impulse-coupled oscillators.**

I. Introduction

Due to its broad applications, synchronization of dynamical systems has received a significant amount of attention recently. Specifically, synchronization is a key property to study in spiking neurons and control of chaotic systems [1], [2], formation control and flocking maneuvers [3], and many others applications.

Synchronization of networked systems has been studied using different approaches and methodologies. Namely, synchronization in both continuous and discrete-time domains has been investigated in [4], [5], and for both linear and nonlinear systems [6]. The network structure in such systems is typically studied using graph theory. Graph theory provides a solid understanding of the connectivity of the network and its effect on the individual dynamics of the systems [7]. On the other hand, the study of stability and attractivity of synchronization is typically done using systems theory tools, like Lyapunov functions [8], contraction theory [9], and incremental input-to-state stability [10], for instance. To the best of our knowledge, there is a distinct lack in the synchronization literature for the case where each agent may contain states that evolve both continuously and discretely.

In this paper, we consider synchronization of multi-agent systems where each agent may be hybrid in nature. The agents considered here are allowed to have states that evolve continuously and, at times, discretely. To allow for such complex behavior, we propose a general framework for the study of synchronization in interconnected hybrid systems. The interconnected model we present allows for states that evolve continuously and, at times, jump; such actions allow information to be transferred across a network to affect the states of a neighboring agent. For such general models, we define notions of asymptotic synchronization for hybrid systems, which require both stable and convergent behavior (in the uniform and nonuniform sense). Using tools for analysis of asymptotic stability of set in hybrid systems, we present results for asymptotic synchronization of interconnected hybrid systems. Namely, we consider the implications of asymptotic stability of a set to the asymptotic synchronization (in both the uniform and nonuniform sense). We illustrate the results in two applications.

The remainder of this paper is organized as follows. In Section II, we introduce the notation along with some preliminaries on hybrid systems and graph theory. Section III introduces the class of interconnected hybrid systems considered, the notion of asymptotic synchronization, and the main results. In Section IV, we introduce two examples of networks of interconnected hybrid systems. Simulations of the examples are included. Proofs of the main results will be published elsewhere.

II. Preliminaries

A. Notation

The set of real and natural numbers including zero are denoted as $\mathbb{R}$ and $\mathbb{N} := \{0, 1, 2, 3, \dots\}$, respectively. The set of nonnegative reals is given by $\mathbb{R}_{\geq 0}$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$, notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given matrices A, B with proper dimensions, we define the operator $\mathrm{He}(A, B) := A^\top B + B^\top A$. The distance from a point $x \in \mathbb{R}^n$ to the closed set $\mathcal{A} \subset \mathbb{R}^n$ is defined as $|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}_\infty$ function, written $\alpha \in \mathcal{K}_\infty$, if α is zero at zero, continuous, strictly increasing, and bounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\mathrm{dom}\, f$, i.e., $\mathrm{dom}\, f := \{x \in \mathbb{R}^m : f(x) \text{ is defined }\}$, and its graph is given by gph $f := \{(x, y) : y \in f(x), x \in \mathbb{R}^m\}$. I_n is an n dimensional identity matrix, $\mathbf{1}$ is the vector with entries equal to one, and Π is defined as $(I_n - \frac{1}{N}\mathbf{1}\mathbf{1}^\top)$.

B. Preliminaries on Graph Theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \dots, N\}$ and the edges are pairs in the set

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$\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two different nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can send information to agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} = (g_{ik}) \in \mathbb{R}^{N \times N}$, where $g_{ik} = 1$ if $(i, k) \in \mathcal{E}$, and $g_{ik} = 0$ otherwise. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (k, i) \in \mathcal{E}\}$.

Definition 2.1: A directed graph is said to be

- *weight balanced* if, at each node $i \in \mathcal{V}$, the out-degree and in-degree are equal; i.e., for each $i \in \mathcal{V}$, $d^{out}(i) = d^{in}(i)$;
- *complete connected* if every pair of distinct vertices is connected by a unique edge; that is $g_{ik} = 1$ for each $i, k \in \mathcal{V}$, $i \neq k$;
- *strongly connected* if and only if any two distinct nodes of the graph can be connected via a path that traverses the directed edges of the digraph. □

C. Preliminaries on Hybrid Systems

A hybrid system $\mathcal{H}$ has data (C, F, D, G) and is given by the hybrid inclusion

$$\begin{aligned} \dot{x} &\in F(z) \qquad x \in C, \\ x^+ &\in G(z) \qquad x \in D, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, F defines the flow map capturing the continuous dynamics and C defines the flow set on which flows are allowed. The set-valued map G defines the jump map and models the discrete behavior, and D defines the jump set which is where jumps are allowed. A solution[1] ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\cup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. The set $\mathcal{S}_{\mathcal{H}}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_{\mathcal{H}}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ.

