Analysis and Design of Event-triggered Control Algorithms using Hybrid Systems Tools

Jun Chai, Pedro Casau, and Ricardo G. Sanfelice

Abstract— **This paper proposes a general framework for analyzing continuous-time systems controlled by event-triggered algorithms. Closed-loop systems resulting from using both static and dynamic output (or state) feedback laws that are implemented via asynchronous event-triggered techniques are modeled as hybrid systems given in terms of hybrid inclusions and studied using recently developed tools for robust stability. Properties of the proposed models, including stability of compact sets, robustness, and Zeno behavior of solutions are addressed. The framework and results are illustrated in several event-triggered strategies available in the literature.**

I. Introduction

Event-triggered control reduces the need to continuously or periodically update the control input by triggering such events only when necessary. Such control strategies can be employed when a continuous-time controller for a continuous-time plant is already available, which is an emulation-based approach, or they can be directly designed by analyzing the closed-loop system that would result from using such a strategy. A wide range of contributions pursuing both types of design methods are available in the literature. Without attempting to cover such a vast and rapidly growing literature, for the case of nonlinear continuous-time systems with static-state feedback laws, an event-triggered strategy is proposed in [1] for scheduling tasks in embedded processors. For linear systems with dynamic output feedback, the stability and $\mathcal{L}_\infty$-performance of event-triggered control strategies are studied in [2]. The survey paper [3] collects many more event-triggered control strategies, classifies them into different categories, such as event-triggered and self-triggered, and highlights key properties they guarantee. Moreover, the recent application of event-triggered control to a plethora of different problems, such as the stabilization of control affine systems [4], [5] attitude control [6] and quadrotor stabilization [7] further highlight the importance of the development of analysis and synthesis tools for event-based control systems.

J. Chai and R. G. Sanfelice are with the Department of Computer Engineering, University of California, Santa Cruz. Email: jchai3, ricardo@ucsc.edu. This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484 and Grant no. CNS-1544396, and by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015.

P. Casau is with the Department of Electrical and Computer Engineering, Faculty of Science and Technology, University of Macau, Taipa, Macau, and also with the Institute for Systems and Robotics, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisbon, Portugal. Email: pcasau@isr.ist.utl.pt. Research by P. Casau was supported by the Macao Science and Technology Development Fund under Grants FDCT/048/2014/A1 and FDCT/026/2017/A1 and by the University of Macau, Macao, China, under Projects MYRG2015-00127-FST and MYRG2016-00097-FST.

Due to the impulsive nature of event-triggered control strategies, it is natural to analyze and design such strategies using tools for hybrid dynamical systems. In this paper (Section III), we propose a rather general formulation of event-triggered control for continuous-time systems within the hybrid inclusions framework developed in [8]. In such a framework, differential and difference inclusions with constraints are used to describe the continuous and discrete behavior of the closed-loop system resulting from employing an event-triggered control law. The proposed formulation captures closed-loop systems resulting from using both static and dynamic output (or state) asynchronous event-triggered feedback laws. It allows for local events triggered by part of the state components [9], [10], which may involve memory states storing the most recent controller and output values.

While the hybrid inclusions framework in [8] has been used in a few instances for the analysis and design of event-triggered control algorithms [3], [11], [12], a complete treatment of general event-triggered control strategies using the tools in [8] has not yet been pursued, and as shown in this article, leads to insightful results. More precisely, in Section IV-A, we provide relaxed Lyapunov-based sufficient conditions for asymptotic stability and convergence of a given set applying tools from [8]. These conditions do not require a decrease in the Lyapunov function both during flows and jumps, but rather allow increases that can be compensated by decrease. Moreover, assuming the objects defining the closed-loop system satisfy the conditions that lead to sequential compactness of solutions, we point out that asymptotic stability of a compact set is robust to small perturbations (Section IV-B). In addition, conditions guaranteeing that solutions exists for arbitrarily long (hybrid) time are provided in Section IV-C. Very importantly, as it is typically desired that event-triggering control algorithms assure that the time in between consecutive events – typically called the *inter-event time* – is (uniformly) lower bounded by a positive constant for each solution, necessary and sufficient conditions for such a property to hold for the resulting well-posed systems are provided in Section IV-D, both through design and as a temporal regularization (Sections IV-E & IV-F). Through an example we show that the existence of a Zeno solution at the attractor of interest could lead to Zeno solutions nearby it when vanishing noise is present.

Notation: Given a vector $x, |x|$ denotes the 2-norm of x. The distance from point x to a closed set K is denoted by $|x|_K = \inf_{\xi \in K} |x - \xi|$. Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, we denote the domain of M as $\operatorname{dom} M = \{x \in \mathbb{R}^m : M(x) \neq \emptyset\}$, and given a set $K \subset \mathbb{R}^n$, the set $M(K) :=$

$\{M(x) : x \in K\} \subset \mathbb{R}^n$ denotes all points that result from evaluating M on the set K. The boundary points of a closed set K is denoted by ∂K; the interior points of a set K is denoted by int K. Given $r \in \mathbb{R}$ and $V : \mathbb{R}^n \to \mathbb{R}$, $V^{-1}(r) = \{x \in \mathbb{R}^n : V(x) = r\}$ denotes the r-level set of V. The closed unit ball around the origin in $\mathbb{R}^n$ is denoted as $\mathbb{B}$. The closure of the convex hull of a set K is denoted by $\overline{\text{co}}K$.

