Notions and a Passivity Tool for Switched DAE Systems

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Abstract— **This paper proposes notions and a tool for passivity properties of non-homogeneous switched Differential Algebraic Equation (DAE) systems and their relationships with stability and control design. Motivated by the lack of results on input-output analysis (such as passivity) for switched DAE systems and their interconnections, we propose to model non-homogeneous switched DAE systems as a class of hybrid systems, modeled here as hybrid DAE systems with linear flows. Passivity and its variations are defined for switched DAE systems and methods relying on storage functions are proposed. The main contributions of this paper are: 1) passivity and detectability concepts for switched DAE systems, 2) links of the aforementioned passivity and detectability properties to stabilization via static output-feedback. Our results are illustrated in a power system, namely, the DC-DC boost converter, whose model involves DAEs and requires feedback control.**

I. Introduction

The characterization of a system behavior based on the relationship between the energy injected and dissipated by a system is known as passivity. A system that stores and dissipates energy without generating energy on its own is said to be passive. The physical interpretation of energy makes passivity an intuitive tool to assert the stability properties of any system with inputs and outputs. There is plenty of literature that documents dissipativity and passivity, from definitions, sufficient conditions for stability, to passivity based control [1]. Passivity using storage functions, which for the case of zero inputs guarantee asymptotic stability. Passivity properties are also very useful when analyzing interconnection of systems.

This paper pertains to the study and design of hybrid and switched Differential Algebraic Equations (DAEs) using passivity tools. In particular, a switched DAE is given as

$$E_\sigma \dot{\xi} = A_\sigma \xi + B_\sigma u \tag{1a}$$

$$y = h_\sigma(\xi, u), \tag{1b}$$

where $\sigma : [0, \infty) \to \Sigma$ is the switching signal and Σ is a finite discrete set. The results in [2] allow us to model homogeneous and autonomous switched DAE systems as hybrid DAE systems. We extend the results in [2] and [3] to allow the analysis of the passivity properties of hybrid DAE systems and switched DAE systems with inputs and outputs[1]. More precisely, we characterize non homogeneous linear switched DAE systems as a class of hybrid DAE systems.

As a motivation for the study of switched DAE systems with inputs and outputs, we employ the DC-DC boost converter [5]. First, we model each mode of operation using the switched DAE representation in (1). We study its passivity properties and solve the problem of set-point tracking of the output voltage, in this case the voltage at the capacitor, using a passivity-based controller. The main contribution of this paper is a tool that allows one to link the passivity properties of a switched DAE system to the asymptotic stability of the system using a static state-feedback controller. Building from the invariance principles for hybrid DAE systems in [2] and the passivity and detectability notions for hybrid systems in [3] and given a static output-feedback control law that satisfies some mild conditions and a (flow- or jump-) passive switched DAE with respect to a set of interest, we show that the control law renders such set asymptotically stable. Due to space constraints, strict and output versions of the passivity properties described in this document and its relationship to zero-input stability of switched and hybrid DAE systems, as well as proofs are not included in this document.

The notation used throughout the paper is as follows. We define $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. Given vectors $\nu \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$, $[\nu^\top \ \omega^\top]^\top$ is equivalent to (ν, ω), where $(\cdot)^\top$ denotes the transpose operation. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m \mid f(x) \text{ is defined}\}$. The range of f is denoted by $\operatorname{rge} f$, i.e., $\operatorname{rge} f := \{f(x) \mid x \in \operatorname{dom} f\}$. The right limit of the function f is defined as $f^+(x) := \lim_{\nu \to 0^+} f(x + \nu)$ if it exists. The notation $f^{-1}(r)$ stands for the $r-$level set of f on $\operatorname{dom} f$, i.e., $f^{-1}(r) := \{z \in \operatorname{dom} f \mid f(z) = r\}$. Given two functions $f : \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^m \to \mathbb{R}^n$, $\langle f(x), h(x) \rangle$ denotes the inner product between f and h at x. We denote the distance from a vector $y \in \mathbb{R}^n$ to a closed set $\mathcal{A} \subset \mathbb{R}^n$ by $|y|_\mathcal{A}$, which is given by $|y|_\mathcal{A} := \inf_{x \in \mathcal{A}} |x - y|$. Given a matrix $P \in \mathbb{R}^{n \times n}$, the determinant of P is denoted by $\det P$. Given $n \in \mathbb{N}$, the matrix $0_n \in \mathbb{R}^{n \times n}$ denotes the zero $n \times n$ matrix, while $I_n \in \mathbb{R}^{n \times n}$ denotes the $n \times n$ identity matrix.

The remainder of this paper is organized as follows. In Section III, the required modeling background is presented.

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[1]It is important to clarify that, for a solution to (1a) to exist, at each change in σ it is required to map the state previous to the switching instant to a point in the space defined by the algebraic conditions of the subsequent mode. These resets of the state can be computed by the so-called consistency projectors in Definition 3.3 [4, Definition 3.7].

In Section IV, a description of hybrid DAE systems with inputs/outputs is presented, which is followed in Section V by the introduction of the passivity and stability definitions for such systems. Also Section V-B revisits the motivational Example II, where the definitions and the results in Sections V through V-B are exercised.

II. Motivational example

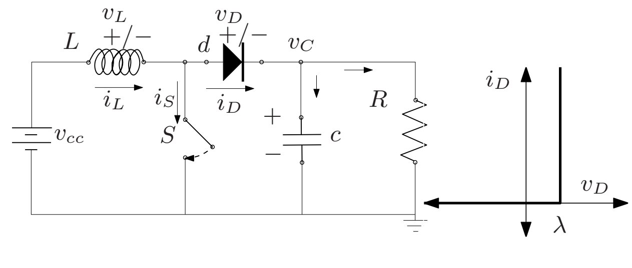
We consider the DC-DC boost converter shown in Figure 1(a) and model it as a switched DAE as in (1). More interestingly, using concepts of passivity, a given set-point for the voltage, and an appropriate selection of inputs and outputs, we will show that a passivity-based control law renders a set of interest asymptotically stable.

