Complex hybrid systems: stability analysis for omega limit sets

Chaohong Cai[1], Rafal Goebel[2], Ricardo G. Sanfelice[1], Andrew R. Teel[1]

1. Center for Control, Dynamical Systems, and Computation, Department of Electrical and Computer Engineering
University of California, Santa Barbara, CA 93106-9560, USA
E-mail: cai, rsanfelice, teel@ece.ucsb.edu

2. 3518 NE 42 St., Seattle, WA 98105, USA
E-mail: rafal.k.goebel@gmail

Abstract: This paper focuses on the asymptotic stability properties of omega limit sets for complex hybrid dynamical systems, which are commonly found in systems and engineering. It spells out specific stability results that follow when a hybrid dynamical system has certain structure, e.g., when it admits a decomposition resembling a cascade of hybrid dynamical systems.

Key Words: Hybrid systems, Asymptotic stability, Omega limit sets

1 HYBRID SYSTEMS

Hybrid systems are dynamical systems the state of which can both change continuously (flow) and change discontinuously (jump). The state may contain logic variables, counters, timers, and physical variables, among other things. Hybrid dynamical systems subsume many useful and important systems such as hybrid automata, switched control systems, rigid mechanical systems with impacts, reset control systems, sampled-data control systems, networked control systems, networked of biological oscillators, etc., (see [16], [10], [6], [5, Section 2.2], [8], [12], [11]).

Usually, a hybrid system can be modeled using four objects: a differential inclusion $\dot{x} \in F(x)$ describing the flows, a set C constraining the flow, a difference inclusion $x^+ \in G(x)$ describing the jumps, and a set D stating from where the jumps can happen. Thus, four objects comprise the data of a hybrid system: the flow map F, the flow set C, the jump map G, and the jump set D. Symbolically, the hybrid system can be written as

$$\mathcal{H}\begin{cases} \dot{x} & \in \quad F(x) \qquad x \in C \\ x^+ & \in \quad G(x) \qquad x \in D\,. \end{cases} \tag{1}$$

The state x, as already noted, may include logic variables, etc. Solutions to $\mathcal{H}$ require a generalized concept of a time domain. A *compact hybrid time domain* is a set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of the form

$$S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of nonnegative numbers $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. The set S is a *hybrid time domain* if for all $(T, J) \in S$,

$$S \cap ([0, T] \times \{0, 1, \ldots J\})$$

is a compact hybrid time domain. A *hybrid arc* is a function $\phi : S \to \mathbb{R}^n$ such that S is a hybrid time domain and $t \mapsto \phi(t, j)$ is locally absolutely continuous for fixed j and $(t, j) \in \text{dom}\ \phi$. The hybrid time domain associated with a hybrid arc ϕ will be denoted by dom ϕ.

A hybrid arc ϕ : dom $\phi \to \mathbb{R}^n$ is a *solution to* $\mathcal{H}$ if $\phi(0, 0) \in C \cup D$ and:

(S1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in$ dom ϕ,

$$\phi(t, j) \in C, \quad \dot{\phi}(t, j) \in F(\phi(t, j));$$

(S2) for all $(t, j) \in$ dom ϕ such that $(t, j + 1) \in$ dom ϕ,

$$\phi(t, j) \in D, \quad \phi(t, j + 1) \in G(\phi(t, j)).$$

A solution is said to be *maximal* if it is not a truncation of another solution ϕ' to some proper subset of dom ϕ'. The notation $\mathcal{S}_{\mathcal{H}}(\mathcal{X})$ indicates the set of maximal solutions to $\mathcal{H}$ from the set of initial conditions $\mathcal{X}$. Note that if (and only if) $x^0 \notin C \cup D$ then $\mathcal{S}_{\mathcal{H}}(x^0) = \emptyset$. The set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *forward pre-invariant* if each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X})$ satisfies $\phi(t, j) \in \mathcal{X}$ for all $(t, j) \in$ dom ϕ.

Standing Assumption 1 (Hybrid Basic Conditions) *The sets C and D are closed; the mappings F and G are outer semicontinuous and locally bounded* [1] *; $F(x)$ is nonempty and convex for all $x \in C$; $G(x)$ is nonempty for all $x \in D$.*

These regularity conditions on the data of $\mathcal{H}$ are needed to guarantee that the set of solutions is sequentially compact and upper semicontinuous with respect to initial conditions

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1. A set-valued mapping G defined is *outer semicontinuous* if for each sequence x_i converging to a point x and each sequence $y_i \in G(x_i)$ converging to a point y, it holds that $y \in G(x)$; equivalently, if the graph of G is closed. It is *locally bounded* if, for each compact set K there exists $\mu > 0$ such that $G(K) := \cup_{x \in K} G(x) \subset \mu\mathbb{B}$, where $\mathbb{B}$ is the open unit ball in $\mathbb{R}^n$. For more details, see [14, Chapter 5].

and system perturbations; see the exposition and motivation in [4] and the results in [5]. Such properties of the set of solutions are important for establishing (LaSalle-like) invariance principles (see [15]), converse Lyapunov theorems for (pre-)asymptotic stability (see [3] and [2]), and robustness of hybrid feedbacks for general asymptotically controllable nonlinear systems (see [13]), etc.

