Existence of Hybrid Limit Cycles and Zhukovskii Stability in Hybrid Systems

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***Abstract*— This work pertains to the study of the existence of hybrid limit cycles for a class of hybrid systems. Necessary conditions, particularly, a condition using a forward invariance notion, for existence of hybrid limit cycles are first presented. Due to its usefulness in continuous-time systems with limit cycles, the notion of Zhukovskii stability, typically stated for continuous-time systems, is extended to hybrid systems given by the combination of continuous dynamics on a flow set and discrete dynamics on a jump set. A sufficient condition using incremental graphical stability is proposed for Zhukovskii stability. Examples illustrate the results and provide insight about the existence (or lack of) of hybrid limit cycles when Zhukovskii stability and incremental graphical stability hold.**

I. Introduction

Nonlinear dynamical systems with periodic solutions are found in many areas, including biological dynamics [1], neuronal systems [2], and population dynamics [3], to name just a few. In recent years, the study of limit cycles in hybrid systems has received substantial attention. One reason is the existence of hybrid limit cycles in many engineering applications, such as walking robots [4], genetic regulatory networks [5], among others.

As a difference to general continuous-time systems, for which the Poincare-Bendixson theorem offers criteria for existence of limit cycles/periodic orbits, the problem of identifying the existence of limit cycles for hybrid systems has been studied for specific classes of hybrid systems. Specific results for existence of hybrid limit cycles include [4], [7]-[13]. In particular, Matveev and Savkin established a criterion for existence of a finite number of limit cycles in a class of hybrid dynamical systems modeled by multivalued differential automata with discrete states depending on the switching time sequence [9]. Grizzle et al. established the existence and stability properties of a periodic orbit of nonlinear systems with impulsive effects via the method of Poincaré sections [4]. Nersesov et al. generalized the Poincaré's method to analyze limit cycles for left-continuous hybrid impulsive dynamical systems [10]. Using the transverse contraction framework, the existence and orbital stability of nonlinear hybrid limit cycles were analyzed for a class of autonomous hybrid dynamical systems with impulse in [12]. More recently, in [13], the existence and stability of limit cycles in reset control systems were investigated via techniques that rely on the linearization of the Poincaré map about its fixed point. We believe that conditions for existence of hybrid limit cycles in general hybrid systems should play a more prominent role in analysis and control of hybrid limit cycles. However, general results on existence or nonexistence of hybrid limit cycles for a class of hybrid systems in [6] are still not available in the literature.

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Building from our previous results in [7], we study conditions guaranteeing existence of hybrid limit cycles in the class of hybrid dynamical systems in [6] and introduce a notion that might be instrumental in guaranteeing such existence. The stated conditions include compactness and transversality of the limit cycle, as well as a continuity of the so-called time-to-impact function. Motivated by the use of Zhukovskii stability methods for periodic orbits, as done in [14], [15], we introduce this notion fot for the class of hybrid systems introduced in [6]. A sufficient condition for Zhukovskii stability that involves the incremental stability notion introduced in [16] is provided. Finally, via examples, we provide insight on the existence of hybrid limit cycles under Zhukovskii stability and incremental stability. The results in this paper pave the road for the developed extensions of the results on existence of limit cycles in [14], [15] to the hybrid case.

The organization of the paper is as follows. Section II gives some preliminaries on hybrid systems and basic properties of hybrid limit cycle. In Section III, the Zhukovskii stability notion and incremental graphical stability notion are introduced. Moreover, the relationship between these two notions is studied. In Section IV, a sufficient condition for existence of hybrid limit cycles is developed. Examples illustrating the results are presented throughout the paper.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geqslant 0} := [0, +\infty)$. $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \cdots\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of h at x in the direction of f is denoted by $L_f h(x) := \langle \nabla h(x), f(x)\rangle$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$. The range of f is denoted by rge$f$, i.e., rge$f := \{f(x) : x \in \operatorname{dom} f\}$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x-y|$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $\overline{\mathcal{A}}$ (respectively, $\overline{\text{con}}\,\mathcal{A}$) denotes its closure (respectively, its closed convex hull). $\mathbb{B}$ denotes a closed unit ball in Euclidean space (of appropriate dimension). Given $\delta > 0$ and $x \in \mathbb{R}^n$, $x + \delta\mathbb{B}$ denotes a closed ball centered at x with radius δ.

II. Definitions and Basic Properties of Hybrid Systems with Hybrid Limit Cycles

A. Hybrid Systems

We consider hybrid systems $\mathcal{H}$ as in [6], given by

$$\mathcal{H}\left\{\begin{array}{rcll} \dot{x} & = & f(x) & x \in C \\ x^{+} & = & g(x) & x \in D \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $\dot{x}$ denotes its derivative with respect to time, and x^+ denotes its value after a jump. The function $f : C \to \mathbb{R}^n$ (respectively, $g : D \to \mathbb{R}^n$) is a single-valued map describing the continuous evolution (respectively, the discrete jumps) while $C \subset \mathbb{R}^n$ (respectively, $D \subset \mathbb{R}^n$) is the set on which the flow map f is effective (respectively, from which jumps can occur). The data of a hybrid system $\mathcal{H}$ is given by (C, f, D, g). The restriction of $\mathcal{H}$ on a set M is defined as $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$. A solution to $\mathcal{H}$ is parameterized by ordinary time t and a counter j for jumps. It is given by a hybrid arc[1] $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ that satisfies the dynamics of $\mathcal{H}$; see [6] for more details. A solution ϕ to $\mathcal{H}$ is said to be complete if $\operatorname{dom}\phi$ is unbounded. It is said to be maximal if it is not a (proper) truncated version of another solution. The set of maximal solutions to $\mathcal{H}$ from the set K is denoted as $\mathcal{S}_{\mathcal{H}}(K)$. We define $t \mapsto \phi^f(t, x_0)$ as a solution of the flow dynamics

$$\dot{x} = f(x) \quad x \in C$$

from $x_0 \in \overline{C}$. A hybrid system $\mathcal{H}$ is said to be well-posed if it satisfies the *hybrid basic conditions*, namely,

A1) The sets $C, D \subset \mathbb{R}^n$ are closed.
A2) The flow map $f : C \to \mathbb{R}^n$ and the jump map $g : D \to \mathbb{R}^n$ are continuous.