II. Preliminaries on Hybrid Systems

A hybrid system $\mathcal{H}$, or more precisely, a hybrid closed-loop system in our setting, can be written as

$$\mathcal{H}\begin{cases} \dot{z} & \in & F(z) & z \in C \\ z^+ & \in & G(z) & z \in D, \end{cases} \tag{1}$$

where C, F, D, and G represent the flow set, the flow map, the jump set, and the jump map, respectively. Solutions to (1) have continuous and/or discrete behavior depending on the system data (C, F, D, G). Following [8], besides the usual time variable $t \in \mathbb{R}_{\geq 0}$, we consider the number of jumps, $j \in \mathbb{N} := \{0, 1, 2, ...\}$, as an independent variable. Thus, hybrid time is parametrized by (t, j). The domain of a solution to $\mathcal{H}$ is given by a hybrid time domain. A hybrid time domain is defined as a subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ can be written as $\cup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_J$. A solution to the hybrid system (1) is given by a hybrid arc ϕ satisfying the dynamics of (1). A hybrid arc ϕ is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is absolutely continuous on the interval $I^j := \{t : (t, j) \in \operatorname{dom} \phi \}$. In addition, we classify the solutions to hybrid system as follows.

Definition 2.1: A solution ϕ to (1) is

- *nontrivial* if $\operatorname{dom} \phi$ has at least two points;
- *complete* if $\operatorname{dom} \phi$ is unbounded;
- *precompact* if it is complete and bounded;
- *Zeno* if it is complete and $\sup\{t : (t, j) \in \operatorname{dom} \phi\} < \infty$;
- *maximal* if there does not exist another pair ϕ' such that ϕ is a truncation of ϕ' to some proper subset of $\operatorname{dom} \phi'$.

The set $\mathcal{S}_\mathcal{H}$ and $\mathcal{S}_\mathcal{H}(K)$ collect all maximal solutions to the hybrid system $\mathcal{H}$ and all maximal solutions with $\phi(0, 0) \in K$, respectively. □

The following regularity conditions on the system data for a hybrid system $\mathcal{H}$ will be needed in the forthcoming results. They guarantee robustness of stability of compact sets with respect to perturbations; see [8, Chapter 6] for details.

Definition 2.2: (hybrid basic conditions) A hybrid system $\mathcal{H}$ with state $z \in \mathbb{R}^n$ is said to satisfy the hybrid basic conditions if its data (C, F, D, G) is such that

(A1) C and D are closed sets;
(A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous (osc), nonempty and locally bounded, and $F(z)$ is nonempty and convex for all $z \in C$;
(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is osc, nonempty and locally bounded, and $G(z)$ for all $z \in D$. □

III. General Formulation

In this paper, using hybrid inclusions introduced in [8], we model the closed-loop system obtained from a continuous-time plant controlled by a dynamic controller implemented via event-triggered mechanisms (ETMs). The plant has state $x_p \in \mathcal{X}_p \subset \mathbb{R}^{n_p}$, input $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$, output $y \in \mathcal{Y} \subset \mathbb{R}^{n_y}$, and is given by

$$\dot{x}_p \in F_p(x_p, u), \quad y \in H_p(x_p). \tag{2}$$

The dynamic controller has state $x_c \in \mathcal{X}_c \subset \mathbb{R}^{n_c}$ and is given by

$$\dot{x}_c \in F_c(x_c, y), \quad u \in H_c(x_c, y). \tag{3}$$

which is reduced to $u \in H_c(y)$ in the static case. The set-valued maps $F_p : \mathcal{X}_p \times \mathcal{U} \rightrightarrows \mathcal{X}_p$ and $F_c : \mathcal{X}_c \times \mathcal{Y} \rightrightarrows \mathcal{X}_c$ describe the continuous dynamics for the plant and the controller, respectively, while the set-valued maps $H_p : \mathcal{X}_p \rightrightarrows \mathcal{Y}$ and $H_c : \mathcal{X}_c \times \mathcal{Y} \rightrightarrows \mathcal{U}$ assign values for u and y, respectively. Defining the state $x = (x_p, x_c)$, the closed-loop system (without ETM in the loop) is given by

$$\dot{x} \in \left\{ \zeta \in \begin{bmatrix} F_p(x_p, u) \\ F_c(x_c, y) \end{bmatrix} : \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} H_p(x_p) \\ H_c(x_c, y) \end{bmatrix} \right\}.$$

Now, we introduce a model for the closed-loop system of the plant in (2) controlled by (3) implemented via event-triggered strategies. When ETMs are in the loop, the closed-loop system has the structure shown in Figure 1. Similar to sample-and-hold systems, the plant and the controller operate with sampled versions of the output y and the input u, denoted $\hat{y} \in \mathcal{Y}$ and $\hat{u} \in \mathcal{U}$, respectively. An auxiliary state $\chi \in X \subset \mathbb{R}^{n_\chi}$ is introduced to capture possible dynamics added for defining ETMs. The state χ may not be involved in the plant or the controller but rather play a significant role in triggering events [13], [14].

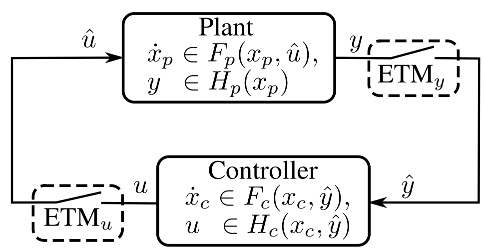

Fig. 1: Closed-loop system with ETM in the loop.