2 STABLE OMEGA LIMIT SETS

2.1 Overview

In the recent paper [1], we have developed results on omega limit sets for hybrid dynamical systems satisfying Standing Assumption 1. These results parallel those for continuous-time dynamical systems, as summarized in [7] for example. In this paper, we focus on characterizing *asymptotically stable* omega limit sets. Moreover, we emphasize results where a decomposition of the state into a pseudo-cascade structure can be used to facilitate the analysis. Henceforth, we use "Ω-limit set" in place of "omega-limit set".

2.2 Ω-limit sets

Consider the hybrid system $\mathcal{H}$. For a given set $\mathcal{X} \subset \mathbb{R}^n$, the Ω-limit set of $\mathcal{X}$ for $\mathcal{H}$ is defined as:

$$\Omega_{\mathcal{H}}(\mathcal{X}) := \big\{ y \in \mathbb{R}^n : y = \lim_{i \to \infty} \phi_i(t_i, j_i),\ \phi_i \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}), \\ (t_i, j_i) \in \text{dom}\,\phi_i\ ,\ t_i + j_i \to \infty \big\}.$$

We also define, for each $i \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) := \big\{ y \in \mathbb{R}^n : y = \phi(t,j),\ \phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}) \\ (t,j) \in \text{dom}\,\phi\ ,\ t + j \geq i \big\}.$$

It was noted in [1], as a simple consequence of the results in [14], that

$$\Omega_{\mathcal{H}}(\mathcal{X}) = \lim_{i \to \infty} \mathcal{R}^i_{\mathcal{H}}(\mathcal{X}) = \bigcap_i \overline{\mathcal{R}^i_{\mathcal{H}}(\mathcal{X})}\ . \tag{2}$$

It is possible for this limit to be empty.

2.3 Pre-asymptotic stability

Pre-asymptotic stability (pre-AS) is a generalization of standard notion of asymptotic stability to the setting where completeness or even existence of solutions is not required. Pre-AS was introduced in [2] as an equivalent characterization of the existence of a smooth Lyapunov function for a hybrid system.

Consider the hybrid system $\mathcal{H}$. Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. We say that

- $\mathcal{A}$ is *pre-stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \text{dom}\,\phi$;
- $\mathcal{A}$ is *pre-attractive* for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is bounded and if it has unbounded domain then $\phi(t,j) \to \mathcal{A}$ as $t + j \to \infty$;
- $\mathcal{A}$ is *uniformly pre-attractive* if there exists $\delta > 0$ and for each $\varepsilon > 0$ there exists $T > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is bounded and $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \text{dom}\,\phi$ satisfying $t + j \geq T$;
- $\mathcal{A}$ is *pre-asymptotically stable* if it is both pre-stable and pre-attractive;
- $\mathcal{A}$ is *uniformly pre-asymptotically stable* if it is both pre-stable and uniformly pre-attractive.

The subset of $C \cup D$ from which all solutions are bounded and the ones with unbounded domain converge to $\mathcal{A}$ is called the *basin of pre-attraction* of $\mathcal{A}$.

Some interesting facts about pre-asymptotic stability, shown in [2], are that

1. it implies uniform pre-asymptotic stability and thus a $\mathcal{KL}$-estimate on the size of solutions,
2. it implies the existence of a smooth Lyapunov function that establishes pre-asymptotic stability,
3. it is robust to sufficiently small perturbations.

3 MAIN RESULTS

Most of the first result given here was reported in [1].

Theorem 1 *Let $\mathcal{X}$ be compact. Suppose that, for the hybrid system $\mathcal{H}$, the set $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is bounded and $\Omega_{\mathcal{H}}(\mathcal{X}) \subset$* int$(\mathcal{X})$. *Then $\Omega_{\mathcal{H}}(\mathcal{X})$ is compact and pre-asymptotically stable with basin of pre-attraction containing $\mathcal{X} \cap (C \cup D)$. Moreover, it is the smallest such set contained in* int$(\mathcal{X})$.

The next result is a corollary of the results in [1], but was not made explicit there.