In this paper, we consider the notion of local asymptotic stability defined in [11, Definition 7.1], and asymptotic stability in the uniform global sense defined in [11, Definition 3.6]. We refer the reader to [11] for more details on these notions and the hybrid systems framework.

III. Interconnected Agents with Hybrid Dynamics

A. Hybrid Modeling and General Properties

Consider a network of N agents connected through a graph. For each $i \in \mathcal{V} := \{1, 2, \ldots, N\}$, the i-th agent is modeled as a of hybrid inclusion as in (1). Let $x_i \in \mathbb{R}^n$ be the state of each agent, $u_i \in \mathbb{R}^p$ the input to each agent, and $y_i \in \mathbb{R}^m$ the output, which is given by $y_i = h(x_i)$. The output y_i may be measured by the agent itself and its neighbors. Moreover, we define the information available to each i-th agent, denoted as $\tilde{y}_i$, as a sequence collecting all neighboring outputs y_k, where $k \in \mathcal{N}(i) \cup \{i\}$, namely, we define $\tilde{y}_i = \{y_k\}_{k \in \mathcal{N}_i \cup \{i\}} \in \mathbb{R}^{m(d_i+1)}$. Then, each i-th agent may evolve continuously by

$$\dot{x}_i \in \widetilde{F}(x_i, u_i) \tag{2}$$

for every $x_i \in \widetilde{C}$, where $\widetilde{C}$ is a subset of $\mathbb{R}^n$. Furthermore, when the state x_i is in a set $\widetilde{D} \subset \mathbb{R}^n$ a self-induced jump in the state of the i-th agent may occur such jump is modeled by the difference inclusion

$$x_i^+ \in \widetilde{G}_{in}(x_i, u_i) \tag{3}$$

Such an event may trigger an abrupt change in the state of its neighbors, in which, for each $i, k \in \mathcal{V}$ $i \neq k$, we have

$$x_k^+ \in (1 - g_{ik}) x_k + g_{ik} G_{ex}(x_k, u_k) =: \widetilde{G}_{ex}^{ik}(x_k, u_k). \tag{4}$$

where g_{ik} is the adjacency matrix. Note that when there is a connection between agents i and k the element in the adjacency matrix is $g_{ik} = 1$ which leads to $\widetilde{G}_{ex}^{ik}(x_k, u_k) = G_{ex}(x_k, u_k)$. Moreover, when there is no connection between such agents, we have that $g_{ik} = 0$ which results in the lack of communication between agents implying that $\widetilde{G}_{ex}^{ik}(x_k, u_k) = x_k$; specifically, there is no change in the state of agent k induced by the jumps of agent i.

We consider state-feedback laws for the control of each agent; the dynamic case can be treated similarly. We define static output feedback controllers by $\kappa_c^i : \mathbb{R}^{m(d_i+1)} \to \mathbb{R}^p$ during the continuous evolution of the state. Depending on the scenario, events due to measurements of information from the neighbors and self-induced jumps may only be available to the agents at different time instances. When there is an instantaneous change in x_i due to a self-induced update, i.e., $x_i \in \widetilde{D}$, we denote the feedback controller by $\kappa_{d,in}^i : \mathbb{R}^m \to \mathbb{R}^p$, which depends on the measured y_i only. When a neighboring system jumps, i.e., when $x_k \in \widetilde{D}$, $k \in \mathcal{N}(i)$, we denote the controller by $\kappa_{d,ex}^i : \mathbb{R}^m \to \mathbb{R}^p$ which depends on the measured y_k which jumped.