There is a fair amount of literature related to the control of DC-DC boost converters from many perspectives. The authors in [6] follow an energy-based hybrid control approach to design controllers for impulsive dynamical systems. In [5], the authors propose a *Control Lyapunov Function* (CLF) approach for the control of the DC-DC boost converter. Following the models therein, the converter is composed by an inductor L, a capacitor c, a resistor R, a voltage source v_{cc}, a switch S, and a diode d. The voltage across the inductor, diode, and capacitor are denoted as v_L, v_D, and v_C, respectively. The current through the inductor, switch, and diode are denoted as i_L, i_S, and i_D, respectively. In this example, we consider the model of the diode depicted in Figure 1(b), where λ is the forward bias voltage of the diode. Now, consider the switching signal $\sigma : [0,\infty) \to \Sigma$, where each element in Σ represents a mode of operation of the boost converter.

The state conditions where each one of the modes is valid are as follows:

- Mode 1: (Switch is open and diode is conducting) In this mode the current through the diode is positive. This requires the voltage v_D to be larger than the threshold voltage $\lambda > 0$.
- Mode 2: (Switch is closed and diode is blocking) In this mode the voltage in the capacitor is larger or equal than the forward bias voltage, i.e., $v_C \geq -\lambda$. In this case, the voltage across the diode is always smaller than λ. Hence the diode is modeled as an open circuit.
- Mode 3: (Switch is open and diode is blocking) When the switch is open, the voltage in the diode could become smaller than λ, in which case the current through the diode is zero.
- Mode 4: (Switch is closed and diode is conducting) In this mode the voltage in the capacitor is smaller than the forward bias voltage of the diode, i.e., $v_C \leq -\lambda$.

More precisely, given the nature of the switch and the diode, the four different modes can be described by



(a) Boost converter circuit. (b) $v_D - i_D$ curve.

Fig. 1. DC-DC boost converter circuit and current-voltage characteristic curve of the diode.

differential-algebraic equations as follows[2]:

Mode 1 ($\sigma = 1$)	Mode 2 ($\sigma = 2$)
$\frac{d}{dt}v_{cc} = 0$ $c\frac{d}{dt}v_C = i_D - \frac{1}{R}v_C$ $L\frac{d}{dt}i_L = v_L$ $0 = i_L - i_S - i_D$ $0 = i_S$ $0 = v_{cc} - v_L - v_D - v_C$ $\frac{d}{dt}v_D = 0$	$\frac{d}{dt}v_{cc} = 0$ $c\frac{d}{dt}v_C = i_D - \frac{1}{R}v_C$ $L\frac{d}{dt}i_L = v_L$ $0 = i_L - i_S - i_D$ $0 = i_D$ $0 = v_{cc} - v_L$ $0 = v_D + v_C$
Mode 3 ($\sigma = 3$)	**Mode 4 ($\sigma = 4$)**
$\frac{d}{dt}v_{cc} = 0$ $c\frac{d}{dt}v_C = i_D - \frac{1}{R}v_C$ $L\frac{d}{dt}i_L = v_L$ $0 = i_L - i_S - i_D$ $0 = i_S$ $0 = v_{cc} - v_L - v_D - v_C$ $0 = i_L$	$\frac{d}{dt}v_{cc} = 0$ $c\frac{d}{dt}v_C = i_D - \frac{1}{R}v_C$ $L\frac{d}{dt}i_L = v_L$ $0 = i_L - i_S - i_D$ $0 = v_D + v_C$ $0 = v_{cc} - v_L$ $\frac{d}{dt}v_D = 0$

The DC-DC boost converter is designed (and controlled) to deliver a desired DC voltage at its output, which is typically the voltage of the capacitor (v_C). The set-point for the voltage is denoted as $u \in \mathbb{R}$ and is treated as a new input. To proceed with a CLF approach, we add an extra state e that is reset to the value of u at switching instants. The dynamics of the state e are given by $\dot{e} = 0$ during flows and by $e^+ = u + \tilde{v}_C$ at switches, where $\tilde{v}_C \in \mathbb{R}_{\geq 0}$ is a constant such that $\tilde{v}_C \geq v_{cc} + \lambda$. Considering the state of the system given by $\xi = [v_{cc}, v_L, v_D, v_C, i_L, i_S, i_D, e]^\top$, the data of the switched DAE system in (1a) is given by $\Sigma = \{1, 2, 3, 4\}$,

$$E_\sigma = \begin{bmatrix} \hat{E}_\sigma & 0 \\ 0 & 1 \end{bmatrix}, \quad A_\sigma = \begin{bmatrix} \hat{A}_\sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad B_\sigma = \begin{bmatrix} \hat{B}_\sigma & 0 \end{bmatrix}^\top \tag{2}$$

where $\hat{E}_\sigma$, $\hat{A}_\sigma$, and $\hat{B}_\sigma$ are given by the differential-algebraic equations describing each mode. We will restrict the state space of this system to the set $\{\xi \in \mathbb{R}^8 \mid v_C \geq 0, i_L \geq 0, v_{cc}^{\min} \leq v_{cc} \leq v_{cc}^{\max}\}$, where $v_{cc}^{\min} > 0$.

