On the Optimality of Lyapunov-based Feedback Laws for Constrained Difference Inclusions

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Abstract— **State-feedback optimal control and cost evaluation problems for constrained difference inclusions are considered. Sufficient conditions, in the form of Lyapunov-like inequalities, are provided to derive an upper bound on the cost associated with the solution to a constrained difference inclusion with respect to a given cost functional. Under additional sufficient conditions, we determine the cost exactly without computing solutions. The proposed approach is extended to study an optimal control problem for discrete-time systems with constraints. In this setting, sufficient conditions for closed-loop optimality are given in terms of a constrained steady-state-like Hamilton-Jacobi-Bellman equation. Applications and examples of the proposed results are presented.**

I. Introduction

A. Background

Although optimal control problems are normally formalized as open-loop control problems, an aspect of fundamental importance in the literature is whether for a given optimal control problem an optimal state-feedback law exists; see, e.g., [4]. When adopting optimal state-feedback laws, the relationship between optimality and stability for the closed-loop system is an important question. In particular, when the horizon of the considered control problem is not bounded, then a natural question that arises is whether the closed-loop control system is asymptotically stable. Typically, an answer to this question can be obtained by relying on the steady-state Hamilton-Jacobi-Bellman equation, which, under some conditions, indirectly provides a Lyapunov function assessing closed-loop asymptotic stability for the closed-loop system. Specifically, interesting results on the connections between Lyapunov theory and optimal control can be found in [1], [7]. Due its importance, this problem has been addressed extensively also in the literature of model predictive control; see [5].

Another problem that is relevant in practice is the problem of cost evaluation, which consists of evaluating the cost associated to the solutions to a dynamical system with respect to a given cost functional. For the class of linear-quadratic problems, i.e., linear dynamics and quadratic costs, closed form expressions of the cost value can be obtained via direct integration of the plant dynamics. Unfortunately, as pointed out in [1], this technique does not extend to nonlinear systems. General results for the solution to this problem are given in [1] for nonlinear systems. However, to the best of the authors knowledge, there is a lack of similar results for constrained discrete-time systems given by equations/inclusions.

B. Contribution

In this paper, motivated by the general ideas presented in [1], we address a class of cost evaluation and optimal control problems for constrained difference inclusions, i.e., difference inclusions in which the input and the state are confined to a given set. The contributions we offer in this paper are as follows. First, we provide sufficient conditions, in the form of Lyapunov inequalities, for the estimation of an upper bound on the cost associated to the solutions to a constrained difference inclusion with respect to a given cost functional. Then, it is shown that under certain conditions on the solutions to the considered difference inclusion, the cost can be perfectly determined via the proposed conditions. Second, we address a specific optimal control problem for discrete-time systems with state and input constraints. More specifically, in the optimal control problem we address, we require limit points of feasible state trajectories to belong to a given closed, potentially unbounded, set. This requirement essentially prescribes an asymptotic constraint on feasible trajectories, which are not necessarily complete and may grow unbounded while approaching the target set. Pursuing this approach, we are able to formulate and solve a meaningful control problem for which completeness of feasible solutions is not required. In this setting, we provide sufficient conditions for a static state-feedback controller to be optimal.

A unique feature of our approach is that we relax the completeness requirement for feasible solutions. This allows us to deal with constrained systems in an elegant way. The price to pay in taking this approach is that the proposed feedback law provides an actual solution to the considered optimal control problem only if trajectories to the closed-loop system satisfy the above mentioned limit point state constraint, condition that needs to be checked *a posteriori*. In future research, we intend to use these results as a baseline to obtain sufficient conditions for optimal control and cost evaluation for hybrid dynamical systems in the framework of [6]. Indeed, for the case of hybrid dynamical systems, the interplay of continuous-time behaviors and instantaneous changes is modeled through constrained hybrid inclusions,

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This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484, Grant no. ECS-1710621, and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, by the Air Force Research Laboratory under Grant no. FA9453-16-1-0053, and by CITRIS and the Banatao Institute at the University of California. This work has been partially done when the first author was with the Department of Computer Engineering, University of California Santa Cruz.

and this makes constrained systems crucial in this context.