A complete model of the network can be obtained by, stacking the agents' states with dynamics as in (2)-(4). The resulting interconnected hybrid system, denoted $\mathcal{H}$, with state $x = (x_1, x_2, \ldots, x_N)$ is given by

$$\begin{aligned} \dot{x} \in (&\widetilde{F}(x_1, \kappa_c^1(\tilde{y}_1)), \widetilde{F}(x_2, \kappa_c^2(\tilde{y}_2)), \\ &\ldots, \widetilde{F}(x_N, \kappa_c^N(\tilde{y}_N))) =: F(x) \\ & x \in C := \widetilde{C} \times \widetilde{C} \times \cdots \times \widetilde{C} \\ x^+ \in \{&\widetilde{G}_i(x) : x_i \in \widetilde{D}, i \in \mathcal{V}\} =: G(x) \\ & x \in D := \{x \in \mathbb{R}^{Nn} : \exists i \in \mathcal{V}, \text{ s.t. } x_i \in \widetilde{D}\}. \end{aligned} \tag{5}$$

The jump map $\widetilde{G}_i$ updates the i-th entry of the full state x, via $\widetilde{G}_{in}$ when $x_i \in \widetilde{D}$, and maps all other $k \in \mathcal{V} \setminus \{i\}$

[1]A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a (proper) truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded.

components by $\widetilde{G}_{ex}^{ik}$, namely,

$$\widetilde{G}_i(x) := (\widetilde{G}_{ex}^{i1}(x_1, \kappa_{d,ex}^1(y_i)), \ldots, \widetilde{G}_{in}(x_i, \kappa_{d,in}^i(y_i)), \ldots, \widetilde{G}_{ex}^{iN}(x_N, \kappa_{d,ex}^N(y_i))). \quad (6)$$

Note that when multiple components of x are in $\widetilde{D}$ then G is the union of more than one jump map $\widetilde{G}_i$.

Remark 3.1: Due to the structure of $\mathcal{H}$, this framework covers the cases of synchronization and consensus protocols for both continuous-time systems and discrete time systems. For example, using the above hybrid model, a purely continuous-time model can be recovered by considering $\widetilde{D} = \emptyset$ and with $\widetilde{G}_{in}$, $\widetilde{G}_{ex}$ arbitrary, and, likewise, a discrete-time model can be obtained using $\mathcal{H}$ by letting $\widetilde{C} = \emptyset$ and with $\widetilde{F}$ arbitrary.

B. Partial Synchronization Notions

We introduce an asymptotic synchronization notion that requires both the synchronization error between the components of solutions on hybrid time domains to converge to zero as well as stable behavior. We call this notion *asymptotic synchronization* and we define it as follows:

Definition 3.2 (asymptotic synchronization for $\mathcal{H}$): Consider the hybrid system $\mathcal{H}$ in (5). For each $i \in \mathcal{V}$, $x_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^r$ and $q_i \in \mathbb{R}^{n-r}$ with integers $n \geq r \geq 1$. The hybrid system $\mathcal{H}$ is said to have

- *stable synchronization* with respect to p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, every solution $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ to $\mathcal{H}$ where $\phi_i = (\phi_i^p, \phi_i^q)$ is such that
$$|\phi_i(0,0) - \phi_k(0,0)| \leq \delta \implies \begin{cases} |\phi_i^p(t,j) - \phi_k^p(t,j)| \leq \varepsilon \\ \forall (t,j) \in \operatorname{dom} \phi \end{cases}$$
for all $i, k \in \mathcal{V}$.
- *locally attractive synchronization* with respect to p if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ is complete and $|\phi_i(0,0) - \phi_k(0,0)| \leq \mu$ implies
$$\lim_{\substack{(t,j) \in \operatorname{dom} \phi \\ t+j \to \infty}} |\phi_i^p(t,j) - \phi_k^p(t,j)| = 0. \quad (7)$$
for all $i, k \in \mathcal{V}$.
- *global attractive synchronization* with respect to p if every maximal solution ϕ to $\mathcal{H}$ is complete and satisfies (7) for all $i, k \in \mathcal{V}$.
- *local asymptotic synchronization* with respect to p if it has both stable and local attractive synchronization with respect to p.
- *global asymptotic synchronization* with respect to p if it has both stable and global attractive synchronization with respect to p. □

Next, we define synchronization in the uniform sense.

Definition 3.3: (uniform global asymptotic synchronization for $\mathcal{H}$): Consider the hybrid system $\mathcal{H}$ in (5). For each $i \in \mathcal{V}$, let $x_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^r$ and $q_i \in \mathbb{R}^{n-r}$ with integers $n \geq r \geq 1$. The hybrid system $\mathcal{H}$ is said to have