At corresponding triggering events, the most recent values of the output y from the plant and the input u are assigned to $\hat{y}$ and $\hat{u}$, respectively. Moreover, χ is updated via the difference inclusion $\chi^+ \in G_\chi(x_p, x_c, \hat{y}, \hat{u}, \chi)$. In between two events, (x_p, x_c) evolves according to F_p and F_c, while $\hat{y}$ and $\hat{u}$ are governed by

$$\dot{\hat{y}} \in \hat{F}_y(x_p, x_c, \hat{y}, \hat{u}, \chi), \quad \dot{\hat{u}} \in \hat{F}_u(x_p, x_c, \hat{y}, \hat{u}, \chi).$$

When simply "zero-order hold" is employed for $\hat{y}$ and $\hat{u}$, we have $\hat{F}_y \equiv 0$ and $\hat{F}_u \equiv 0$. Also in between two events, the auxiliary state χ has dynamics $\dot{\chi} \in F_\chi(x_p, x_c, \hat{y}, \hat{u}, \chi)$.

We also consider *local triggering events (LTE)*, which trigger updates in individual components of the output memory state and the input memory state. In particular, the presence of each individual LTE permits selective updates of the components of $\hat{y}$ and $\hat{u}$. To this end, the vectors y and $\hat{y}$ are partitioned into N_y subcomponents; while u and $\hat{u}$ are partitioned into N_u subcomponents, i.e.,

$$y = (y_1, y_2, ..., y_{N_y}), \qquad \hat{y} = (\hat{y}_1, \hat{y}_2, ..., \hat{y}_{N_y}),$$
$$u = (u_1, u_2, ..., u_{N_u}), \qquad \hat{u} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_{N_u}).$$

With $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, we define *triggering event functions* as $\gamma^y_{i_y} : \Xi \to \mathbb{R}^{m_y}$ and $\gamma^u_{i_u} : \Xi \to \mathbb{R}^{m_u}$ with the argument for these event functions given as $\xi := (y, u, \hat{y}, \hat{u}, \chi) \in \Xi$ with $\Xi := \mathcal{Y} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{U} \times X$. When $\gamma^y_{i_y}(\xi) = 0$, only the i_y-th component of $\hat{y}$ is updated according to the local output, i.e., $\hat{y}^+_{i_y} = y_{i_y}$ and $\hat{y}^+_k = \hat{y}_k$ for every $k \in \{1, 2, ..., N_y\}, k \neq i_y$. Similarly, when $\gamma^u_{i_u}(\xi) = 0$, only the i_u-th component of $\hat{u}$ is updated according to the local input, i.e., $\hat{u}^+_{i_u} = u_{i_u}$ and $\hat{u}^+_k = \hat{u}_k$ for every $k \in \{1, 2, ..., N_u\}, k \neq i_u$.

Due to the impulsive nature of ETM, we propose to model the closed-loop system as a hybrid system given in (1). We assume that when $\gamma^y_{i_y}(\xi) \geq 0$ (or $\gamma^u_{i_u}(\xi) \geq 0$) the update of each corresponding component y_{i_y} (or y_{i_y}, respectively) is triggered.[1] Defining the state

$$z = (x_p, x_c, \hat{y}, \hat{u}, \chi) \in \mathcal{Z} := \mathcal{X}_p \times \mathcal{X}_c \times \mathcal{Y} \times \mathcal{U} \times X,$$

the closed-loop system resulting from the mechanism described above has a jump set given by

$$D := D_y \cup D_u, \tag{4}$$

where $D_y := \bigcup_{i_y=1}^{N_y} D^y_{i_y}, D_u := \bigcup_{i_u=1}^{N_u} D^u_{i_u}$, $D^y_{i_y} := \{z \in \mathcal{Z} : \gamma^y_{i_y}(\xi) \geq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$ and $D^u_{i_u} := \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) \geq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$. The flow set is given by

$$C := C_y \cap C_u, \tag{5}$$

where $C_y := \bigcap_{i_y=1}^{N_y} C^y_{i_y}, C_u := \bigcap_{i_u=1}^{N_u} C^u_{i_u}, C^y_{i_y} := \{z \in \mathcal{Z} : \gamma^y_{i_y}(\xi) \leq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$ and $C^u_{i_u} := \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) \leq 0, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$. Without much loss of generality, we assume that $C \cup D = \mathcal{Z}$. Observe that when each $\gamma^u_{i_u}$ and $\gamma^u_{i_u}$ is defined for every $\xi \in \Xi$, by (5), (4) and the construction of maps H_p and H_c, $C \cup D = \mathcal{Z}$ holds. Then, for each $z \in C$, the flow map is given by

$$F(z) := (F_p(x_p, \hat{u}), F_c(x_c, \hat{y}), \hat{F}_y(z), \hat{F}_u(z), F_\chi(z)) \tag{6}$$