The remainder of this paper is structured as follows. Section II-A presents some preliminaries on constrained difference inclusions. Section II-B presents our main results concerning the considered cost evaluation problems. Section III is dedicated to the formulation and solution to the optimal control problem we consider. Due to space constraints proofs are omitted and will be published elsewhere.

Notation: The set $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{R}_{\geq 0}$ represents the set of nonnegative real scalars, $\mathbb{S}_+^n$ denotes the set of real symmetric positive definite matrices of dimension n. In partitioned symmetric matrices, the symbol $\bullet$ stands for symmetric blocks. The matrix $\operatorname{diag}\{A_1,A_2,\dots,A_n\}$ is the block-diagonal matrix having $A_1,A_2,\dots,A_n$ as diagonal blocks. Given $x\in\mathbb{R}$, $\lceil x\rceil$ is the smallest integer larger than x. Given a matrix $A\in\mathbb{R}^{n\times n}$, we say that A is Schur if the eigenvalues of A are contained in the open unit circle. For a vector $x\in\mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x, x_i denotes the i-th entry of x, and x^T denotes the transpose of x. The symbol $\mathbf{1}_n$ denotes the vector in $\mathbb{R}^n$ whose entries are equal to one. Given two vectors x,y, we denote $(x,y)=[x^\mathsf{T}\ y^\mathsf{T}]^\mathsf{T}$, while we say that $x\preceq y$, if for every entry x_i and y_i of respectively, x and y, one has $x_i\leq y_i$. Given a vector $x\in\mathbb{R}^n$ and a closed set $\mathcal{A}$, the distance of x to $\mathcal{A}$ is defined as $|x|_\mathcal{A}=\inf_{y\in\mathcal{A}}|x-y|$. Given a set S, we denote by $\overline{S}$ the closure of S. The set $\mathbb{B}$ denotes the closed unit ball, of appropriate dimension, in the Euclidean norm. Given a function $f\colon X\to Y$, $\operatorname{rge} f$ denotes the range of f. We say that a function $\alpha\colon\mathbb{R}\geq 0\to\mathbb{R}_{>0}$ is said to belong to the class $\mathcal{K}_\infty$, i.e., $\alpha\in\mathcal{K}_\infty$, if $\alpha(0)=0$, α is strictly increasing, continuous, and $\lim_{s\to\infty}\alpha(s)=\infty$. The notation $F\colon X\rightrightarrows Y$ indicates that F is a set-valued mapping with $F(x)\subset Y$ for all $x\in X$.

II. Cost evaluation for Constrained Difference Inclusions

A. Preliminaries on Constrained Difference Inclusions

In this section, we consider unforced constrained difference inclusions of the form

$$\mathcal{D}_0\colon\quad x^+\ \in\ G(x)\qquad x\in D \tag{1}$$

where $x\in\mathbb{R}^n$, $D\subset\mathbb{R}^n$, and $G\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^n$.

Let $\mathcal{X}$ be the set of functions $x\colon\operatorname{dom}x\to\mathbb{R}^n$, with $\operatorname{dom}x=\mathbb{N}_0\cap\{0,1,\dots,J\}$ for some $J\in\mathbb{N}_0\cup\{\infty\}$. A solution ϕ to $\mathcal{D}_0$ is any $\phi\in\mathcal{X}$ for which $\phi(0)\in D$, and for all $j\in\operatorname{dom}\phi$ such that $j+1\in\operatorname{dom}\phi$

$$\begin{aligned}&\phi(j)\in D\\&\phi(j+1)\in G(\phi(j))\end{aligned}$$

In particular, a solution ϕ to $\mathcal{D}_0$ is said to be *complete* if its domain $\operatorname{dom}\phi$ is unbounded and *maximal* if it is not the truncation of another solution. Given a set M, we denote by $\mathcal{S}(M)$ the set of all maximal solutions ϕ to $\mathcal{D}_0$ with $\phi(0)\in M$.