For more details about this hybrid systems framework, we refer the readers to [6].

B. Hybrid Limit Cycles

Before revealing their basic properties, we define hybrid limit cycles. For this purpose, we consider the following notion of flow periodic solutions.

Definition 2.1: (flow periodic solution) A complete solution ϕ^* to $\mathcal{H}$ is *flow periodic with period T^* and one jump in each period* if there exists $T^* \in (0, \infty)$ such that $\phi^*(t + T^*, j + 1) = \phi^*(t, j)$ for all $(t, j) \in \operatorname{dom}\phi^*$.

The definition of a flow periodic solution ϕ^* with period $T^* > 0$ and one jump per period above implies that if $(t, j) \in \operatorname{dom}\phi^*$, then $(t + T^*, j + 1) \in \operatorname{dom}\phi^*$. For a notion allowing for multiple jumps in a period, see [8]. A flow periodic solution to $\mathcal{H}$ as in Definition 2.1 generates a hybrid limit cycle.

Definition 2.2: (hybrid limit cycle) A flow periodic solution ϕ^* with period T^* and one jump in each period defines a *hybrid limit cycle* $\mathcal{O} := \{x \in \mathbb{R}^n : x = \phi^*(t, j), (t, j) \in \operatorname{dom}\phi^*\}$.

Example 2.3: Consider a timer with state $\chi \in [0, 1]$ and hybrid dynamics

$$\mathcal{H}_{\mathrm{T}}\left\{\begin{array}{rcll} \dot{\chi} & = & 1 & \chi \in [0,1] \\ \chi^{+} & = & 0 & \chi = 1 \end{array}\right. \tag{2}$$

Its unique maximal solution from $\xi \in [0, 1]$ is given by $\phi(t, j) = \xi + t - j$ for each $(t, j) \in \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ such that $t \in [\max\{0, j - \xi\}, j + 1 - \xi]$. The hybrid orbit generated by ϕ is $\mathcal{O}=\{\chi \in \mathbb{R} : \chi=\phi(t, 1), t \in [1 - \xi, 2 - \xi]\} = [0, 1]$. △

C. Necessary Conditions for Existence of Hybrid Limit Cycles in a Class of Hybrid Systems

In this section, we derive several necessary conditions for the existence of hybrid limit cycles for a class of hybrid systems $\mathcal{H}$ satisfying the following properties.

Assumption 2.4: For a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$, there exists a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ such that

1) the flow set is $C = \{x \in \mathbb{R}^n : h(x) \geqslant 0\}$ and the jump set is $D = \{x \in \mathbb{R}^n : h(x) = 0, L_f h(x) \leqslant 0\}$;
2) the flow map f is continuously differentiable on an open neighborhood of $M \cap C$, and the jump map g is continuous on $M \cap D$;
3) $L_f h(x) < 0$ for all $x \in M \cap D$ and $g(M \cap D) \cap (M \cap D) = \emptyset$;

Remark 2.5: By items 1) and 2) of Assumption 2.4, the data of $\mathcal{H}_M := (M \cap C, f, M \cap D, g)$ satisfies the hybrid basic conditions [6, Assumption 6.5]. Then, using item 3) of Assumption 2.4, by [18, Lemma 2.7], for any bounded and complete solution ϕ to $\mathcal{H}_M$, there exists $r > 0$ such that $t_{j+1} - t_j \geqslant r$ for all $j \geqslant 1$, (t_j, j), $(t_{j+1}, j) \in \operatorname{dom}\phi$ (i.e., the elapsed time between two consecutive jumps is uniformly bounded below by a positive constant).

It can be shown that a hybrid limit cycle generated by periodic solutions as in Definition 2.2 is closed and bounded, as established in the following result.

Lemma 2.6: Given a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 2.4, suppose that $\mathcal{H}$ has a hybrid limit cycle $\mathcal{O}$. Then, $\mathcal{O}$ is compact.

Remark 2.7: Since a hybrid limit cycle $\mathcal{O}$ to $\mathcal{H}_M$ is compact, for any solution ϕ to $\mathcal{H}_M$, the distance $|\phi(t, j)|_{\mathcal{O}}$ is well-defined for all $(t, j) \in \operatorname{dom}\phi$.

The following result establishes a transversality property of any hybrid limit cycle for $\mathcal{H}$ restricted to M.[2]

Lemma 2.8: Given a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 2.4, suppose that $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ has a hybrid

[1] A hybrid arc is a function ϕ defined on a hybrid time domain and for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous. A *compact hybrid time domain* is a set $\mathcal{E} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ of the form $\mathcal{E} = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_J$; the set $\mathcal{E}$ is a *hybrid time domain* if for all $(T, J) \in \mathcal{E}$, $\mathcal{E} \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain.