The jump map captures the dynamics at events. The memory states $\hat{y}, \hat{u}$ are updated via *local reset functions*. More precisely, for every $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, we define

$$g^y_{i_y}(y, \hat{y}) := \begin{cases} (\hat{y}_1, ..., \hat{y}_{i_y-1}, y_{i_y}, \hat{y}_{i_y+1}, ..., \hat{y}_{N_y}) & \text{if } z \in D^y_{i_y} \\ \emptyset & \text{otherwise,} \end{cases}$$

$$g^u_{i_u}(u, \hat{u}) := \begin{cases} (\hat{u}_1, ..., \hat{u}_{i_u-1}, u_{i_u}, \hat{u}_{i_u+1}, ..., \hat{u}_{N_u}) & \text{if } z \in D^u_{i_u} \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence, the union of these reset functions captures the LTE dynamics. At triggering events, the state x remains unaltered, the auxiliary state χ resets according to G_χ, and the components of states $\hat{y}$ and $\hat{u}$ are either kept the same or are updated according to the local reset functions. Then, the jump map is given by

$$G(z) := \left\{ \begin{bmatrix} x \\ \bigcup_{i_y=1}^{N_y} g^y_{i_y}(y, \hat{y}) \\ \bigcup_{i_u=1}^{N_u} g^u_{i_u}(u, \hat{u}) \\ G_\chi(z) \end{bmatrix} : \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} H_p(x_p) \\ H_c(x_c, \hat{y}) \end{bmatrix} \right\}. \tag{7}$$

Therefore, the closed-loop system resulting from event-triggered control is given by (1). Next, we provide conditions guaranteeing $\mathcal{H}$ to satisfy the hybrid basic conditions.

Lemma 3.1: The closed-loop hybrid system $\mathcal{H}$ in (1) *satisfies the hybrid basic conditions in Definition 2.2 if*

(A1') The set C and D given in (5) *and* (4) *are closed;*
(A2') The maps $F_p, F_c, \hat{F}_y, \hat{F}_u$ and F_χ are osc, nonempty, and locally bounded relative to the respective sets of definition, and convex valued;
(A3') The maps H_p, H_c, and G_χ are osc, nonempty, and locally bounded.

Remark 3.2: When H_p and H_c satisfy (A3'), item (A1') in Lemma 3.1 is guaranteed if $\mathcal{Z}$ is closed and for each $i_y \in \{1, 2, ..., N_y\}$ and $i_u \in \{1, 2, ..., N_u\}$, $\gamma^y_{i_y}$ and $\gamma^u_{i_u}$ are continuous.

Next, we show event-triggered controlled systems in the literature that fit in the framework (1) with data (4)-(7).

Example 3.3: (ETM for output-feedback in [2]) An ETM is designed for a continuous-time LTI plant given by[2]

$$\dot{x}_p = A_p x_p + B_p u, \quad y = C_p x_p$$

which is controlled by a dynamic controller given by

$$\dot{x}_c = A_c x_c + B_c y, \quad u = C_c x_c + D_c y,$$

where matrices $A_p, B_p, C_p, A_c, B_c, C_c, D_c$ have appropriate size. The ETM introduced in [2] leads to $N_y = N_u = 1$ and $\gamma^y(\xi) = \gamma^u(\xi) = \min\{|y - \hat{y}|^2 - \sigma_y|y|^2 - \varepsilon_y, |u - \hat{u}|^2 - \sigma_u|u|^2 - \varepsilon_u\}$ with $\xi = (y, u, \hat{y}, \hat{u})$, where $\sigma_y, \sigma_u, \varepsilon_y, \varepsilon_u$ are constants to be designed. With $z = (x_p, x_c, \hat{y}, \hat{u}) \in \mathcal{Z} := \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u}$, the closed-loop system is given by

$$\mathcal{H} \begin{cases} \dot{z} = F(z) := (A_p x_p + B_p \hat{u}, A_c x_c + B_c \hat{y}) & z \in C, \\ z^+ = G(z) := (x_p, x_c, C_p x_p, C_c x_c + D_c \hat{y}) & z \in D, \end{cases}$$

where the flow set C and jump set D are given as in (5) and (4), respectively. Note that the formulation in (1) could be exploited to extend the ETM in [2] to the case of asynchronous events for input and output. △

The following examples illustrate (1) for the state-feedback case with $y = x_p$ and $\hat{y} = \hat{x}_p$.

Example 3.4: (ETM for state-feedback in [14]) A framework is proposed for nonlinear continuous-time plants $\dot{x}_p = F_p(x_p, u)$ controlled by the dynamic state-feedback controller $\dot{x}_c = F_c(x_c, x_p), u = H_c(x_c, x_p)$ that is implemented via ETMs. Such a model in [14] leads to $N_y = N_u = 1$, state $z = (x_p, x_c, \hat{x}_p, \hat{u}, \chi) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_\chi}$, and the closed-loop system modeled as (1) with data defined as follows. For each $z \in C$, flow map is given by $F(z) := (F_p(x_p, e_u + H_c(x_p, x_c)), F_c(x_c, e_{x_p} + x_p), \hat{F}_y(z), \hat{F}_u(z), F_\chi(z))$ and the jump map is given by

[1]Note that two independent sets of event functions are considered in this model to allow $\hat{y}$ and $\hat{u}$ to be updated via asynchronous events. However, this general case can be simplified by setting $\gamma^y_{i_y}(\xi) = \gamma^u_{i_u}(\xi), N_y = N_u$ and $i_u = i_y$ for every $\xi \in \Xi$.

[2]The unknown disturbances w in [2] are ignored. See Section IV-B for robustness analysis.

$G(z) := (x_p, x_c, x_p, H_c(x_p, x_c), G_\chi(z))$, where $e_u = \hat{u} - u$ and $e_{x_p} = \hat{x}_p - x_p$. However, the flow set C and jump set D in [14] are only provided specifically for each of the ETMs in the five given examples, among which, all can be written in form (5) and (4); in particular, see Example 3.5 for the strategy in [14, Section V.C]. △

According to [14, Section V.C], the ETM developed for systems with static state-feedbacks in [1] can be adapted to the framework in [14]. Thus, it also fits (1).