- *uniform stable synchronization* with respect to p if there exists a class-$\mathcal{K}_\infty$ function α such that any solution $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ to $\mathcal{H}$ where $\phi_i = (\phi_i^p, \phi_i^q)$ satisfies $|\phi_i^p(t,j) - \phi_k^p(t,j)| \leq \alpha(|\phi_i(0,0) - \phi_k(0,0)|)$ for all $(t,j) \in \operatorname{dom} \phi$ and for all $i, k \in \mathcal{V}$.
- *uniform global attractive synchronization* with respect to p if every maximal solution to $\mathcal{H}$ is complete and for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that for any solution ϕ to $\mathcal{H}$ with $|\phi_i(0,0) - \phi_k(0,0)| \leq r$, for each $i, k \in \mathcal{V}$, $(t,j) \in \operatorname{dom} \phi$ satisfying $t + j \geq T$ imply $|\phi_i^p(t,j) - \phi_k^p(t,j)| \leq \varepsilon$ for each $i, k \in \mathcal{V}$.
- *uniform global asymptotic synchronization* if it has both uniform stable and uniform global attractive synchronization.

Remark 3.4: If $r = n$, then the notions in Definitions 3.2 and 3.3 can be considered as full-state notions, while if $r < n$ it can be considered to be a partial state notions. Note that stable synchronization requires solutions ϕ_i, for each $i \in \mathcal{V}$, to start close; while, only the components ϕ_i^p, $i \in \mathcal{V}$ remain close over their solution domain. Similarly, local attractive synchronization with respect to p only requires the distance between each ϕ_i^p to approach zero, while the other component is left unconstrained.

C. Results for Partial Synchronization

Our first observation is that stable synchronization with respect to p in a hybrid system leads to uniqueness in the p components of solutions to $\mathcal{H}$.

Lemma 3.5: If the hybrid system $\mathcal{H}$ as in (5) *has stable synchronization with respect to p, then the p component in the t direction*[2] *of every maximal solution $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ from $\phi(0,0) \in \mathcal{A}$, with $\mathcal{A}$ as in* (8)*, is unique.*

Next, we recast synchronization as a set stabilization problem with $x_i = (p_i, q_i)$ for each $i \in \mathcal{V}$. We define the synchronization set as

$$\mathcal{A} = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{Nn} : p_1 = p_2 = \ldots = p_N\} \quad (8)$$

Note that when $r = n$, the synchronization set $\mathcal{A}$ reduces to the diagonal set $\{x \in \mathbb{R}^{nN} : x_1 = x_2 = \ldots = x_N\}$.

By using the properties of solutions ϕ for the hybrid system $\mathcal{H}$ in (5), we levy the fact that given a solution, the distance of ϕ to the set $\mathcal{A}$ results in $|\phi(t,j)|_{\mathcal{A}}^2 = \sum_{i=1}^{N} |\phi^p(t,j) - \bar{\phi}^p(t,j)|$ where $\bar{\phi}^p$ is the average value of ϕ_k^p for each $k \in \mathcal{V}$, i.e., $\bar{\phi}^p = \sum_{k=1}^{N} \phi_k^p$. Using this equivalence and the triangle inequality with the definition of global asymptotic stability in Definition [11, Definition 7.1] leads to the implication that $\mathcal{H}$ has global asymptotic synchronization. Next, we present our sufficient conditions for global asymptotic synchronization with respect to p for $\mathcal{H}$.

Theorem 3.6: Given a hybrid system $\mathcal{H}$ as in (5) *with data (C, F, D, G), if the set $\mathcal{A}$ in* (8) *is globally asymptotically stable for $\mathcal{H}$, then $\mathcal{H}$ has global asymptotic synchronization with respect to p.*

[2]The projection of the p component in the t direction is defined as $t \mapsto \phi^p(t) := \lim_{h \searrow 0, (t+h,j) \in \operatorname{dom} \phi} \phi^p(t+h, j)$.

Remark 3.7: The local case of the result in Theorem 3.6 can be considered similarly, specifically, if the hybrid system $\mathcal{H}$ has the set $\mathcal{A}$ locally asymptotically stable then $\mathcal{H}$ has local asymptotic synchronization with respect to p.

The next result establishes the sufficient conditions for uniform global attractive synchronization with respect to p.

Theorem 3.8: Given a hybrid system $\mathcal{H}$ as in (5) *with data (C, F, D, G), if the set $\mathcal{A}$ in* (8) *is uniformly globally attractive for $\mathcal{H}$, then $\mathcal{H}$ has uniform global attractive synchronization with respect to p.*

Remark 3.9: Note that we cannot achieve uniform stable synchronization for the hybrid system $\mathcal{H}$ from uniform global asymptotic stability of the synchronization set $\mathcal{A}$ for $\mathcal{H}$. At this time, it is not evident how to recover the $\mathcal{K}_\infty$ function in terms of the synchronization error on the right-hand side of the inequality in the definition of uniform stable synchronization in Definition 3.3.