First we need the following bit of notation: given a hybrid system $\mathcal{H}$ and a closed set $\mathcal{Y}$, we define the hybrid system $\mathcal{H}_{|\mathcal{Y}}$ as

$$\mathcal{H}_{|\mathcal{Y}} \left\{ \begin{array}{cccl} \dot{x} & \in & F(x) & x \in C \cap \mathcal{Y} \\ x^+ & \in & G(x) & x \in D \cap \mathcal{Y}\ . \end{array} \right. \tag{3}$$

Theorem 2 *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact. Suppose, for $\mathcal{H} = (F, G, C, D)$ that*

$$G(D \cap \mathcal{Y}) \subset \mathcal{Y}\ ,$$

$\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is bounded, $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y}$ and $\mathcal{A} := \Omega_{\mathcal{H}_{|\mathcal{Y}}}(\mathcal{Y}) \subset$ int$(\mathcal{X})$. *Then, for $\mathcal{H}$, the set $\mathcal{A}$ is compact and pre-asymptotically stable with basin of pre-attraction containing $\mathcal{X} \cap (C \cup D)$.*

Proof: First, the assumption that $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is bounded verifies [1, Assumption 1]. Using the compactness of $\mathcal{Y}$ and the assumption $G(D \cap \mathcal{Y}) \subset \mathcal{Y}$, we infer that $\mathcal{A} = \Omega_{\mathcal{H}_{|\mathcal{Y}}}(\mathcal{Y}) \subset \mathcal{R}^0_{\mathcal{H}_{|\mathcal{Y}}}(\mathcal{Y}) \cup \mathcal{Y}$. Then [1, Theorem 2] says that $\mathcal{A}$ is compact and forward pre-invariant for $\mathcal{H}$.

Second, combining the assumption $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y}$ and [1, Corollary 1] gives $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{A}$. This, as well as the assumption $\mathcal{A} \subset$ int$(\mathcal{X})$, implies $\Omega_{\mathcal{H}}(\mathcal{X}) \subset$ int$(\mathcal{X})$. Then [1, Theorem 5] says that $\Omega_{\mathcal{H}}(\mathcal{X})$ is a compact pre-asymptotically stable set with basin of pre-attraction containing $\mathcal{X} \cap (C \cup D)$. Since $\mathcal{A}$ contains $\Omega_{\mathcal{H}}(\mathcal{X})$, we conclude that $\mathcal{A}$ is uniformly attractive with basin of pre-attraction containing $\mathcal{X} \cap (C \cup D)$.

In summary, the set $\mathcal{A}$ is compact and forward pre-invariant for $\mathcal{H}$ and $\mathcal{A}$ is also uniformly attractive with basin of pre-attraction containing $\mathcal{X} \cap (C \cup D)$ for $\mathcal{H}$. By [2, Proposition 3.2], we conclude that for $\mathcal{H}$, the set $\mathcal{A}$ is compact and pre-asymptotically stable with basin of pre-attraction containing $\mathcal{X} \cap (C \cup D)$. ■

The examples in the next section will illustrate Theorem 2 in settings that are very similar to cascade interconnections of differential equations.

4 EXAMPLES

4.1 State reset

The nonlinear control system

$$F(x) := \begin{bmatrix} -x_1 + x_1^2 x_2 \\ u \end{bmatrix},$$

where $x = (x_1, x_2) \in \mathcal{X} = \mathbb{R}^2$, cannot be globally stabilized with the control law $u = -kx_2$ with $k > 0$. Only semiglobal stabilization is possible (see [9, Example 13.16]). The control law $u = -kx_2 - x_1^3$ with $k > 0$ (via backstepping design) can achieve global stabilization (see [9, Example 14.11]).

Theorem 2 can be used to show how the state reset (discrete dynamics) can help achieve global stabilization with the control law is $u = -kx_2$ with $k > 0$. Consider a hybrid system (1) with the data

$$\begin{aligned} F(x) &:= \begin{bmatrix} -x_1 + x_1^2 x_2 \\ -kx_2 \end{bmatrix}, \\ G(x) &:= \begin{bmatrix} x_1 \\ \lambda x_2 \end{bmatrix}, \\ C &:= \{x : |x_1 x_2| \le 0.5\}, \\ D &:= \{x : |x_1 x_2| \ge 0.5\}, \end{aligned}$$

where $\lambda \in [0, 1)$. It is not difficult to check that, with $V(x) = x^T x$, we have $\langle \nabla V(x), F(x) \rangle \le 0$ for all $x \in C$ and $V(G(x)) \le V(x)$ for all $x \in D$. It follows that, for each compact $\mathcal{X} \subset \mathbb{R}^2$, $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is bounded. (Moreover, since $C \cup D = \mathbb{R}^2$, each solution is complete.) Let $\mathcal{X}$ be an arbitrary compact set containing the origin. Define

$$\mathcal{Y} := \overline{\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})} \cap \{x : x_2 = 0\} .$$