To show that a passivity property holds for this system, consider the functions $V_e : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ and $\widetilde{V} : \mathbb{R}^8 \to \mathbb{R}_{\geq 0}$

[2]Notice that the voltage source v_{cc} does not change, thus it can be either modeled as a state or a constant. We choose to model it as a state to maintain the switched DAE model structure.

defined as

$$V_e(e, v_{cc}) = \frac{1}{2}(e - \tilde{v}_C)^2 + \frac{1}{2}\left(\frac{e^2}{v_{cc}R} - \frac{\tilde{v}_C^2}{v_{cc}R}\right)^2 \quad (3)$$

$$\widetilde{V}(\xi) = \frac{1}{2}(v_C - e)^2 + \frac{1}{2}\left(i_L - \frac{e^2}{v_{cc}R}\right)^2 + V_e(e, v_{cc}) \quad (4)$$

Notice that for each (ξ, σ^*) such that the algebraic constraints on ξ for the mode of operation σ^* are met, we can always pick σ such that $\langle \nabla\widetilde{V}(\xi), \tilde{f}(\xi, u)\rangle =: \gamma_\sigma(\xi) \leq 0$, where the equivalent vector field $\tilde{f}(\xi, u) := \Pi_\sigma^{\text{diff}} A_\sigma \xi + \Pi_\sigma^{\text{diff}} B_\sigma u$ is given as in [7, Theorem 6.4.4], Π_σ^{diff} is given in Definition 3.3. More precisely, we have[3]

- if $\sigma = 1$, $\langle \nabla\widetilde{V}(\xi), \tilde{f}(\xi, u)\rangle = (v_C - e)\left(\frac{i_L}{c} - \frac{v_C}{cR}\right) + \left(i_L - \frac{e^2}{v_{cc}R}\right)\left(\frac{v_{cc}-\lambda}{L} - \frac{v_C}{L}\right) =: \gamma_1(\xi)$.
- if $\sigma = 2$, $\langle \nabla\widetilde{V}(\xi), \tilde{f}(\xi, u)\rangle = (v_C - e)\left(-\frac{v_C}{cR}\right) + \left(i_L - \frac{e^2}{v_{cc}R}\right)\left(\frac{v_{cc}}{L}\right) =: \gamma_2(\xi)$.
- if $\sigma = 3$, $\langle \nabla\widetilde{V}(\xi), \tilde{f}(\xi, u)\rangle = (v_C - e)\left(-\frac{v_C}{cR}\right) =: \gamma_3(\xi)$.
- if $\sigma = 4$, $\langle \nabla\widetilde{V}(\xi), \tilde{f}(\xi, u)\rangle = \left(i_L - \frac{e^2}{v_{cc}R}\right)\left(\frac{v_{cc}}{L}\right) =: \gamma_4(\xi)$.

In addition, notice that by picking the output of the system as $y = \sqrt{-\gamma_\sigma(\xi)}$ and assigning the input $u = -ky$, where $0 < k \leq 1$, we have that $\langle \nabla\widetilde{V}(\xi), \tilde{f}(\xi, u)\rangle \leq uy$, which describes a passivity property of the system during the continuous regime. At switching instants, by making use of the consistency projectors in Definition 3.3, as we show in Example 5.4, we have that $v_C^+ = v_C$ and $i_L^+ = i_L$ for each $\sigma, \sigma^+ \in \Sigma$. By choosing $u = -ky$ and $y = \tilde{v}_C - e$ it can be shown that the following holds: $\widetilde{V}([v_{cc}^+, v_L^+, v_D^+, v_C^+, i_L^+, i_S^+, i_D^+, -ky + \tilde{v}_C]) \leq \widetilde{V}(\xi)$, which gives $\widetilde{V}(\xi^+) - \widetilde{V}(\xi) \leq 0$ and corresponds to a passive property at jumps. This motivates the development of passivity-based control techniques for such kind of switching DAE systems.

The question that remains open is whether the passivity-like property shown above can be exploited to design a static output-feedback control law that renders asymptotically stable a given set of interest. In the upcoming sections, we propose a static output-feedback control law for the switched DAE system in the previous example. Using concepts of passivity, a given set-point for the voltage $\tilde{v}_C$, and an appropriate selection of inputs and outputs, we will show that such control law renders a set of interest asymptotically stable. To this end, we develop a general set of notions and results for linking the passivity properties presented here with the asymptotic stability of a set of interest.

III. PRELIMINARIES

In this paper, we consider the class of switched DAE systems with linear flows given by (1), where $\xi \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, $y \in \mathbb{R}^q$ is the output, for each $\sigma \in \Sigma$, $h_\sigma : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$ is the output map, $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is the switching signal, Σ is a finite discrete set, $E_\sigma, A_\sigma \in \mathbb{R}^{n\times n}$, and $B_\sigma \in \mathbb{R}^{n\times p}$. Solutions to (1a) are typically given by (right or left) continuous functions (see [8] and references therein). Definition 3.5 below introduces the notion of solution to (1a) employed here.

Definition 3.1: (DAE regularity [9, Definition 1-2.1]) *The collection (E_σ, A_σ) is regular if for each $\sigma \in \Sigma$ the matrix pencil $sE_\sigma - A_\sigma \in \mathbb{R}^{n\times n}$ $(s \in \mathbb{C})$ is regular. The matrix pencil $sE_\sigma - A_\sigma$ is called regular if there exists a constant $s \in \mathbb{C}$ such that $det(sE_\sigma - A_\sigma) \neq 0$, or $det(sE_\sigma - A_\sigma)$ is not the zero polynomial.*

To define a switched DAE system as in [4], we recall first some concepts regarding the linear subspaces where solutions to (1a) belong. Due to the algebraic constraints in (1a), the solutions to (1a) evolve within a linear subspace (or manifold) called the *consistency* space.

Definition 3.2: (Consistency set) *Given $\sigma \in \Sigma$, the consistency set for* (1a) *is given by*

$$\mathfrak{O}_\sigma := \{\xi_0 \in \mathbb{R}^n \mid \exists \text{ Lebesgue measurable functions } \xi : [0, \tau) \to \mathbb{R}^n, u : [0, \tau) \to \mathbb{R}^p \text{ s.t. } E_\sigma\dot{\xi}(t) = A_\sigma\xi(t) + B_\sigma u(t)\, \forall t \in (0, \tau),\ \xi(0) = \xi_0,\ \tau > 0\}$$

For a linear switched DAE system as in (1a), for each $\sigma \in \Sigma$, the consistency set is given by a linear subspace (see more in [4]).