Our final result establishes a $\mathcal{KL}$ characterization for uniform global asymptotic synchronization.

Theorem 3.10: A hybrid system $\mathcal{H}$ as in (5) *has uniform global asymptotic synchronization with respect to p if and only if there exists a function $\beta \in \mathcal{KL}$ such that any maximal solution $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ where $\phi_i = (\phi_i^p, \phi_i^q)$, satisfies*

$$|\phi_i^p(t,j) - \phi_k^p(t,j)| \leq \beta(|\phi_i(0,0) - \phi_k(0,0)|, t+j) \quad (9)$$

for all $(t,j) \in \operatorname{dom}\phi$ and for all $i, k \in \mathcal{V}$.

Remark 3.11: With global asymptotic stability, synchronization can be studied using (nonuniform and uniform) asymptotic stability tools for hybrid systems. Namely, sufficient conditions for such notions can be formulated in terms of an appropriately defined Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the conditions in [11, Definition 3.16] for $\mathcal{A}$. Such conditions require V to satisfy a bound of the form $\langle \nabla V(x), f \rangle < 0$ for all $x \in C \setminus \mathcal{A}$ and $f \in F(x)$, and $V(g) - V(x) < 0$ for all $x \in D \setminus \mathcal{A}$ and $g \in G(x)$. Then, integration of V over a solution ϕ leads to a strict decrease in V for all points in the flow and jump set, respectively. At times, however, strict inequalities might be hard to obtain. For such cases, when the system $\mathcal{H}$ in (5) satisfies the hybrid basic conditions, we get stability and, via the invariance principle in [11, Theorem 8.2], every maximal and complete solution to $\mathcal{H}$ converges to the largest weakly invariant subset where V does not change. Furthermore, at times, V may not necessarily decrease during flows but strictly decreases at jumps, or vice versa, in which case we can utilize the relaxed conditions in [11, Proposition 3.24, Proposition 3.27, Proposition 3.29, Proposition 3.30].

IV. Applications

A. Distributed Consensus of Continuous Agents

Consider the problem of achieving consensus of the state of multiple continuous-time agents over a graph Γ with first-order integrator dynamics given by

$$\dot{z}_i = u_i \quad (10)$$

where $z_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the input of the i-th agent. The value of the state of the agents' neighbors is only received at sporadic, isolated time instances. We define these times as the sequence $\{t_s\}_{s=1}^{\infty}$ such that $t_{s+1} - t_s \in [T_1, T_2]$ for each $s > 1$, and $t_1 \leq T_2$. The positive scalar values T_1 and T_2 define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive transmission instances.

Inspired by [12], we model every possible sequence of update times $\{t_s\}_{s=1}^{\infty}$ by defining a timer state $\tau_i \in [0, T_2]$ for each agent. We accomplish this by letting τ_i decrease with ordinary time and when $\tau_i = 0$, it gets reset to any value in the interval $[T_1, T_2]$. Namely, we define the dynamics of τ_i as follows

$$\begin{aligned} \dot{\tau}_i &= -1 & \tau_i &\in [0, T_2] \\ \tau_i^+ &\in [T_1, T_2] & \tau_i &= 0. \end{aligned} \quad (11)$$

To achieve consensus, we define a controller variable η_i assigned to the input of each agent, i.e., $u_i = \eta_i$, such that when the i-th agent has access to information from its neighbors, at communication events, η_i is updated impulsively as

$$\eta_i^+ = -\gamma \sum_{k \in \mathcal{N}(i)} (z_i - z_k) \qquad \tau_i = 0 \quad (12)$$

where $\gamma > 0$. To compensate for lack of information between communication events, we update η_i between such events by using the dynamics

$$\dot{\eta}_i = -h\eta_i \qquad \tau_i \in [0, T_2] \quad (13)$$

where $h > 0$.