Example 3.5: (ETM for ISS static state-feedback in [1]) For a real-time scheduling problem, [1] develops an ETM for the continuous-time system $\dot{x}_p = F_p(x_p, u)$ controlled by a static state-feedback controller $u = H_c(x_p)$. The controller is assumed to render the closed-loop system $\dot{x}_p = F_p(x_p, H_c(x_p + e))$ Input-to-State Stable (ISS) with respect to e, namely, it is assumed that there exists a smooth function $\widetilde{V} : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ and $\overline{\alpha}, \underline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x_p|) \leq \widetilde{V}(x_p) \leq \overline{\alpha}(|x_p|) \quad (8a)$$
$$\langle \nabla \widetilde{V}(x_p), F_p(x_p, H_c(x_p + e)) \rangle \leq -\alpha(|x_p|) + \gamma(|e|)$$

for each $(x, e) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$. The ETM memorizes x_p from the most recent event; hence, $e = \hat{x}_p - x_p$. Such ETM leads to (1) with $N_y = N_u = 1$ and the event-functions given as $\gamma^y(x_p, \hat{x}_p) = \gamma^u(x_p, \hat{x}_p) = \gamma(|\hat{x}_p - x_p|) - \sigma\alpha(|x_p|)$ where $\sigma \in (0, 1)$. With $z = (x_p, \hat{x}_p) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$, the resulting system is given by

$$\mathcal{H} \begin{cases} \dot{z} = F(z) = (F_p(x_p, H_c(\hat{x}_p)), 0) & z \in C \\ z^+ = G(z) = (x_p, x_p) & z \in D. \end{cases} \quad (9)$$

where D and C are given as in (4) and (5), respectively. △

IV. Properties of General Formulation

In this section, we study the properties of the closed-loop system $\mathcal{H}$ with ETMs modeled as (1).

A. Stability and Convergence Analysis

The results in [8] for certifying asymptotic stability for general hybrid systems can be employed to design the ETMs in the closed-loop system $\mathcal{H}$ in (1). In [8, Chapter 3], uniform pre-asymptotic stability of a set is defined as the property that, in particular, solutions starting close to $\mathcal{A}$ stay close to it, and maximal solutions that are complete converge to it, uniformly in hybrid time over compact sets; see [8, Definition 3.6]. In [8, Chapter 7], an invariance principle to locate the ω-limit set of maximal and complete solutions is given. The following theorem conveniently summarizes these results.

Theorem 4.1: Let $\mathcal{H}$ be the hybrid system with data (C, F, D, G) given by (5)*,* (6)*,* (4)*, and* (7)*, respectively, and let $\mathcal{A}$ be closed. Suppose that there exists a continuous function V that is Lipschitz continuous on an open set containing C such that*

$$\alpha_1(|z|_\mathcal{A}) \leq V(z) \leq \alpha_2(|z|_\mathcal{A}) \quad \forall z \in \mathcal{Z} \quad (10a)$$
$$V^o(z; f) \leq \alpha_{3,c}(z) \quad \forall z \in C, f \in F(z) \quad (10b)$$
$$V(g) - V(z) \leq \alpha_{3,d}(z) \quad \forall z \in D,\ g \in G(z) \quad (10c)$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_{3,c} : C \to \mathbb{R}$, $\alpha_{3,d} : D \to \mathbb{R}$, where $V^o(z; f)$ denotes Clarke's generalized derivative (see [15]). Then, the following hold:

a) If $\alpha_{3,c}(z) = \lambda_c V(z)$ and $\alpha_{3,d}(z) = (\exp(\lambda_d) - 1)V(z)$ with $\lambda_c, \lambda_d \in \mathbb{R}$ and there exists $M, \gamma > 0$ such that, for each solution ϕ to $\mathcal{H}$, $(t, j) \in \operatorname{dom}\phi \implies \lambda_c t + \lambda_d j \leq M - \gamma(t + j)$ then $\mathcal{A}$ is uniformly globally pre-asymptotically stable;

b) If $\mathcal{H}$ satisfies the hybrid basic conditions, $\alpha_{3,c}$ is a negative definite function relative to $\mathcal{A}$ and $\alpha_{3,d}(z) \leq 0$ for each $z \in D$, then the set $\mathcal{A}$ is stable and each precompact solution to $\mathcal{H}$ approaches the largest weakly invariant subset[3] *of $V^{-1}(r) \cap ((\mathcal{A} \cap C) \cup (\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0))))$ for some $r \in V(\mathcal{Z})$;*

c) If $\mathcal{H}$ satisfies the hybrid basic conditions, $\alpha_{3,d}$ is a negative definite function relative to $\mathcal{A}$ and $\alpha_{3,c}(z) \leq 0$ for each $z \in C$, then the set $\mathcal{A}$ is stable and each precompact solution to $\mathcal{H}$ approaches the largest weakly invariant subset of $V^{-1}(r) \cap (\overline{\alpha_{3,c}^{-1}(0)} \cup (\mathcal{A} \cap D))$ for some $r \in V(\mathcal{Z})$;

d) If $\mathcal{H}$ satisfies the hybrid basic conditions, $\alpha_{3,d}(z) \leq 0$ for each $z \in D$ and $\alpha_{3,c}(z) \leq 0$ for each $z \in C$, then the set $\mathcal{A}$ is stable and each precompact solution to $\mathcal{H}$ approaches the largest weakly invariant subset of

$$V^{-1}(r) \cap (\overline{\alpha_{3,c}^{-1}(0)} \cup (\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0))))$$

for some $r \in V(\mathcal{Z})$.