[2] A hybrid limit cycle $\mathcal{O}$ to a hybrid system $\mathcal{H}$ satisfying Assumption 2.4 is transversal to $M \cap D$ if its closure intersects $M \cap D$ at exactly one point $\bar{x} := \mathcal{O} \cap (M \cap D)$ with the property $L_f h(\bar{x}) \neq 0$.

limit cycle $\mathcal{O} \subset M \cap (C \cup D)$. Then, $\mathcal{O}$ is transversal to $M \cap D$.

To state our next result, let us introduce the *time-to-impact function* and the Poincaré map for hybrid dynamical systems as in $\mathcal{H}$. Following the definition in [4], for a hybrid system $\mathcal{H} = (C, f, D, g)$, the *time-to-impact function with respect to D* is defined by $T_I : \overline{C} \cup D \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where[3]

$$T_I(x) := \inf\{t \geqslant 0 : \phi(t, j) \in D,\ \phi \in \mathcal{S}_{\mathcal{H}}(x)\} \quad (3)$$

for each $x \in \overline{C} \cup D$.

Inspired by [4, Lemma 3], we show that the function $x \mapsto T_I(x)$ is continuous on a subset of $\mathcal{O}$.

Lemma 2.9: Given a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 2.4, suppose that $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ has a unique hybrid limit cycle $\mathcal{O} \subset M \cap (C \cup D)$ defined by the flow periodic solution ϕ^*. Then, T_I is continuous on $\mathcal{O} \setminus \{\phi^*(t, 0)\}$, where t is such that $(t, 0), (t, 1) \in \operatorname{dom} \phi^*$.

Proof: By [7, Lemma 4.15], T_I is continuous at points in $\mathcal{X} := \{x \in M \cap (C \cup D) : 0 < T_I(x) < \infty\}$. Note that $\mathcal{O} \subset M \cap (C \cup D)$ and for all $x \in \mathcal{O} \setminus \{\phi^*(t, 0)\}$, where $\phi^*(t, 0) \in D$, we have that $0 \leqslant T_I(x) < \infty$ since $(t, 0)$ is such that $\phi^*(t, 0)$ has a jump; namely, $(t, 0), (t, 1) \in \operatorname{dom} \phi^*$. Then, T_I is continuous on $\mathcal{O} \setminus \{\phi^*(t, 0)\}$. ■

D. A Necessary Condition via Forward Invariance

Following the spirit of the necessary condition for existence of limit cycles in nonlinear continuous-time systems in [19], we have the following necessary condition for general hybrid systems with a hybrid limit cycle given by the zero-level set of a smooth enough function.

Proposition 2.10: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ satisfying the hybrid basic conditions with f continuously differentiable. Suppose every solution $\phi \in \mathcal{S}_{\mathcal{H}}$ is unique and there exists a hybrid limit cycle $\mathcal{O}$ for $\mathcal{H}$ with period $T^* > 0$ and one jump per period that is forward invariant[4] and defined as

$$\mathcal{O} := \{x \in \mathbb{R}^n : p(x) = 0\},$$

where $p : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on an open neighborhood $\mathcal{U}$ of $\mathcal{O}$. Then, there exists a function $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ that is twice continuously differentiable on $\mathcal{U}$ and

$$W(x) \geqslant 0 \ \ \forall x \in \mathcal{O}, \quad (4)$$

$$\langle \nabla W(x), f(x) \rangle = 0 \ \ \forall x \in \mathcal{O} \cap C, \quad (5)$$

$$\langle \nabla \langle \nabla W(x), f(x) \rangle, f(x) \rangle = 0 \ \ \forall x \in \mathcal{O} \cap C, \quad (6)$$

$$W(g(x)) - W(x) = 0 \ \ \forall x \in \mathcal{O} \cap D. \quad (7)$$

Proof (sketch): The stated properties can be shown to hold using the function W as $W(x) = (p(x) - p(\bar{x}))^{\bar{n}}$ where $\bar{x} \in \mathcal{U} \setminus \mathcal{O}$ and $\bar{n} \in \mathbb{N} \setminus \{0\}$ is an arbitrary positive even integer, and forward invariance of the hybrid limit cycle $\mathcal{O}$. ■

[3] In particular, when there does not exist $t \geqslant 0$ such that $\phi^f(t, x) \in D$, we have $\{t \geqslant 0 : \phi^f(t, x) \in D\} = \emptyset$, which gives $T_I(x) = \infty$.

[4] Every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$ is complete and satisfies $\operatorname{rge} \phi \subset \mathcal{O}$; see [21, Definition 2.6]

Proposition 2.10 provides a necessary condition that can be used, by seeking for such a function W with the properties therein, the existence of a hybrid limit cycle with period T^* and one jump per period. In addition, as exploited in [19, Theorem 1], it can be used to determine the stability of limit cycles; see [7] for tools to study stability properties of hybrid limit cycles.

The following example illustrates the results in Lemma 2.6, Lemma 2.8, Lemma 2.9 as well as Proposition 2.10.

Example 2.11: Consider a hybrid system $\mathcal{H}_{\mathrm{S}} = (C_{\mathrm{S}}, f, D_{\mathrm{S}}, g)$ with state $x = (x_1, x_2)$ and data

$$\mathcal{H}_{\mathrm{S}} \begin{cases} \dot{x} = f(x) := b \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} & x \in C_{\mathrm{S}} \\ x^+ = g(x) := \begin{bmatrix} c \\ 0 \end{bmatrix} & x \in D_{\mathrm{S}} \end{cases} \quad (8)$$

where $C_{\mathrm{S}} := \{x \in \mathbb{R}^2 : x_1 \geqslant 0\}$ and $D_{\mathrm{S}} := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leqslant 0\}$. The two parameters b and c satisfy $b \geqslant 1$ and $c > 0$. Since C_{S} and D_{S} are closed, and the flow and jump maps are continuous with f continuously differentiable, the hybrid system $\mathcal{H}_{\mathrm{S}}$ satisfies the hybrid basic conditions. Note that every solution $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}}}$ is unique. The flow dynamics characterizes an oscillatory behavior. In fact, a maximal solution ϕ^* to $\mathcal{H}_{\mathrm{S}}$ from $\phi^*(0, 0) = (c, 0)$ is a unique flow periodic solution with period $T^* = \frac{\pi}{2b}$.