Next, the consistency, differential, and impulse projectors are defined.

Definition 3.3: (Consistency, differential, and impulse projectors [7, Definition 6.4.1]) *For the quasi-Weierstrass transformation in [10, Theorem 3.4] define*[4]

- *Consistency projector:* $\Pi_\sigma := T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & 0_{n_2^\sigma} \end{bmatrix} T_\sigma^{-1}$
- *Differential projector:* $\Pi_\sigma^{\text{diff}} := T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & 0_{n_2^\sigma} \end{bmatrix} S_\sigma$
- *Impulse projector:* $\Pi_\sigma^{\text{imp}} := T_\sigma \begin{bmatrix} 0_{n_1^\sigma} & 0 \\ 0 & I_{n_2^\sigma} \end{bmatrix} S_\sigma$

Before introducing Definition 3.5, were we define a solution to a non homogeneous switched DAE system, consider the following assumption.

Assumption 3.4: For each $\sigma \in \Sigma$, $\Pi_\sigma^{\text{imp}} B_\sigma = 0$.

Now we can define a solution to a switched DAE system.

Definition 3.5: (Solution to a non-homogeneous switched DAE system) *A solution pair (ϕ, u) to the switched DAE system* (1a) *that satisfies Assumption 3.4, where $\phi = (\phi_\xi, \sigma)$, consists of a piecewise constant function $t \mapsto \sigma(t) \in \Sigma$, a piecewise continuously differentiable function $t \mapsto \phi_\xi(t) \in \mathfrak{O}_{\sigma(t)}$, and $t \mapsto u(t) \in \mathbb{R}^p$, all right continuous, such that $E_{\sigma(t)}\dot{\phi}_\xi(t) = A_{\sigma(t)}\phi_\xi(t) + B_{\sigma(t)}u(t)$ for almost all $t \in \operatorname{dom}\phi_\xi$, with $\operatorname{dom}\phi = \operatorname{dom}\phi_\xi = \operatorname{dom}\sigma = \operatorname{dom}u$.*

IV. SWITCHED DAE SYSTEMS WITH INPUTS AND OUTPUTS

In this section, we introduce a class of hybrid systems that model switched DAE systems with inputs and outputs, and state-triggered jumps. A simplified version of this model was introduced in [2], [11], and [10]. We build from the

[3]A more detailed discussion is provided in Example 5.4

[4] For each $\sigma \in \Sigma$, T_σ, S_σ, and n_1^σ and n_2^σ are given in [10].

aforementioned results and augment the model to consider inputs and outputs.

For the remainder of this paper, we assume the following.

Assumption 4.1: (Switched DAE regularity conditions) *Given the collection $\{(E_\sigma, A_\sigma, B_\sigma)\}_{\sigma\in\Sigma}$, we have that the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ is regular (see Definition 3.1) and for each $\sigma \in \Sigma$, $\Pi_\sigma^{imp} B_\sigma = 0$.*

The assumption upon the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ assures existence and uniqueness of the solutions of (1a) [8] and the assumption on B_σ assures that for each $\sigma \in \Sigma$ solutions for each DAE system belongs to the consistency set $\mathfrak{O}_\sigma$, which is given by a linear subspace.

The state vector is given by

$$x = (\xi, \sigma) \in \mathbb{R}^n \times \Sigma,$$

where ξ is the state component associated with the switched DAE system and σ is the state component associated with the switching signal, where Σ is a finite discrete set as defined in Section III. The input is denoted by $u = (u_c, u_d) \in \mathbb{R}^{m_c} \times \mathbb{R}^{m_d}$ in which $u_c \in \mathbb{R}^{m_c}$ and $u_d \in \mathbb{R}^{m_d}$ are respectively the inputs for flows and jumps. The output is denoted by $y = (y_c, y_d) \in \mathbb{R}^{m_c} \times \mathbb{R}^{m_d}$, where y_c, y_d are assigned via functions of the state $h_{c,\sigma} : \mathbb{R}^n \times \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$ and $h_{d,\sigma} : \mathbb{R}^n \times \mathbb{R}^{m_d} \to \mathbb{R}^{m_d}$.

Then, following [2], a switched DAE system is given by the hybrid inclusion

$$\mathcal{H}_{\mathrm{DAE}}^{\mathcal{SW}} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_\sigma \xi + B_\sigma u_c \\ 0 \end{bmatrix} & (x, u_c) \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma} \in \varphi(x,u_d)} \begin{bmatrix} g(x, \tilde{\sigma}, u_d) \\ \tilde{\sigma} \end{bmatrix} & (x, u_d) \in D \\ y_c = h_{c,\sigma}(\xi, u_c) \\ y_d = h_{d,\sigma}(\xi, u_d) \end{cases} \tag{5a}$$

where

$$C := \bigcup_{\sigma\in\Sigma} (C_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\} \times \mathbb{R}^{m_c})) \tag{5b}$$

$$D := \bigcup_{\sigma\in\Sigma} \big((D_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\} \times \mathbb{R}^{m_d})) \cup ((\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \{\sigma\} \times \mathbb{R}^{m_d})\big) \tag{5c}$$