Furthermore, if $\mathcal{H}$ is such that C and F satisfy (A1) and (A2) in Definition 2.2 then the above statements hold with (10b) *replaced by*

$$V^o(z; f) \leq \alpha_{3,c}(z) \quad \forall z \in C, f \in F(z) \cap T_C(z). \quad (11)$$

Remark 4.2: A local version of Theorem 4.1 also holds by restricting the system to the set of interest. Item a) provides good flexibility in the search for a Lyapunov function as, in particular, covers the case where V grows during flows ($\lambda_c > 0$) but decreases at jumps ($\lambda_d < 0$), which seems natural in event-trigger control as the control input is only updated at events, likely leading to a decrease of V, while in between events V may grow continuously. In addition, item a) covers [14, Theorem 1], which, since assumes that $\mathcal{H}$ satisfies the hybrid basic conditions, (11) can be used instead of (10b). Furthermore, Theorem 4.1 pertains to stability and convergence only. The issue of completeness, lower bound on the inter-event times, and robustness are addressed in the next sections.

Next, we revisit Example 3.5 to illustrate the use of Theorem 4.1.

Example 4.3: (Example 3.5 revisited) In this example, Theorem 4.1 is applied to show pre-asymptotic stability of $\mathcal{A} := \{(x_p, \hat{x}_p) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} : x_p = 0\}$ for (9). Consider $V(z) = \widetilde{V}(x_p)$ for every $z = (x_p, \hat{x}_p) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$. Condition (10a) follows from assumption (8a). Then, (10b) holds with $\alpha_{3,c}(z) = -(1 - \sigma)\alpha(|x_p|)$ for each $z \in C$. The inequality in (10c) holds with $\alpha_{3,d}(z) = 0$ for each $z \in D$ because V is constant during jumps. Hence, it follows from Theorem 4.1.b) that the set $\mathcal{A}$ is globally pre-asymptotically stable since $\alpha_{3,d}^{-1}(0) \cap G(\alpha_{3,d}^{-1}(0)) \subset \mathcal{A}$. △

[3] See [8, Definition 6.19].

B. Robustness Analysis

When the closed-loop system given as in (1) satisfies the properties in Lemma 3.1, $\mathcal{H}$ is nominally well-posed [8, Definition 6.2]. Moreover, given a compact set that is (uniformly) pre-asymptotically stable for such $\mathcal{H}$, the stability property is robust to small perturbations. In particular, perturbations on y, u, and the plant dynamics leads to

$$\dot{x} \in \left\{ \zeta \in \begin{bmatrix} F_p(x_p, u + d_u) + d_p \\ F_c(x_c, y + d_y) \end{bmatrix} : \begin{bmatrix} y \\ u \end{bmatrix} \in \begin{bmatrix} H_p(x_p) \\ H_c(x_c, y + d_y) \end{bmatrix} \right\},$$

where d_y, d_u correspond to the noise and d_p captures unmodeled dynamics. Hence, with $\widetilde{d}_1 := (0, 0, d_y, d_u, 0) \in \mathcal{Z}$, $\widetilde{d}_2 := (d_p, 0, 0, 0, 0)$, the closed-loop system $\mathcal{H}$ given as in (1) with such perturbations, which is denoted by $\widetilde{\mathcal{H}}$, has state $z = (x_p, x_c, \hat{y}, \hat{u}, \chi)$ and dynamics[4]

$$\widetilde{\mathcal{H}} \begin{cases} \dot{z} \in F(z + \widetilde{d}_1) + \widetilde{d}_2 & z + \widetilde{d}_1 \in C \\ z^+ \in G(z + \widetilde{d}_1) & z + \widetilde{d}_1 \in D. \end{cases}$$

The following result establishes that stability is robust to small measurement noise and unmodeled dynamics.

Theorem 4.4: Suppose $\mathcal{H}$ satisfies the hybrid basic conditions and there exists a compact set $\mathcal{A} \subset \mathcal{Z}$ that is pre-asymptotically stable for $\mathcal{H}$ with basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$.[5] Then, there exists $\beta \in \mathcal{KL}$ such that, for each $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}^p_{\mathcal{A}}$, there exists $\delta > 0$ such that for any two measurable functions $\widetilde{d}_1, \widetilde{d}_2 : \mathbb{R}_{\geq 0} \mapsto \delta\mathbb{B}$, every solution $\widetilde{\phi} \in \mathcal{S}_{\widetilde{\mathcal{H}}}(K)$ satisfies

$$|\widetilde{\phi}(t, j)|_{\mathcal{A}} \leq \beta(|\widetilde{\phi}(0, 0)|_{\mathcal{A}}, t + j) + \varepsilon \quad \forall (t, j) \in \operatorname{dom} \widetilde{\phi}.$$

C. Completeness of Maximal Solutions

Conditions to guarantee completeness of every maximal solution to (1) are proposed next using [8, Proposition 6.10].