Define the function $h : \mathbb{R}^2 \to \mathbb{R}$ as $h(x) = x_1$ for all $x \in \mathbb{R}^2$. Then, the sets C_{S} and D_{S} can be written as $C_{\mathrm{S}} = \{x \in \mathbb{R}^2 : h(x) \geqslant 0\}$ and $D_{\mathrm{S}} = \{x \in \mathbb{R}^2 : h(x) = 0, L_f h(x) \leqslant 0\}$. Define a compact set $M := \{x \in \mathbb{R}^2 : |x| \geqslant c, x_2 \leqslant 0\}$. Then, we obtain that for all $x \in M \cap D_{\mathrm{S}}$, $L_f h(x) = b x_2 < 0$ and $\mathcal{O} \subset M \cap (C_{\mathrm{S}} \cup D_{\mathrm{S}})$. In addition, it is found that the invariant set defined by points (x_1, x_2) such that $x_1^2 + x_2^2 = c^2$ represents a hybrid limit cycle $\mathcal{O}$, i.e.,

$$\mathcal{O} := \left\{(x_1, x_2) \in M \cap (C_{\mathrm{S}} \cup D_{\mathrm{S}}) : x_1^2 + x_2^2 = c^2\right\},$$

along which the state vector $x = (x_1, x_2)$ moves clockwise. Using the flow and jump maps, it is verified that $\mathcal{O}$ is forward invariant. Define the continuously differentiable function $p(x) := x_1^2 + x_2^2 - c^2$. To validate Proposition 2.10, pick the point $\bar{x} := (0, 0) \notin \mathcal{O}$ satisfying $p(\bar{x}) = -c^2 \neq 0$ and define a continuously differentiable function $W : \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ satisfying (5)-(7). In fact, for all $x \in \mathcal{O}$, a function defined by $W(x) = (x_1^2 + x_2^2)^2 \geqslant 0$ satisfies (5)-(7) using the fact $\langle \nabla p(x), f(x) \rangle = [2x_1 \ \ 2x_2] f(x) = 0$ for all $x \in C_{\mathrm{S}}$, in particular, $\forall x \in \mathcal{O} \cap M_{\mathrm{S}} \cap C_{\mathrm{S}}$,

$$\langle \nabla W(x), f(x) \rangle = 2(x_1^2 + x_2^2)[2x_1 \ 2x_2] f(x) = 0$$
$$\langle \nabla \langle \nabla W(x), f(x) \rangle, f(x) \rangle = 0$$

and $\forall x \in \mathcal{O} \cap M_{\mathrm{S}} \cap D_{\mathrm{S}}$, $W(g(x)) - W(x) = 0$ where the condition $x = (0, -c)$ for $\mathcal{O} \cap D_{\mathrm{S}}$ is applied.

Note that the hybrid limit cycle $\mathcal{O} := \left\{(x_1, x_2) \in M \cap (C_{\mathrm{S}} \cup D_{\mathrm{S}}) : x_1^2 + x_2^2 = c^2\right\}$ is bounded, otherwise a flow periodic solution ϕ^* will escape to infinity in finite time which leads to a contradiction with the definition of a flow periodic solution. Moreover, due to the closedness of $M \cap (C_S \cup D_S)$, $\mathcal{O}$ is closed; hence, $\mathcal{O}$ is compact, which illustrates Lemma 2.6. In addition, since for all $x \in M \cap D_S$, $L_f h(x) = bx_2 < 0$ and $\bar{x} = \mathcal{O} \cap (M \cap D_S) = (-c, 0)$, $\mathcal{O}$ is transversal to $M \cap D_S$, which illustrates Lemma 2.8. Finally, for the hybrid limit cycle $\mathcal{O}$ defined by a flow periodic solution ϕ^* and for any $x \in \mathcal{O} \setminus \{\phi^*(t,0)\}$, $T_I(x) \in [0, \frac{\pi}{2b}]$ is continuous which illustrates Lemma 2.9. △

III. Zhukovskii Stability for Hybrid Systems

In this section, we introduce a version of Zhukovskii stability for general hybrid systems that extends the one in the literature for continuous-time systems; see, e.g., [17], [14]. In the next section, we establish links to the existence of hybrid limit cycles for a class of hybrid systems.

A. Definition

Following [14], [15] and [20], we employ the family of maps $\mathcal{T}$ defined by

$$\mathcal{T} = \{\tau(\cdot,\cdot) | \tau : \mathbb{R}_{\geqslant 0} \times \mathbb{N} \to \mathbb{R}_{\geqslant 0}, \tau(0,0) = 0\}.$$

Definition 3.1: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. A maximal solution ϕ_1 to $\mathcal{H}$ is said to be