$g(x, \tilde{\sigma}, u_d) := g_D(x, \tilde{\sigma}, u_d) \cup g_{\mathfrak{O}}(x, \tilde{\sigma}, u_d)$ for all $(x, u_d) \in D$, $\tilde{\sigma} \in \varphi(x, u_d)$, and

$$g_D(x, \tilde{\sigma}, u_d) := \begin{cases} \emptyset & \text{if } (x, u_d) \in S_\sigma^1 \\ \Pi_{\tilde{\sigma}} g_\sigma(\xi, u_d) & \text{if } (x, u_d) \in S_\sigma^2, \end{cases} \tag{5d}$$

$$g_{\mathfrak{O}}(x, \tilde{\sigma}, u_d) := \begin{cases} \Pi_{\tilde{\sigma}} \xi & \text{if } (x, u_d) \in S_\sigma^1 \\ \emptyset & \text{if } (x, u_d) \in S_\sigma^2, \end{cases} \tag{5e}$$

where $\mathfrak{O}_\sigma$ are the consistency sets and Π_σ is the consistency projectors in Definition 3.2 and Definition 3.3, respectively, $S_\sigma^1 := (\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \{\sigma\} \times \mathbb{R}^{m_d}$, and $S_\sigma^2 := D_\sigma \cap (\mathfrak{O}_\sigma \times \{\sigma\} \times \mathbb{R}^{m_d})$. At jumps, the map g defines the changes of ξ, where g_σ models changes not related to algebraic restrictions. The set-valued map φ determines the changes of σ. For each $\sigma \in \Sigma$, the sets C_σ and D_σ are subsets of $\mathbb{R}^n \times \Sigma \times \mathbb{R}^{m_c}$ and $\mathbb{R}^n \times \Sigma \times \mathbb{R}^{m_d}$, respectively, that, together with the consistency sets $\mathfrak{O}_\sigma$, define where the evolution of the system according to the continuous and discrete dynamics are possible. Namely, the sets C and D model where the state of the system can change according to the differential algebraic equation or the difference inclusion in (5), respectively. The model in (5) is actually a hybrid DAE system [2] and its data can be defined to model switched DAE systems under a variety of switching signals. For example, a switched DAE system under arbitrary switching signals can be captured by a hybrid DAE system with inputs $u_c := u_d := u$, outputs $y_c := y_d := y$, output maps $h_{c,\sigma} := h_{d,\sigma} := h_\sigma$, $C_\sigma = D_\sigma = \mathbb{R}^n \times \{\sigma\} \times \mathbb{R}^p$, $g_\sigma(\xi, u_d) = \xi$, and $\varphi_\sigma(\xi, u_d) = \Sigma \setminus \{\sigma\}$, leading to

$$\mathcal{H}_{\mathrm{DAE}}^{\mathcal{SW}} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_\sigma \xi + B_\sigma u \\ 0 \end{bmatrix} & (x, u) \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma} \in \Sigma \setminus \{\sigma\}} \begin{bmatrix} \Pi_{\tilde{\sigma}} \xi \\ \tilde{\sigma} \end{bmatrix} & (x, u) \in D \\ y = h_\sigma(\xi, u) \end{cases}$$

A switched DAE system under dwell time switching signals can be similarly modeled; for more details and complete examples, see [2] and [11].

As in the framework in [12], we define solutions to hybrid DAEs using hybrid time domains. Therefore, during flows, solutions are parametrized by $t \in \mathbb{R}_{\geq 0}$, while at jumps they are parametrized by $j \in \mathbb{N}$.

V. Passivity and Stability for Switched DAEs

Building from previous results on passivity for hybrid systems [3], [13], which employ smooth storage functions, we consider storage functions that are locally Lipschitz functions as those permit analysis of passivity properties in hybrid DAE systems. Given a locally Lipschitz function V, to be able to take derivatives, we employ the generalized directional gradient (in the sense of Clarke) of V at x in the direction v, which is given by $V^\circ(x, v) = \max_{\zeta \in \partial V(x)} \langle \zeta, v \rangle$, where $\partial V(x)$ is a closed, convex, and nonempty set equal to the convex hull of all limits of sequences $\nabla V(x_i)$ with x_i converging to x. For more information about Clarke's derivative, see [14].

A. Passivity Notions

For the class of hybrid DAE systems in (5), inspired by [3], we consider the following concept of passivity. Below, the functions $h_{c,\sigma}$, $h_{d,\sigma}$, and a compact set $\mathcal{A} \subset \mathbb{R}^n \times \Sigma$ satisfy $h_{c,\sigma}(\xi, 0) = h_{d,\sigma}(\xi, 0) = 0$ for each $(\xi, \sigma) \in \mathcal{A}$.

Definition 5.1: (Passivity for switched DAE systems) *A switched DAE system as in* (1) *for which, for each $\sigma \in \Sigma$, there exist a continuous and locally Lipschitz function $V_\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\omega_{c,\sigma} : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ and $\omega_{d,\sigma} : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$, such that*

$$V_\sigma^\circ(\xi, \mu) \leq \omega_{c,\sigma}(u_c, \xi) \quad \forall (\xi, \sigma, u_c) \in C, \quad \mu \in \tilde{F}(x, u_c) \tag{6}$$

$$V_{\tilde{\sigma}}(\eta) - V_\sigma(\xi) \leq \omega_{d,\sigma}(u_d, \xi) \quad \forall (\xi, \sigma, u_d) \in D, \quad (\eta, \tilde{\sigma}) \in \hat{G}(x, u_d) \tag{7}$$

holds with $x = (\xi, \sigma)$, $\tilde{F} : \cup_{\sigma\in\Sigma} (\mathfrak{O}_\sigma \times \{\sigma\} \times \mathbb{R}^p) \rightrightarrows \mathbb{R}^n$ defined as $\tilde{F}(x, u) = \Pi_\sigma^{diff} A_\sigma \xi + \Pi_\sigma^{diff} B_\sigma u$, $\hat{G} : \mathbb{R}^n \times \Sigma \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n \times \Sigma$ defined as $\hat{G}(x, u_d) := \bigcup_{\tilde{\sigma} \in \varphi(x,u_d)} \begin{bmatrix} g_D(x, \tilde{\sigma}, u_d) \\ \tilde{\sigma} \end{bmatrix}$, and

C and D given as in (5), is said to be passive *with respect to a compact set $\mathcal{A} \subset \mathbb{R}^n \times \Sigma$ if*

$$(u_c, x) \mapsto \omega_{c,\sigma}(u_c, x) = {u_c}^\top y_c \quad (8)$$

$$(u_d, x) \mapsto \omega_{d,\sigma}(u_d, x) = {u_d}^\top y_d. \quad (9)$$

It is then called flow-passive (respectively, jump-passive) if it is passive with $\omega_{d,\sigma} \equiv 0$ (respectively, $\omega_{c,\sigma} \equiv 0$).