Since $\dot{x}_2 = -kx_2$ and $x_2^+ = \lambda x_2$, we infer $\Omega_{\mathcal{H}}(\mathcal{X}) \subset \mathcal{Y}$. Since $\dot{x}_1 = -x_1 + x_1^2 \cdot 0$ for $x \in \mathcal{Y} \cap C$, we verify that $\Omega_{\mathcal{H}_{|\mathcal{Y}}}(\mathcal{Y})$ equals the origin. Theorem 2 says that the origin is pre-asymptotically stable (in fact, asymptotically stable) with basin of pre-attraction (in fact, basin of attraction) containing $\mathcal{X}$. This is true for any $\mathcal{X}$, and so the origin is globally asymptotically stable.

4.2 A cart and a spring

Consider a hybrid system (1) with the system data defined as follows

$$\begin{aligned} F(x) &:= \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix}, \\ G(x) &:= \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \end{bmatrix}, \\ C &:= \{x : x_1 \ge 0\}, \\ D &:= \{x : x_1 = 0, x_2 \le 0\}, \end{aligned}$$

where u is the control input and $\lambda \in [0, 1)$ is a parameter. Note that $G(x) = \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix}$ for all $x \in D$. This model describes the control of a cart that is located to the right of the equilibrium and where every time the cart tries to move left past the equilibrium, it collides with a rigid obstacle at the equilibrium resulting in the sign of the cart's velocity being toggled and the absolute value of the cart's velocity decreasing. The control goal is to steer the cart to the equilibrium via the information of the cart's position. In particular, we associate the output $y = x_1$ to the above hybrid system, and we are interested in stabilization by output feedback. We observe that the origin is pre-asymptotically stable when $u = 0$, but the convergence could be extremely slow when $\lambda \nearrow 1$. We verify that the state feedback control law $u = -x_2$ can "quickly" stabilize the hybrid system to the origin, but the information of x_2 is not contained in the output y. So we have to design an observer, and then we use Theorem 2 to analyze the closed-loop system behavior. In order to design an observer, we use the fact that the value of the output x_1 can be used to detect jumps. In particular, we assume we can take the following observer:

$$\mathcal{H}_o \begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 + y - z_1 \\ -z_1 - z_2 \end{bmatrix} & x \in C \\ \begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} = \begin{bmatrix} -\lambda z_1 \\ -\lambda z_2 \end{bmatrix} & x \in D \end{cases}$$

and we make the control choice $u = -z_2$. The hybrid system $\mathcal{H}$, with this u, combines easily with the observer $\mathcal{H}_o$ to form a new hybrid system $\widehat{\mathcal{H}}$ with state $\xi = (x, z)$, flow map $\widehat{F}$, flow set $\widehat{C} := \{(x, z) : x \in C\}$, jump map $\widehat{G}$, and jump set $\widehat{D} := \{(x, z) : x \in D\}$.

Taking

$$V(\xi) := (x_1 - z_1)^2 + (x_2 - z_2)^2 - (x_1 - z_1)(x_2 - z_2),$$

it can be verified that

$$\begin{aligned} &\langle \nabla V(\xi), \widehat{F}(\xi) \rangle = -V(\xi) && \forall \xi \in \widehat{C}, \\ &V(\widehat{G}(\xi)) = \lambda^2 V(\xi) && \forall \xi \in \widehat{D}. \end{aligned} \tag{4}$$

It also can be established that the x component of ξ remains uniformly bounded if $x - z$ is uniformly bounded. In turn, the properties of V given in (4) establish that, for each compact set $\mathcal{X}$, the set $\mathcal{R}^0_{\widehat{\mathcal{H}}}(\mathcal{X})$ is bounded.

Let $\mathcal{X}$ be an arbitrary compact set containing the origin. Define

$$\mathcal{Y} = \overline{\mathcal{R}^0_{\widehat{\mathcal{H}}}(\mathcal{X})} \cap \{\xi : x = z\} .$$

It follows from the properties of V in (4) that $\Omega_{\widehat{\mathcal{H}}}(\mathcal{X}) \subset \mathcal{Y}$. Since the state feedback control law $u = -x_2$ stabilizes the original hybrid system $\mathcal{H}$ to the origin, it follows that $\Omega_{\widehat{\mathcal{H}}_{|\mathcal{Y}}}(\mathcal{Y})$ is equal to the origin. Using that $\mathcal{X}$ was arbitrary and using Theorem 2 we conclude that the origin of $\widehat{\mathcal{H}}$ is asymptotically stable with the basin of pre-attraction $\widehat{C} \cup \widehat{D}$.

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