1) *Zhukovskii stable* (ZS) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\phi_2 \in \mathcal{S}_{\mathcal{H}}(\phi_1(0,0) + \delta\mathbb{B})$ there exists $\tau \in \mathcal{T}$ such that for each $(t,j) \in \operatorname{dom} \phi_1$ we have
$$(\tau(t,j), j) \in \operatorname{dom} \phi_2 \tag{9}$$
$$|\phi_1(t,j) - \phi_2(\tau(t,j), j)| \leqslant \varepsilon, \tag{10}$$
2) *Zhukovskii locally attractive* (ZLA) if there exists $\mu > 0$ such that for each $\phi_2 \in \mathcal{S}_{\mathcal{H}}(\phi_1(0,0) + \mu\mathbb{B})$ there exists $\tau \in \mathcal{T}$ such that for each $\varepsilon > 0$ there exists $T > 0$ for which we have that
$$(t,j) \in \operatorname{dom} \phi_1, \qquad t + j \geqslant T \tag{11}$$
implies
$$(\tau(t,j), j) \in \operatorname{dom} \phi_2 \tag{12}$$
$$|\phi_1(t,j) - \phi_2(\tau(t,j), j)| \leqslant \varepsilon, \tag{13}$$
3) *Zhukovskii locally asymptotically stable* (ZLAS) if it is both ZS and ZLA.

Remark 3.2: The map τ in Definition 3.1 reparameterizes the flow time of the solution ϕ_2. In particular, the ZS notion only requires that solutions ϕ_2 stay close to the solution ϕ_1 for the same value of the jump counter j but potentially different flow times. When for some δ, τ can be chosen to be equal to t then the notion reduces to stability of the solution ϕ_1. Note that τ in the ZS notion may depend on the initial condition of ϕ_2; see, e.g., [15, Definition 2.1]. For simplicity, the ZLA notion is written as a uniform property (in hybrid time, and over the compact set of initial conditions defined by μ). Note that τ in the ZLA notion may also depend on the initial condition of ϕ_2. When ϕ_1 and each ϕ_2 are complete, the nonuniform version of that property would require

$$\lim_{(t,j) \in \operatorname{dom} \phi_1, t+j \to \infty} |\phi_1(t,j) - \phi_2(\tau(t,j), j)| = 0$$

which resembles the notion defined in the literature of continuous-time systems.

Next, the ZLAS notion in Definition 3.1 is illustrated in an example with a hybrid limit cycle.

Example 3.3: Consider the timer system in Example 2.3. Note that every maximal solution to the timer system is unique and complete. Consider $\phi_1 \in \mathcal{S}_{\mathcal{H}_T}$. For a given $\varepsilon > 0$, let $0 < \delta < \varepsilon$. Then, for each $\phi_2 \in \mathcal{S}_{\mathcal{H}_T}(\phi_1(0,0) + \delta\mathbb{B})$, $T_I(\phi_2(0,0)) = 1 - \phi_2(0,0)$. Without loss of generality, we further suppose $\phi_1(0,0) > \phi_2(0,0)$. Then, solution ϕ_1 jumps before ϕ_2. For each $j \in \mathbb{N}\backslash\{0\}$, let $\bar{t}_j = \max_{(t,j-1) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$ and $\bar{t}'_j = \min_{(t,j) \in \operatorname{dom} \phi_1 \cap \operatorname{dom} \phi_2} t$. Then, following the idea in [16, Example 3.4], by constructing the map τ as follows

$$\tau(t,j) = \begin{cases} t & t \in [0, \bar{t}_1], \quad j = 0, \\ \bar{t}'_j & t \in [\bar{t}_j, \bar{t}'_j], \quad j > 0, \\ t & t \in [\bar{t}'_j, \bar{t}_{j+1}], \quad j > 0, \end{cases} \tag{14}$$

we have that for each $(t,j) \in \operatorname{dom} \phi_1$, $(\tau(t,j), j) \in \operatorname{dom} \phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t,j), j)| \leqslant \varepsilon$.

In fact, for $j = 0$, for each $t \in [0, \bar{t}_1]$ with $\bar{t}_1 = 1 - \phi_1(0,0)$, define τ as $\tau(t,0) = t$, which satisfies $(\tau(t,0), 0) \in \operatorname{dom} \phi_2$ and $|\phi_1(t,0) - \phi_2(\tau(t,0), 0)| = |\phi_1(0,0) + t - \phi_2(0,0) - t| \leqslant \delta < \varepsilon$. When $j = 1$ and for each $t \in [\bar{t}_1, \bar{t}'_1]$, define $\tau(t,1) = \bar{t}'_1$, which satisfies $|\phi_1(t,1) - \phi_2(\bar{t}'_1, 1)| = |t - \bar{t}'_1| \leqslant \delta < \varepsilon$. Due to the constructions of $\bar{t}_1$ and $\bar{t}'_1$, $|t - \bar{t}'_1| \leqslant \delta < \varepsilon$ holds for $t \in [\bar{t}_1, \bar{t}'_1]$. Similar analysis can be applied for all $j > 1$, for each $(t,j) \in \operatorname{dom} \phi_1$, $(\tau(t,j), j) \in \operatorname{dom} \phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t,j), j)| \leqslant \varepsilon$. Therefore, the solution $\phi_1 \in \mathcal{S}_{\mathcal{H}_T}$ is ZS. Moreover, note that for $\mu > 0$ and for each $\phi_2 \in \mathcal{S}_{\mathcal{H}_T}(\phi_1(0,0) + \mu\mathbb{B})$, there exists $\tau \in \mathcal{T}$ such that $(t,j) \in \operatorname{dom} \phi_1$ implies $(\tau(t,j), j) \in \operatorname{dom} \phi_2$ and $\lim_{t \to \infty, (t,j) \in \operatorname{dom} \phi_1} |\phi_1(t,j) - \phi_2(\tau(t,j), j)| = 0$. Therefore, solution $\phi_1 \in \mathcal{S}_{\mathcal{H}_T}$ is ZLA; hence, it is also ZLAS. △

B. A Sufficient Condition via Incremental Graphical Stability

In this section, we establish a link between the ZS notion in Definition 3.1 and incremental graphical stability as introduced in [16]. The later notion is presented for self-containedness.