In the next section, we show how the definitions of passivity previously introduced can be used to link passivity to stability. Next, we define detectability for switched DAE systems with inputs set to zero and the required mild regularity properties on the data of the hybrid DAE $\mathcal{H}^{\mathcal{SW}}_{\mathrm{DAE}}$ in (5). Strict and input/output passivity notions can be also defined following [3]. These notions permit establishing a link between passivity and stability as in [3]. These are not included here due to space constraints.

Definition 5.2: (Detectability for switched DAE systems) *Consider the switched DAE system in* (5). *Given sets $\mathcal{A}, K \subset \mathbb{R}^n \times \Sigma$ with $\mathcal{A}$ closed, the distance to $\mathcal{A}$ is 0-input detectable relative to K for the switched DAE system if every complete solution pair $(\phi, 0)$, as in Definition 3.5, is such that $\phi(t) \in K$ for all $t \in \operatorname{dom}\phi$ implies $\lim_{t\to\infty,\ t\in\operatorname{dom}\phi} |\phi(t)|_{\mathcal{A}} = 0$.*

B. Passivity-based control for switched DAEs

Next, following [3] we link the properties of flow and jump passivity in Definition 5.1, together to the concept of detectability in Definition 5.2, to asymptotic stability via static output-feedback[5].

Theorem 5.3: (Passivity-based control for switched DAE systems). *Given a switched DAE system as in* (1), *suppose that the collection $\{(E_\sigma, A_\sigma, B_\sigma)\}_{\sigma\in\Sigma}$ satisfies Assumption 4.1. Given a compact set $\mathcal{A} \subset \mathbb{R}^n \times \Sigma$, the vector state $x = (\xi, \sigma)$, and the continuous output map $\xi \mapsto h_\sigma(\xi)$ the following hold: if the switched DAE system is flow- or jump-passive with respect to $\mathcal{A}$ with a storage function[6] V that is positive definite on $\mathcal{X} := \cup_{\sigma\in\Sigma}(\mathfrak{D}_\sigma \times \{\sigma\})$ with respect to $\mathcal{A}$ and for each $\sigma \in \Sigma$ there exists a continuous function $\kappa_\sigma : \mathbb{R}^p \to \mathbb{R}^p$, with $y^\top\kappa_\sigma(y) > 0$ for all $y \neq 0$ having defined $u_c := u_d := u$, $y_c := y_d := y$, $h_{c,\sigma} := h_{d,\sigma} := h_\sigma$, $y = h_\sigma(\xi)$, such that the resulting closed-loop system with $u = -\kappa_\sigma(y)$ has the following properties:*

1) *the distance to $\mathcal{A}$ is 0-input detectable relative to $\{(\xi,\sigma) \in \mathcal{X} : h_\sigma(\xi)^\top\kappa_\sigma(h_\sigma(\xi)) = 0\}$;*
2) *every complete solution ϕ is such that for some $\theta > 0$ and some $J \in \mathbb{N}$ we have $t_{j+1} - t_j \geq \theta$ for all $j \geq J$;*

then the control law $u = -\kappa_\sigma(y)$ renders $\mathcal{A}$ pre-asymptotically stable. Furthermore, if there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$ for all $x \in \mathcal{X}$, the stability properties of $\mathcal{A}$ hold globally.

It is important to notice that condition 2) in Theorem 5.3 does not allow solutions with multiple jumps (or switching instants) at a single time instant t, which is a commonly used assumption in switching DAE modeling.

[5]Due to space constraints, the proof will be published elsewhere.

[6]Notice that, the storage function V can be explicitly indexed by σ. For sake of simplicity, the common storage function case is presented.

Next, we revisit the motivational example in Section II. We model the switched DAE system capturing the dynamics of the boost converter as a hybrid DAE system as in (5). In Proposition 5.5, by selecting the appropriate inputs and output functions, we show that the system is flow-passive relative to a compact set of interest.

Example 5.4: (Motivational example revisited) Recalling Example II, first we show how use $\widetilde{V}$ as a Control Lyapunov Function (CLF). Later in the example, we use the properties of the CLF $\widetilde{V}$ and the passivity properties of the system to design a passivity-based control law using the result in Theorem 5.3. To show that $\widetilde{V}$ can be a used as a CLF, we need show that for each $\xi \in \mathbb{R}^8$ there exists σ such that the derivative of $\widetilde{V}$ is negative along the solution to the switched DAE system. To do so, we compute the derivative of $\widetilde{V}$ along the equivalent vector field $\tilde{f}(\xi, u)$ for each $\sigma \in \Sigma$. It is easy to see that it is always possible to pick some $\gamma_s(\xi)$ such that $\gamma_s(\xi) \leq 0$, where $s \in \Sigma$. For the sake of simplicity, it is possible to use $\gamma_1^{-1}(0)$ and $\gamma_2^{-1}(0)$ as switching surfaces for modes $\{1, 3\}$ and $\{2, 4\}$, respectively, this is due to the algebraic restrictions on modes 3 and 4. For a more detailed derivation of this assertion, please consider γ_1 and γ_2 and apply [5, Lemma III.1] with $p_{11} = p_{22} = 1$, $v_c^* = u > v_{cc} + \lambda$, $i_L^* = u^2/(v_{cc}R)$. We have that for each $(\xi, u) \in \mathbb{R}^8 \times \mathbb{R}_{\geq 0}$, such that $u > v_{cc} + \lambda$, $\gamma_s(\xi) \leq 0$ holds for some $s \in \{1, 2\}$.