We fit the construction above in the hybrid system model in (5). Let $x_i = (z_i, \eta_i, \tau_i) \in \mathbb{R} \times \mathbb{R} \times [0, T_2]$. It follows that, for each i-th agent, we have that,

$$\begin{aligned} \dot{x}_i = \left(\begin{bmatrix} I & 0 \\ 0 & -hI \end{bmatrix} \begin{bmatrix} z_i \\ \eta_i \end{bmatrix}, -1 \right) &=: \widetilde{F}(x) \\ x_i \in \mathbb{R}^2 \times [0, T_2] &=: \widetilde{C} \end{aligned} \quad (14)$$

which implicitly defines κ_c^i for each $i \in \mathcal{V}$ as $\kappa_c^i(y_i) = \eta_i$, $y_i = x_i$. Moreover, at jumps, i.e., we have that

$$\begin{aligned} x_i^+ = \left(\begin{bmatrix} z_i \\ -\gamma \sum_{k \in \mathcal{N}(i)} (z_i - z_k) \\ [T_1, T_2] \end{bmatrix} \right) &=: \widetilde{G}_{in}(x) \\ x_i \in \widetilde{D} &=: \mathbb{R}^2 \times \{0\} \end{aligned} \quad (15)$$

Furthermore, for such a case, all other agents are updated by the identity, i.e., for each $k \in \mathcal{V} \setminus \{i\}$

$$x_k^+ = x_k =: \widetilde{G}_{ex}^{ik}(x). \quad (16)$$

Due to the continuous-time nature of the consensus problem, the feedback laws $\kappa_{d,in}^k$ and $\kappa_{d,in}^k$ are unused.

The definition of the timer dynamics results in the timers never converging to synchronization. Therefore, we are interested in a partial notion of global asymptotic synchronization, wherein, for each $i \in \mathcal{V}$, we consider $x_i = (p_i, q_i)$ where $p_i = (z_i, \eta_i)$ and $q_i = \tau_i$. Namely, we are interested in showing that the hybrid system $\mathcal{H}$ in (5) with data as in (14)–(16) has global asymptotic synchronization with respect to (z, η) where $z = (z_1, z_2, \ldots, z_N)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$.

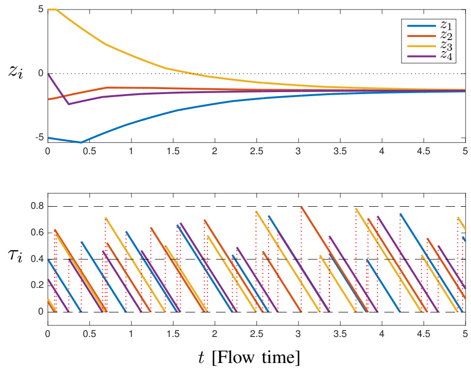


Fig. 1: The z, τ components of a solution $\phi = (\phi_z, \phi_\eta, \phi_\tau)$ to a hybrid system with with agent dynamics (10), communication governed by local timers τ_i with dynamics in (11), and distributed controller (12) and (13).

Then, by invoking Theorem 3.6 and [13, Proposition 4.5], we have the following result.

Theorem 4.1: Given positive scalars $T_1 \leq T_2$ and the hybrid system $\mathcal{H}$ with agent data as in (14)-(16)*, suppose the digraph Γ is weight balanced and strongly connected. The resulting hybrid system $\mathcal{H}$ has global asymptotic synchronization with respect to (z, η) if there exist diagonal matrices $P > 0$, $Q > 0$, and scalars $\gamma, h \in \mathbb{R}$ and $\sigma > 0$ such that*

$$\begin{bmatrix} \gamma He(P, \mathcal{L}) & -P\Pi + \gamma \mathcal{K}_2 \mathcal{L} Q E(\tau) \\ \star & -\sigma Q E(\tau) - He(QE(\tau), \mathcal{K}_2) \end{bmatrix} \leq 0 \qquad (17)$$

for every $\tau \in [0, T_2]$, where $\Pi = I - \frac{1}{N} 1_N 1_N^\top$, $\mathcal{K}_2 = \gamma \mathcal{L} - hI$, and $E(\tau) = diag(e^{\sigma \tau_1}, e^{\sigma \tau_2}, \ldots, e^{\sigma \tau_N})$.

Remark 4.2: Note that condition (17) guaranteeing global asymptotic synchronization in Theorem 4.1 is a bilinear matrix in P, Q, γ, h, and σ. To design γ and h, it is possible to decompose and linearize the inequality in (17) in a manner similar to the approach in [12].