Definition 3.4: [16, Definition 3.2] Consider a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^n$. The hybrid system $\mathcal{H}$ is said to be

1) *incrementally graphically stable* (δS) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any two maximal solutions ϕ_1, ϕ_2 to $\mathcal{H}$, $|\phi_1(0,0) - \phi_2(0,0)| \leqslant \delta$ implies that, for each $(t,j) \in \operatorname{dom} \phi_1$, there exists s such that $(s,j) \in \operatorname{dom} \phi_2$, $|t - s| \leqslant \varepsilon$ and
$$|\phi_1(t,j) - \phi_2(s,j)| \leqslant \varepsilon, \tag{15}$$

2) *incrementally graphically locally attractive* (δLA) if there exists $\mu > 0$ such that, for every $\varepsilon > 0$, for any two maximal solutions ϕ_1, ϕ_2 to $\mathcal{H}$, $|\phi_1(0,0) - \phi_2(0,0)| \leqslant \mu$ implies that there exists $T > 0$ such that for each $(t,j) \in \operatorname{dom}\phi_1$ such that $t + j > T$, there exists $(s,j) \in \operatorname{dom}\phi_2$ satisfying $|t-s| \leqslant \varepsilon$ and
$$|\phi_1(t,j) - \phi_2(s,j)| \leqslant \varepsilon, \quad (16)$$
3) *incrementally graphically locally asymptotically stable* (δLAS) if it is both δS and δLA.

The following theorem establishes a sufficient condition for ZS and ZLA.

Theorem 3.5: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 2.4. Suppose every maximal solution ϕ to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then

- *If the hybrid system $\mathcal{H}_M$ is δS, each $\phi \in \mathcal{S}_{\mathcal{H}_M}$ is ZS;*
- *If the hybrid system $\mathcal{H}_M$ is δLA, each $\phi \in \mathcal{S}_{\mathcal{H}_M}$ is ZLA.*

Proof (sketch): If the system is δS, a function $\tau \in \mathcal{T}$ does exist. In fact, it can be chosen as $\tau(t,j) := s$ where s is given in (15), which satisfies $(\tau(t,j), j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t,j),j)| \leqslant \varepsilon$. Then, we have that if the hybrid system $\mathcal{H}_M$ is δS, every $\phi \in \mathcal{S}_{\mathcal{H}_M}$ is ZS. Similarly, the claim "δLA implies ZLA" can also be proved. ■

The following example is provided to illustrate the sufficient condition for ZS in Theorem 3.5.

Example 3.6: Consider the hybrid system $\mathcal{H}_{\mathrm{S}_M} = (M \cap C_{\mathrm{S}}, f, M \cap D_{\mathrm{S}}, g)$ in Example 2.11, with M given therein. We shall verify the property δS rather than δLA for $\mathcal{H}_{\mathrm{S}_M}$. Items 1) and 3) of Assumption 2.4 have been shown to hold in Example 2.11. Moreover, f and g are continuously differentiable, and $g(M \cap D_{\mathrm{S}}) \cap (M \cap D_{\mathrm{S}}) = \emptyset$. Therefore, items 1)-3) of Assumption 2.4 hold. Furthermore, due to the definition of g in (8), for each $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}_M}}(M \cap (C_{\mathrm{S}} \cup D_{\mathrm{S}}))$, $|\phi(t,j)|_{\mathcal{O}}$ converges to zero in finite time, namely, $|\phi(t,j)|_{\mathcal{O}} = 0$ for all $t + j \geqslant 1 + \frac{\pi}{2b}$, $(t,j) \in \operatorname{dom}\phi$. Note that every $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}_M}}$ is complete.

Consider $\phi_1 = (\phi_{1,x_1}, \phi_{1,x_2}), \phi_2 = (\phi_{2,x_1}, \phi_{2,x_2}) \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}_M}}$ and denote
$$t_\Delta = \arctan\left(\frac{\phi_{2,x_1}(0,0)}{-\phi_{2,x_2}(0,0)}\right) - \arctan\left(\frac{\phi_{1,x_1}(0,0)}{-\phi_{1,x_2}(0,0)}\right).$$
For a given $\varepsilon > 0$, let $0 < \delta < \varepsilon$ such that $|\phi_1(0,0) - \phi_2(0,0)| \leqslant \delta$ and $\max\{t_\Delta/b, \sqrt{2}c\sqrt{1-\cos t_\Delta}\} \leqslant \varepsilon$. Without loss of generality, assume ϕ_1 jumps first. For each $j \in \mathbb{N}\backslash\{0\}$, let $\bar{t}_j = \max_{(t,j-1)\in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2} t$ and $\bar{t}'_j = \min_{(t,j)\in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2} t$. Then, we have that for each $t \in [0, \bar{t}_1]$, there exists $(s,0) \in \operatorname{dom}\phi_2$ such that $s = t$ and $|\phi_1(t,0) - \phi_2(s,0)| = |e^{At}\phi_1(0,0) - e^{At}\phi_2(0,0)| \leqslant \delta < \varepsilon$, where $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$.