Defining $\mathcal{X} := \mathbb{R}^8 \times \Sigma \times \mathbb{R}_{\geq v_{cc}+\lambda}$ and the set of interest

$$\mathcal{A} := \{(\xi, \sigma) \in \mathbb{R}^8 \times \Sigma \mid \widetilde{V}(\xi) \leq 2\delta, e = \tilde{v}_C\}, \quad (10)$$

where $\delta \in \mathbb{R}_{>0}$ and $V_e(e, v_{cc}) = 0$ for each $(\xi, \sigma) \in \mathcal{A}$, consider the control problem of steering each solution to $\mathcal{A}$ in finite time and stay in it. To do so, let us recast the switched DAE system in Example II as in (5) with input $u = u_c = u_d$ and data in (11) and

$$g_\sigma(\xi, u_d) := [v_{cc}, v_L, v_D, v_C, i_L, i_S, i_D, u_d + \tilde{v}_C]^\top \ \forall\sigma \in \Sigma,$$

$\varphi(x, u_d)$ is given by the outer semicontinuous set-valued map returning 1 at points $x \in \{(x, u_d) \in \mathcal{X} \mid \sigma \in \{2, 4\}, \gamma_2(\xi) \geq \epsilon, \widetilde{V}(\xi) - V_e(e, v_{cc}) \geq \delta\}$, 2 at points $x \in \{(x, u_d) \in \mathcal{X} \mid \sigma \in \{1, 3\}, \gamma_1(\xi) \geq \epsilon, \widetilde{V}(\xi) - V_e(e, v_{cc}) \geq \delta\}$, 3 at points $x \in \{(x, u_d) \in \mathcal{X} \mid \sigma = 1, i_D = 0, v_C \geq v_{cc} - \lambda - v_L, \gamma_1(\xi) \leq \epsilon\}$, 4 at points $x \in \{(x, u_d) \in \mathcal{X} \mid \sigma = 2, v_C = \lambda, \gamma_2(\xi) \leq \epsilon\}$, where the constant $\tilde{v}_C$ is such that $\tilde{v}_C \geq v_{cc} + \lambda$ and $-1 \leq \epsilon \leq 0$. Next, by considering the storage function in (4), we choose the inputs and outputs of the switched DAE system such that the passivity properties in Definition 5.1 hold during flows and jumps. To properly define the output of the system, let us define the function v_p which returns v that minimizes the square of the distance between (v_C, i_L) and the curve $\{(v_C, i_L) \in \mathbb{R} \times \mathbb{R} : 0 = i_L - v_C^2/(v_{cc}R)\}$, namely,

$$v_p(\xi) := \left\{ v \in \mathbb{R} \,\middle|\, \operatorname*{argmin}_v \left\{ \frac{1}{2}(v_C - v)^2 + \frac{1}{2}\left(i_L - \frac{v^2}{v_{cc}R}\right)^2 \right\} \right\} \quad (12)$$

which is equal to

$$\left\{ v \in \mathbb{R} \;\middle|\; \frac{2v^3}{v_{cc}^2R^2} + v\left(1 - \frac{2i_L}{v_{cc}R}\right) - v_C = 0 \right\}$$

$$
\begin{aligned}
C_1 &:= \{(x,u_c) \in \mathcal{X} \mid \sigma = 1, v_D = \lambda, i_D \geq 0, \gamma_1(\xi) \leq \epsilon\} \cup \{(x,u_c) \in \mathcal{X} \mid \sigma = 1, \widetilde{V}(\xi) - V_e(e, v_{cc}) \leq \delta\},\\
C_2 &:= \{(x,u_c) \in \mathcal{X} \mid \sigma = 2, v_C \leq \lambda, \gamma_2(\xi) \leq \epsilon\} \cup \{(x,u_c) \in \mathcal{X} \mid \sigma = 2, \widetilde{V}(\xi) - V_e(e, v_{cc}) \leq \delta\},\\
C_3 &:= \{(x,u_c) \in \mathcal{X} \mid \sigma = 3, v_D \leq \lambda, \gamma_1(\xi) \leq \epsilon\} \cup \{(x,u_c) \in \mathcal{X} \mid \sigma = 3, \widetilde{V}(\xi) - V_e(e, v_{cc}) \leq \delta\},\\
C_4 &:= \{(x,u_c) \in \mathcal{X} \mid \sigma = 4, v_C = \lambda, i_D \geq 0, \gamma_2(\xi) \leq \epsilon\} \cup \{(x,u_c) \in \mathcal{X} \mid \sigma = 4, \widetilde{V}(\xi) - V_e(e, v_{cc}) \leq \delta\},\\
D_1 &:= \{(x,u_d) \in \mathcal{X} \mid \sigma = 1, i_D = 0, v_C \geq v_{cc} - \lambda - v_L\} \cup \{(x,u_c) \in \mathcal{X} \mid \sigma = 1, \gamma_1(\xi) \geq \epsilon, \widetilde{V}(\xi) - V_e(e, v_{cc}) \geq \delta\},\\
D_2 &:= \{(x,u_d) \in \mathcal{X} \mid \sigma = 2, v_C = \lambda\} \cup \{(x,u_c) \in \mathcal{X} \mid \sigma = 2, \gamma_2(\xi) \geq \epsilon, \widetilde{V}(\xi) - V_e(e, v_{cc}) \geq \delta\},\\
D_3 &:= \{(x,u_d) \in \mathcal{X} \mid \sigma = 3, \gamma_1(\xi) \geq \epsilon, \widetilde{V}(\xi) - V_e(e, v_{cc}) \geq \delta\},\\
D_4 &:= \{(x,u_d) \in \mathcal{X} \mid \sigma = 4, \gamma_2(\xi) \geq \epsilon, \widetilde{V}(\xi) - V_e(e, v_{cc}) \geq \delta\},
\end{aligned}
\tag{11}
$$

Given the state space defined for this system with positive v_C and i_L, and $v \in \mathbb{R}$, it can be shown that v_p is well defined and single valued.