Proposition 4.5: Suppose the hybrid system $\mathcal{H}$ in (1) *with system data given as in* (4)-(7) *satisfies the hybrid basic conditions. Then, there exists a nontrivial solution to $\mathcal{H}$ from every initial point in $C \cup D = \mathcal{Z}$ if*

(VC') For every $z \in \{z \in \mathcal{Z} : \gamma^u_{i_u}(\xi) < 0, \gamma^y_{i_y}(\xi) < 0, i_u \in \{1, 2, ..., N_u\}, i_y \in \{1, 2, ..., N_y\}, y \in H_p(x_p), u \in H_c(x_c, \hat{y})\}$, $F(z) \cap T_C(z) \neq \emptyset$.

Moreover, every $\phi \in \mathcal{S}_{\mathcal{H}}$ is complete if

(b') case (b) in [8, Proposition 6.10] does not hold for every $\phi \in \mathcal{S}_{\mathcal{H}}$;

(c') $G_\chi(D) \subset X$.

Remark 4.6: When $\mathcal{Z} = \mathbb{R}^n$, the set $C \setminus D$ is open. Since for every $z \in \operatorname{int}(C \setminus D)$, $F(z) \subset T_C(z) = \mathbb{R}^n$, condition (VC') holds trivially. In principle, condition (b') is a solution-dependent property, which can be guaranteed when either C is compact or F is bounded on C. All maximal solutions to the closed-loop in (9) in Example 3.5 are complete when $F_p(x_p, H_c(\hat{x}_p))$ is locally Lipschitz.

D. Lower Bound on Inter-Event Times by Design

In this section, we present conditions on the system data to guarantee a uniform positive lower bound on *inter-event time* for all solutions to (1). By guaranteeing such a bound, the jumps do not happen arbitrarily close in time. Moreover, the proposed conditions ensure a lower bound on the time between events for systems with small perturbations, for which we impose the hybrid basic conditions on the system of interest. As the following example shows, when such a lower bound is not guaranteed, a "vanishing" perturbation leads to Zeno solutions.

Example 4.7: (Example 3.5 revised) Consider the ETM presented in [1] applied to a dynamical system with state $x_p \in \mathbb{R}$ given by $\dot{x}_p = F_p(x_p, u) := u$, $u = H_c(x_p) = -x_p$. Then, the closed-loop system is given as in (9) with $z = (x_p, \hat{x}_p)$, $F(z) := [-x_p \ 0]^\top$ and $G(z) := [x_p \ x_p]^\top$. According to [1], we pick triggering event $\gamma^u(x_p, \hat{x}_p) = \gamma^y(x_p, \hat{x}_p) = |\hat{x}_p - x_p| - \sigma|x_p|$ with $\sigma \in (0, 1)$. Suppose u is effected by a disturbance d_u. Then, the resulting perturbed system has flow map defined as $\widetilde{F}(z) := [-x_p + d_u \ 0]^\top$ for every $z \in C := \{z \in \mathbb{R}^2 : |\hat{x}_p - x_p| - \sigma|x_p| \leq 0\}$, the jump map remains the same as in (9), and the jump set is given as $z \in D := \{z \in \mathbb{R}^2 : |\hat{x}_p - x_p| - \sigma|x_p| \geq 0\}$.

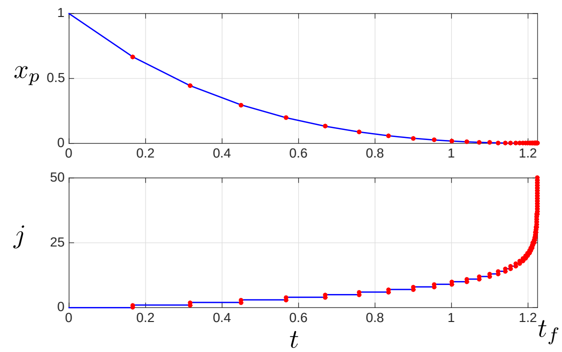

Fig. 2: Simulation of closed-loop system in Example 3.5 with vanishing perturbation d_u on u.

Figure 2 illustrates solution to the system under influence of the vanishing perturbation $d_u = -\hat{x}_p|\hat{x}_p|^{b-1}$ with $\sigma = b = 1/2$ and initial conditions $\hat{x}_p(0, 0) = x_p(0, 0) = 1$. The resulting solution induces Zeno behavior with accumulation point in the time domain given by $t_f \approx 1.22$.[6] △

By construction of (1), jumps are triggered by either "event type y," i.e., $\gamma^y_{i_y}(\xi) = 0$ with $i_y \in \{1, 2, ..., N_y\}$, or "event type u," i.e., $\gamma^u_{i_u}(\xi) = 0$ with $i_u \in \{1, 2, ..., N_u\}$. Since these events are asynchronous, it suffices to guarantee a uniform positive lower bound for each type of event. To this end, for a given solution $\phi \in \mathcal{S}_{\mathcal{H}}$, let E be the set of all points in $\operatorname{dom} \phi$ at which a jump occurs ($J := \sup_j \operatorname{dom} \phi$ can be finite or infinite). Moreover, we denote the collection of points in $\operatorname{dom} \phi$ at which a jump is triggered by "event type y" as E_y, while E_u denotes the collection of points in $\operatorname{dom} \phi$ at which a jump is triggered by "event type u." Note that $E_u \cup E_y = E$. Then, given a solution $\phi \in \mathcal{S}_{\mathcal{H}}$, the minimum inter-event time for "event type y" is given by

$$\Delta t_y = \inf\{t'' - t' : (t', j'), (t'', j'') \in E_y, j' < j''\}. \quad (12)$$

Similarly, the minimum inter-event time for "event type u" is given by

$$\Delta t_u = \inf\{t'' - t' : (t', j'), (t'', j'') \in E_u, j' < j''\}. \quad (13)$$

[4]Perturbations on C and D, in particular, in the ETM, is also allowed.