For each $t \in [\bar{t}_1, \bar{t}'_1]$, $|\phi_1(\bar{t}_1, 0) - \phi_2(s,0)| \leqslant |e^{A\bar{t}_1}\phi_1(0,0) - e^{A\bar{t}_1}\phi_2(0,0)| \leqslant \delta < \varepsilon$. Note that $\bar{t}_1 = \arctan\left(\frac{\phi_{1,x_1}(0,0)}{-\phi_{1,x_2}(0,0)}\right)/b$, and $\bar{t}'_1 = \arctan\left(\frac{\phi_{2,x_1}(0,0)}{-\phi_{2,x_2}(0,0)}\right)/b$. Then, for each $t \in [\bar{t}_1, \bar{t}'_1]$, $|t - \bar{t}_1| \leqslant |\bar{t}'_1 - \bar{t}_1| \leqslant \varepsilon$. By definition of g, we have $\phi_2(\bar{t}'_1, 1) = (c, 0)$. For each $t \in [\bar{t}_1, \bar{t}'_1]$, $\phi_1(t,1) = e^{A(t-\bar{t}_1)}(c,0)$. Therefore, $|\phi_1(t,1) - \phi_2(\bar{t}'_1, 1)| = c\left|(1 - \cos b(t-\bar{t}_1), \sin b(t - \bar{t}_1))\right| = \sqrt{2}c\sqrt{1-\cos b(t-\bar{t}_1)} \leqslant \sqrt{2}c\sqrt{1-\cos t_\Delta} \leqslant \varepsilon$, where we used the fact that $0 \leqslant b(t - \bar{t}_1) \leqslant \pi/2$ with $b \geqslant 1$.

In fact, for each $t \in [\bar{t}_j, \bar{t}'_j]$, where $j \in \mathbb{N}\backslash\{0\}$, $|t - \bar{t}_j| \leqslant |\bar{t}'_j - \bar{t}_j| \leqslant \varepsilon$. Moreover, for each $t \in [\bar{t}'_j, \bar{t}_{j+1}]$, where $j \in \mathbb{N}\backslash\{0\}$, there exists $(s,j) \in \operatorname{dom}\phi_2$ such that $s = t$ and
$$\begin{aligned} & |\phi_1(t,j) - \phi_2(s,j)| \\ = \; & |e^{At}\phi_1(\bar{t}'_j, j) - e^{At}\phi_2(\bar{t}'_j, j)| \\ \leqslant \; & |\phi_1(\bar{t}'_j, j) - \phi_2(\bar{t}'_j, j)| \\ \leqslant \; & \sqrt{2}c\sqrt{1-\cos t_\Delta} \leqslant \varepsilon, \end{aligned} \quad (17)$$
where we used the facts that $\phi_1(\bar{t}'_j, j) = e^{A(\bar{t}'_j - \bar{t}_j)}\phi_1(\bar{t}_j, j)$ and $\phi_1(\bar{t}_j, j) = \phi_2(\bar{t}'_j, j) = (c, 0)$.

For each $t \in [\bar{t}_{j+1}, \bar{t}'_{j+1}]$, where $j \in \mathbb{N}\backslash\{0\}$, there exists $(s,j) \in \operatorname{dom}\phi_2$ such that $s = t$ and
$$\begin{aligned} & |\phi_1(\bar{t}_{j+1}, j) - \phi_2(s,j)| \\ \leqslant \; & |\phi_1(\bar{t}_{j+1}, j) - \phi_2(\bar{t}_{j+1}, j)| \\ = \; & |e^{A(\bar{t}_{j+1} - \bar{t}'_j)}(\phi_1(\bar{t}'_j, j) - \phi_2(\bar{t}'_j, j))| \\ \leqslant \; & \sqrt{2}c\sqrt{1-\cos t_\Delta} \leqslant \varepsilon. \end{aligned} \quad (18)$$
Therefore, the system is δS. By applying Theorem 3.5, every $\phi \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}_M}}$ is ZS. △

IV. Remarks on Zhukovskii and Incremental Graphical Stability, and Hybrid Limit Cycles

For continuous-time systems, the notion of ZLAS was employed in [14] to assure the existence of a limit cycle. Though such an extension is not trivial, the results in Section III pave the road for such a result for hybrid systems as in (1). In fact, the following example indicates that the notion in Section III is suitable to assure the existence of a hybrid limit cycle.

Consider the hybrid system $\mathcal{H}_{\mathrm{S}_M} = (M \cap C_{\mathrm{S}}, f, M \cap D_{\mathrm{S}}, g)$ in Example 3.6. Note that the ZS notion is illustrated in Example 3.6. To verify the ZLA notion, let us consider a maximal solution $\phi_1 = (\phi_{1,x_1}, \phi_{1,x_2})$ to $\mathcal{H}_{\mathrm{S}_M}$. For each $j \in \mathbb{N}\backslash\{0\}$, let $\bar{t}_j = \max_{(t,j-1)\in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2} t$ and $\bar{t}'_j = \min_{(t,j)\in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2} t$. Let $\mu > 0$. Then, for each $\varepsilon > 0$ and for each $\phi_2 = (\phi_{2,x_1}, \phi_{2,x_2}) \in \mathcal{S}_{\mathcal{H}}(\phi_1(0,0) + \mu\mathbb{B})$, $T_I(\phi_2(0,0)) = \arctan\left(\frac{\phi_{2,x_1}(0,0)}{-\phi_{2,x_2}(0,0)}\right)/b$. Without loss of generality, assume ϕ_1 jumps first. Then, $\bar{t}_1 = T_I(\phi_1(0,0))$ and $\bar{t}'_1 = T_I(\phi_2(0,0))$. Denote $t_b = \bar{t}'_1 - \bar{t}_1 = \left[\arctan\left(\frac{\phi_{2,x_1}(0,0)}{-\phi_{2,x_2}(0,0)}\right) - \arctan\left(\frac{\phi_{1,x_1}(0,0)}{-\phi_{1,x_2}(0,0)}\right)\right]/b$. Note that $\phi_1(\bar{t}_1, 1) = \phi_2(\bar{t}'_1, 1) = (c, 0)$. Then, for $j = 1$, for each $t \in [\bar{t}'_1, \bar{t}_2]$, we can define $\tau \in \mathcal{T}$ as $\tau(t,1) = t + t_b$, which satisfies $(\tau(t,1),1) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,1) - \phi_2(\tau(t,1),1)| = 0 < \varepsilon$. For $j = 2$, for each $t \in [\bar{t}_2, \bar{t}'_2]$, we can define $\tau \in \mathcal{T}$ as $\tau(t,2) = t + t_b$, which satisfies $(\tau(t,2),2) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,2) - \phi_2(\tau(t,2),2)| = 0 < \varepsilon$. For $j = 2$, for each $t \in [\bar{t}'_2, \bar{t}_3]$, we can define $\tau \in \mathcal{T}$ as $\tau(t,2) = t + t_b$, which satisfies $(\tau(t,2),2) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,2) - \phi_2(\tau(t,2),2)| = 0 < \varepsilon$. In fact, for all $t \geqslant \bar{t}'_1$ and $j \geqslant 1$, we can define $\tau \in \mathcal{T}$ as $\tau(t,1) = t + t_b$ and have that
$$(t,j) \in \operatorname{dom}\phi_1, \quad t + j \geqslant T = \bar{t}'_1 + 1 \quad (19)$$