Example 4.3: Consider four agents with dynamics as in (10) over a digraph with an adjacency matrix

$$\mathcal{G} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

which is strongly connected and weight balanced. Let $T_1 = 0.4$. For $T_2 = 0.8$, and parameters $\gamma = -0.4$, $h = -0.4$, and $\sigma = 2$, we find that the matrices $P = 5.06I$ and $Q = 1.58I$ satisfy condition (17) in Theorem 4.1. Figure 1 shows a solution ϕ to a hybrid system with agent dynamics (10), communication governed by local timers τ_i with dynamics in (11), and distributed controller (12) and (13) from initial conditions $\phi_z(0,0) = (-5, -2, 5, 0)$, $\phi_\eta(0,0) = (-1, 1, 0, -10)$, and $\phi_\tau(0,0) = (0.4, 1, 0.1, 0.25)$.[3] $\triangle$

[3] Code at `https://github.com/HybridSystemsLab/ConsAsyncTimes`

B. Synchronization of Impulse-coupled Oscillators

Consider a hybrid system model of impulse-coupled oscillators[4]. The oscillators are considered to be completely connected and each i-th oscillator is defined by a timer state $x_i \in [0, T]$, where $T > 0$. The timer state increases monotonically and continuously toward a threshold T with a natural frequency given by $\omega > 0$ and, upon reaching a threshold T, is impulsively reset to zero. Furthermore, upon reseting its own state to zero, the agent releases an impulse which excites the connected agents and updates their timer state to $x_i + \gamma(x_i)$, where γ satisfies the following assumption.

Assumption 4.4: Let the single-valued mapping $\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be locally bounded on the interval $[0, T]$ and such that $\gamma(0) = 0$ and $\gamma(s)$ is increasing for $s > 0$.

With the hybrid system model introduced in Section III, we model each agent as a hybrid system with state $x_i \in [0, T]$. The flow map and flow set are defined as

$$\dot{x}_i = \omega =: \widetilde{F}(x_i) \qquad \dot{x}_i \in [0, T] =: \widetilde{C} \qquad (18)$$

respectively, Furthermore, the jump map is defined as $\widetilde{G}_{in}(s) = 0$ when $s = T$ and

$$\widetilde{G}_{ex}^{ik}(x_i) = \begin{cases} x_i + \gamma(x_i) & x_i + \gamma(x_i) < T \\ \{0, T\} & x_i + \gamma(x_i) = T \\ 0 & x_i + \gamma(x_i) > T \end{cases}, \qquad \widetilde{D} := \{T\}$$
$$\widetilde{G}_{in}(s) = 0 \qquad (19)$$

for each $i, k \in \mathcal{V}$. Note that G_{ex}^{ik} for each $i, k \in \mathcal{V}$ is set valued at $x_i + \gamma(x_i) = T$, at which value it updates x_i to either the threshold T or to zero. Then, we use the above data to build an interconnected hybrid system as in (5), and study the synchronization properties for the case of two impulse-coupled oscillators.

We are interested in showing that the hybrid system in (5) with data as in (18)-(19) has local asymptotic synchronization. We will show asymptotic synchronization of the timer states x_i for $i \in \{1, 2\}$. Namely, through the asymptotic stability of the synchronization set given by

$$\mathcal{A} = \{x \in [0, T]^2 : x_1 = x_2\}. \qquad (20)$$

we use Theorem 3.6 to show that $\mathcal{H}$ with the above data has local asymptotic synchronization.

The basin of attraction for asymptotic stability is the set of points where attractivity to a set $\mathcal{A}$ holds. Likewise, the basin of attraction for asymptotic synchronization, denoted as $\mathcal{B}_s$, is the set of points where $\mathcal{H}$ has the attractive synchronization property. It excludes the set of points, denoted by $\mathcal{X}$, where solutions may never converge to the set $\mathcal{A}$. We define $\mathcal{X}$ as the set of points away from $\mathcal{A}$ where $|x_1 - x_2|$ remains constant. Following [15] and using the definitions of $\widetilde{G}_{in}$ and $\widetilde{G}_{ex}^{ik}$ above, we have that $\mathcal{X}$ is defined by

$$\mathcal{X} = \{x \in [0, T] : |x_1 - x_2| = T - T^\star, \; T - 2T^\star = \gamma(T^\star)\}.$$

[4] Also referred to as a firefly model, pulse-coupled oscillator or integrate-and-fire oscillator [11], [14].

Lemma 4.5: Given $T^\star$ satisfying $T - 2T^\star = \gamma(T^\star)$ for $T > 0$, where T is the timer threshold and γ satisfies Assumption 4.4, the function

$$V(x) = \min\{|x_1 - x_2|, 2(T^\star + \gamma(T^\star)) - |x_1 - x_2|\} \quad (21)$$

is a Lyapunov function candidate for $\mathcal{H}$ on $x \in C \setminus \mathcal{X}$, namely, for all $x \in \{s \in [0,T]^2 : V(s) < T^*\}$.