[5]See [8, Definition 7.3].

[6]Code at github.com/HybridSystemsLab/EventTriggerScalarZeno

Following [16, Lemma 2.7], we provide a necessary and sufficient condition for the existence of a positive uniform lower bound on inter-event time.

Proposition 4.8: (positive lower bound on inter-event times) Suppose $\mathcal{H}$ satisfies the hybrid basic conditions and that every $\phi \in \mathcal{S}_{\mathcal{H}}$ is precompact. Then, for every $\phi \in \mathcal{S}_{\mathcal{H}}$

1) *there exists $\lambda_y > 0$ such that Δt_y given as in* (12) *satisfies $\Delta t_y \geq \lambda_y$ iff $D_y \cap G(D_y) = \emptyset$;*
2) *there exists $\lambda_u > 0$ such that Δt_u given as in* (13) *satisfies $\Delta t_u \geq \lambda_u$ iff $D_u \cap G(D_u) = \emptyset$;*

Note that some assumptions on system data and solutions in Proposition 4.8 are not "necessary" in the sense that without these assumptions,[7] the necessary and sufficient condition for the existence of the lower bound is still valid. We impose these conditions because they guarantee nominal well-posedness, which, as seen in Section IV-B, is crucial in the robustness and stability analysis for hybrid systems.

E. Lower Bound on Inter-Event Time via Temporal Regularization

The conditions in Proposition 4.8 guarantee a lower bound on inter-event times. When those conditions are not enforced at the design stage, the closed-loop system may have Zeno solutions from initial conditions in $\mathcal{A}$ or from nearby it. A way to guarantee such a lower bound is to temporally regularize the closed-loop system by adding a timer to each ETM with dynamics that allow events to occur only after a particular positive amount of time has elapsed after every respective event. To this end, let τ be a timer with positive threshold $T \in [0, T^*)$, where T^* is a fixed positive parameter.[8] The augmented version of the closed-loop system $\mathcal{H} = (C, F, D, G)$ in (1) is denoted $\widetilde{\mathcal{H}}$, has state $\widetilde{z} = (z, \tau) \in \mathcal{Z} \times \mathbb{R}_{\geq 0}$, and dynamics

$$\begin{aligned} \dot{\widetilde{z}} &\in F(z) \times \rho(\tau) & \widetilde{z} &\in (C \times \mathbb{R}_{\geq 0}) \cup (\mathcal{Z} \times [0, T]) \\ \widetilde{z}^+ &\in G(z) \times \{0\} & \widetilde{z} &\in D \times [T, \infty) \end{aligned}$$

where ρ is designed to have τ converge to $[0, T^*]$. A particular choice is $\rho(\tau) = 1$ for each $\tau \in [0, T^*)$, $\rho(\tau) = [0, 1]$ for $\tau = T^*$, and $\rho(\tau) = -\tau + T^*$ for each $\tau > T^*$. Note that when $T = 0$ the z component of $\widetilde{\mathcal{H}}$ matches that of $\mathcal{H}$. We have the following result.

Theorem 4.9: Suppose the set $\mathcal{A}$ is compact and pre-asymptotically stable for the closed-loop system $\mathcal{H}$ in (1) *with basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$. Then, the set $\mathcal{A} \times [0, T^*]$ has the following semiglobal practical (in the parameter T) stability property: there exists a class-$\mathcal{KL}$ function $\widetilde{\beta}$ such that for each compact set $K_z \times K_\tau \subset \mathcal{B}^p_{\mathcal{A}} \times \mathbb{R}_{\geq 0}$ and each $\varepsilon > 0$ there exists $\overline{T} \in (0, T^*)$ such that for each $T \in (0, \overline{T}]$, every solution $\widetilde{\phi}$ to $\widetilde{\mathcal{H}}$ with $\widetilde{\phi}(0,0) \in K_z \times K_\tau$ satisfies*

$$|\widetilde{\phi}(t,j)|_{\mathcal{A}\times[0,T^*]} \leq \widetilde{\beta}(|\widetilde{\phi}(0,0)|_{\mathcal{A}\times[0,T^*]}, t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} \widetilde{\phi}.$$

F. Zeno Stability

Though not recommended due to the reasons illustrated in Example 4.7, if Zeno solutions from $\mathcal{A}$ are acceptable, one might be interested in determining if solutions starting nearby $\mathcal{A}$ are also Zeno. [17, Proposition 4.5] provides a set of conditions for Zeno solutions to exist from nearby $\mathcal{A}$.

V. Conclusion

A general framework is proposed to model the closed-loop system resulting from event-triggered control of a continuous-time system as hybrid systems. Multiple existing event-triggered strategies fit the proposed model. Recent developed tools for hybrid systems are applied to analyze its stability, convergence, and robustness properties. Moreover, conditions are proposed to check completeness of maximal solutions, more importantly, to guarantee a uniform positive lower bound on inter-event times. The Zeno behavior of solutions can be also avoided by constructively designing a temporal regularization of the proposed model.

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[7] The convexity of $F(x)$ required by (A2) in Definition 2.2 is one such assumption.

[8] The threshold could be function of the augmented state.