implies $(\tau(t,j),j) \in \operatorname{dom}\phi_2$ and $|\phi_1(t,j) - \phi_2(\tau(t,j),j)| = 0 < \varepsilon$. Therefore, $\phi_1 \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}_M}}(M \cap (C_\mathrm{S} \cup D_\mathrm{S}))$ is ZLA; hence it is also ZLAS. $\mathcal{H}_{\mathrm{S}_M}$ has a nonempty ω-limit set $\Omega(\phi_1) := \{x \in \mathbb{R}^2 : |x| = c, x_1 \geqslant 0, x_2 \leqslant 0\}$, which is a hybrid limit cycle for $\mathcal{H}_{\mathrm{S}_M}$ with period $T^* = \frac{\pi}{2b}$ and one jump per period.

In light of this example, one may wonder if incremental graphical asymptotic stability would serve as a necessary condition for the existence of a hybrid limit cycle. Unfortunately, the fact that incremental graphical asymptotic stability is a property for all solutions starting in a neighborhood makes it difficult to allow for the existence of a hybrid limit cycle. We illustrate this in the following example.

Consider the system in Example 3.6 and two maximal solutions $\phi_1, \phi_2 \in \mathcal{S}_{\mathcal{H}_{\mathrm{S}_M}}$ where ϕ_1, ϕ_2 are two flow periodic solutions with $\phi_i(t,j) = \phi_i(t + T^*, j+1)$ for all $(t,j) \in \operatorname{dom}\phi_i,\ i = 1,2$. Without loss of generality, for any $\mu > 0$, we can pick $\phi_1(0,0) \in M \cap D_\mathrm{S}$ and $\phi_2(0,0) \in M \cap C_\mathrm{S}$ satisfying $|\phi_1(0,0) - \phi_2(0,0)| \leqslant \mu$. Then, ϕ_1 hits the jump map before ϕ_2 as $\phi_1(0,0)$ belongs to the jump set. For each $j \in \mathbb{N}$, let $t_j = \max_{(t,j)\in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2} t$ and $t'_j = \min_{(t,j+1)\in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2} t$. Then, it follows that $([t_j, t'_j], j+1) \in \operatorname{dom}\phi_1$ and $([t_j, t'_j), j+1) \not\subset \operatorname{dom}\phi_2$ since ϕ_1 and ϕ_2 are two flow periodic solutions that share the same hybrid limit cycle with period T^* and one jump per period. Moreover, we have $t'_j - t_j = t'_1 - t_1$ for all $j \in \mathbb{N}$. Now let $\varepsilon = \frac{1}{4}(t'_1 - t_1) > 0$, and for any $T > 0$, pick $t^* = \frac{1}{2}(t_{j^*} + t'_{j^*})$ at some $j^* \in \mathbb{N}$ such that $(t^*, j^*+1) \in \operatorname{dom}\phi_1$ and $t^* + j^* + 1 > T$. Then, it is impossible to find $(s, j^*+1) \in \operatorname{dom}\phi_2$ such that $|\phi_1(t^*, j^*+1) - \phi_2(s, j^*+1)| \leqslant \varepsilon$ with $|t^* - s| \leqslant \varepsilon$ since $([t^* - \varepsilon, t^* + \varepsilon], j^*+1) \subset ([t_{j^*}, t'_{j^*}), j^*+1) \not\subset \phi_2$. This prevents that the hybrid system $\mathcal{H}_{\mathrm{S}_M}$ is δLAS. Potentially, one can use such an approach to rule out the existence of hybrid limit cycles in some cases.

V. Conclusion

Notions and tools for the analysis of existence of hybrid limit cycles in hybrid dynamical systems were proposed. Necessary conditions were established for the existence of hybrid limit cycles. The Zhukovskii stability notion for hybrid systems was introduced and a relationship between Zhukovskii stability and the incremental graphical stability was presented. Examples suggest that, such as in the continuous-time case, Zhukovskii stability might be a necessary condition for the existence of a hybrid limit cycle, while incremental graphical stability may not. Future work includes extending the conditions presented to the case of hybrid limit cycles with more than one jump per period (as in [8]) and to apply the results to examples with higher dimension and more intricate dynamics.

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