Proposition 5.5: Consider the switched DAE system modeled as in (5) *with the data above. For each $\tilde{v}_C \geq v_{cc} + \lambda$, let us consider the storage function*

$$V(\xi) := \max\left\{\widetilde{V}(\xi) - \delta, 0\right\} \tag{13}$$

where $\delta \in \mathbb{R}_{>0}$. Also, let us consider the function v_p defined in (12) *and define the function $d_p : \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ as*

$$d_p(\xi) := V_e(v_p(\xi), v_{cc}) \max\left\{\widetilde{V}(\xi) - V_e(e, v_{cc}) - \delta, 0\right\}$$

for each $\xi \in \mathbb{R}^8$ such that $v_C > 0$ and $i_L \geq 0$. Then, the switched DAE system is flow-passive with respect to the compact set $\mathcal{A}$ in (10) *with storage function V in* (13) *and the control law using the feedback $v_C^* = -ky_c$, with $0 < k \leq 1$ with the output map*

$$h_\sigma(x,u) = \begin{cases} \min\{(\tilde{v}_C - e), d_p(\xi, \tilde{v}_C), \sqrt{-\gamma_\sigma}\} & v_p \geq \tilde{v}_C \geq e \\ -\min\{(e - \tilde{v}_C), d_p(\xi, \tilde{v}_C), \sqrt{-\gamma_\sigma}\} & v_p < \tilde{v}_C < e \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

renders the set $\mathcal{A}$ in (10) *pre-asymptotically stable.*

In Figure 2, a numerical simulation[7] of the hybrid DAE system is presented. Notice that the level sets $\gamma_1^{-1}(0)$ and $\gamma_2^{-1}(0)$ are also depicted at each jump (change in σ). □

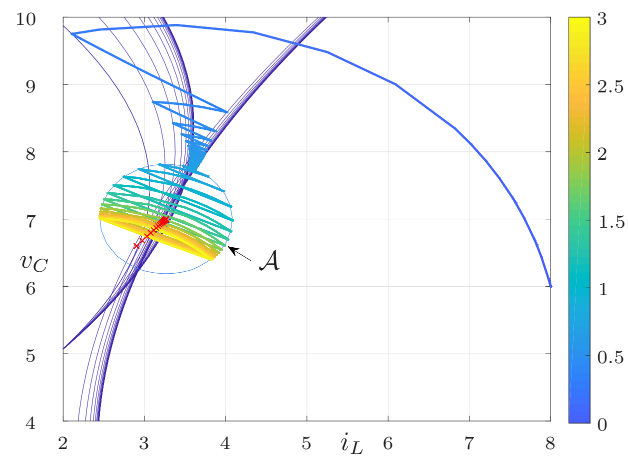


Fig. 2. Simulation of the solutions of the switched DAE in Example 5.4 for the initial condition $(v_C, i_L) = (6, 8)$. Color gradient represents time and the red $\times$ symbol represents the state e. The level sets $\gamma_1^{-1}(\epsilon)$ and $\gamma_2^{-1}(\epsilon)$ that triggers the changes in σ are shown.

REFERENCES

[1] B. Brogliato, *Dissipative systems analysis and control: theory and applications*, 2nd ed. London: Springer, 2007.

[2] P. Nanez, R. G. Sanfelice, and N. Quijano, "On an invariance principle for differential-algebraic equations with jumps and its application to switched differential-algebraic equations," *Mathematics of Control, Signals, and Systems*, vol. 29, no. 1, p. 5, 2017.

[3] R. Naldi and R. G. Sanfelice, "Passivity-based control for hybrid systems with applications to mechanical systems exhibiting impacts," *Automatica*, vol. 49, no. 5, pp. 1104–1116, May 2013.

[4] D. Liberzon and S. Trenn, "Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability," *Automatica*, vol. 48, no. 5, pp. 954 – 963, 2012.

[5] T. A. F. Theunisse, J. Chai, R. G. Sanfelice, and M. Heemels, "Robust global stabilization of the dc-dc boost converter via hybrid control," *IEEE Transactions on Circuits and Systems I*, vol. 62, pp. 1052–1061, April 2015.

[6] W. M. Haddad and Q. Hui, "Energy dissipating hybrid control for impulsive dynamical systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3232 – 3248, 2008.

[7] S. Trenn, "Switched differential algebraic equations," in *Dynamics and Control of Switched Electronic Systems*, ser. Advances in Industrial Control, F. Vasca and L. Iannelli, Eds. Springer London, 2012, pp. 189–216.

[8] ——, "Distributional differential algebraic equations," Ph.D. dissertation, Universitätsverlag Ilmenau, March 2009.

[9] L. Dai, *Singular Control Systems. Number 118 in Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1989.

[10] P. Nanez and R. G. Sanfelice, "An invariance principle for differential-algebraic equations with jumps," in *Proceedings of the 2014 American Control Conference*, 2014, pp. 1426–1431.

[11] P. Nanez, R. G. Sanfelice, and N. Quijano, "Invariance principles for switched differential-algebraic systems under arbitrary and dwell-time switching," in *Proceedings of the 2015 American Control Conference*. IEEE, 2015, pp. 1788–1793.

[12] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[13] R. G. Sanfelice, "Input-output-to-state stability tools for hybrid systems and their interconnections," *IEEE Transactions on Automatic Control*, vol. 59, no. 5, pp. 1360–1366, May 2014.

[14] F. H. Clarke, *Optimization and Nonsmooth Analysis*. SIAM's Classic in Applied Mathematics, 1990, vol. 5.

[7]Code at github.com/HybridSystemsLab/BoostConverterPBC