By virtue of Theorem 3.6 and [16, Proposition 24], we have the following synchronization result.

Theorem 4.6: Let $T > 0$ and let γ satisfy Assumption 4.4. Given a hybrid system $\mathcal{H}$ with data as in (18)-(19) *with $N = 2$, and $T > 0$, the resulting hybrid system has local asymptotic synchronization on the basin of attraction $\mathcal{B}_s = C \setminus \mathcal{X}$.*

Remark 4.7: Results similar to Theorem 3.6 for the interconnected hybrid system $\mathcal{H}$ have been presented in the literature for specific choices of γ. In [16], an asymptotic stability result for the case of $\gamma(s) = \varepsilon s$ for $\varepsilon > 0$ and the threshold $T = 1$ is given. It may be possible to relax the assumptions on γ in Assumption 4.4. For instance, in [15], $\gamma(s) = h\sin(s)$ was shown to have the synchronization set locally asymptotically stable. Moreover, also in [15], it was shown through a Lyapunov based analysis that the synchronization set is attractive for the case $\gamma(s) = \varepsilon(1 - \cos(s))$ when $T = 2\pi$. In [17], the case when $\gamma(s) = 2\pi - s$ when $s \geq \pi$ and $\gamma(s) = -s$ when $s \leq \pi$ is considered. Note that this choice is an example of a discontinuous function γ.

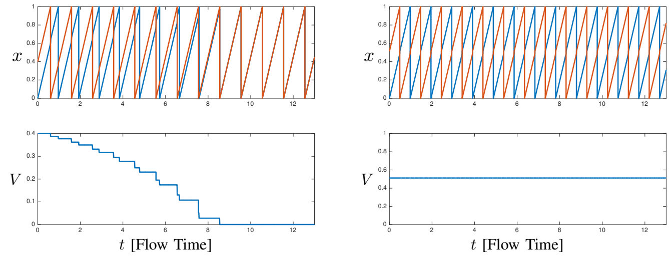
Example 4.8: Consider the case of a hybrid system $\mathcal{H}$ with $\widetilde{F}(x_i) = \omega$, $\widetilde{C} = [0, T]$, $\widetilde{G}_{in}$ and G_{ex}^{ik} in (19), $N = 2$ with γ defined as $\gamma(s) = \varepsilon s^2$, with $\varepsilon > 0$. Note that γ satisfies Assumption 4.4. Moreover, the value $T^\star$ can be found by solving for the roots of $\varepsilon s^2 + 2s - T = 0$, which leads to $T^\star = \frac{\sqrt{1+\varepsilon T} - 1}{\varepsilon}$. Figure 2 shows two numerical solutions[5] showcasing the cases when a) solutions converge to synchrony asymptotically and b) never converge; namely, we show a solution to $\mathcal{H}$ that is initialized in the basin of synchronization $\mathcal{B}_s = C \setminus \mathcal{X}$, and another initialized outside of $\mathcal{B}_s$. The parameters used are $T = 1$ and $\varepsilon = 0.1$. Figure 2(a) has initial conditions $\phi_{x_1}(0, 0) = 0$ and $\phi_{x_2}(0, 0) = 0.4$ that converge to synchronization. In Figure 2(b), for a solution with initial conditions $\phi_{x_1}(0, 0) = 0$ and $\phi_{x_2}(0, 0) \approx 0.5119$, the solution does not converge to synchronization.

$\triangle$

V. Conclusion

In this paper, we introduced a generic model for studying synchronization in interconnected agents with hybrid dynamics. We defined notions of partial state asymptotic synchronization in the sense of both stable and attractive synchronization which leads to an asymptotic synchronization property. Sufficient conditions for synchronization through the stability of a synchronization set were presented Lyapunov based tools for hybrid systems to certify asymptotic synchronization for two applications. Current efforts include determining constructive conditions for asymptotic synchronization in special classes of hybrid systems.

[5] Code at `https://github.com/HybridSystemsLab/FireFly`



(a) A solution $\mathcal{H}$ that is asymptotically synchronizes after several jumps.

(b) A solution $\mathcal{H}$ that never synchronizes. Note V stays constant over the entire solution domain

Fig. 2: Examples of solutions to $\mathcal{H}$ for the impulse-coupled oscillator with $\gamma(s) = \varepsilon s^2$, $T = 1$, and $\varepsilon = 0.1$. The function V over the respective solution is also given.

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