B. Upper bounds

By following the general ideas proposed in [1], in this section we investigate how a Lyapunov function can be used to provide estimates of nonlinear cost functionals for a given constrained difference inclusion. For each initial condition $\xi\in D$ to $\mathcal{D}_0$ in (1), consider the following cost:

$$\mathcal{J}(\xi)=\sup_{\phi\in\mathcal{S}(\xi)}\sum_{j=1}^{\sup\operatorname{dom}\phi}q_d(\phi(j-1)) \tag{2}$$

where $q_d\colon\mathbb{R}^n\to\mathbb{R}_{\geq 0}$. Given a solution ϕ to (1), having defined the cost (2) up to $(\sup\operatorname{dom}\phi)-1$ implies that, when $\operatorname{dom}\phi$ is bounded, the final state is not included in the computation of the cost; obviously, this does not lead to any difference in the cost value when ϕ is complete. On the one hand, as it will be clear in the following, such a formulation turns out to be convenient for our analysis. On the other hand, notice that if $\operatorname{dom}\phi$ is bounded, then $\phi(\sup\operatorname{dom}\phi)\notin D$, that is ϕ leaves the set D in finite time. Therefore, discarding the cost associated with the last value of such a ϕ is reasonable.

Throughout the paper, given a solution ϕ to (1), we denote

$$J_\phi:=\lim_{j\to\sup\operatorname{dom}\phi}\sum_{i=1}^{j}q_d(\phi(i-1))$$

The following result can be established.

Proposition 1: Let $\xi\in D$ and $q_d\colon\mathbb{R}^n\to\mathbb{R}_{\geq 0}$. Assume that there exists $V\colon\mathbb{R}^n\to\mathbb{R}$ such that

$$\sup_{g\in G(x)}V(g)-V(x)+q_d(x)\leq 0\qquad\forall x\in D \tag{3}$$

Let $\phi\colon\operatorname{dom}\phi\to\mathbb{R}^n$ be a solution to (1) from ξ. Assume that $V\circ\phi$ is bounded. Then J_ϕ is finite and in particular

$$J_\phi+\limsup_{j\to\sup\operatorname{dom}\phi}V(\phi(j))\leq V(\xi) \tag{4}$$

□

By building on a suitable function V, Proposition 1 provides an upper bound on the cost J_ϕ. This bound depends on the solution chosen from ξ. Next, by relying on further assumptions, for a given initial condition $\xi\in D$, we provide an upper bound on the cost $\mathcal{J}(\xi)$ that is independent on the solution chosen from ξ.

Corollary 1: Let $\mathcal{A}\subset\mathbb{R}^n$ be closed, $\xi\in D$ and $q_d\colon\mathbb{R}^n\to\mathbb{R}_{\geq 0}$. Assume that there exists a function $V\colon\mathbb{R}^n\to\mathbb{R}$ uniformly continuous on a neighborhood of $\mathcal{A}$ such that $V(\mathcal{A})=\{0\}$ and (3) holds. Furthermore, assume that for each $\phi\in\mathcal{S}(\xi)$

$$\lim_{j\to\sup\operatorname{dom}\phi}|\phi(j)|_\mathcal{A}=0 \tag{5}$$

Then

$$\mathcal{J}(\xi)\leq V(\xi) \tag{6}$$

□

Remark 1: To get a solution independent upper bound on the cost, in the above result we assumed V to be uniformly continuous on a neighborhood of $\mathcal{A}$. Indeed, since $V(\mathcal{A})=\{0\}$, one can show that uniform continuity of V on a neighborhood of $\mathcal{A}$ ensures that for any $j\mapsto\phi(j)$ such that ϕ approaches $\mathcal{A}$, $V \circ \phi$ approaches zero[1]. On the other hand, observe that when $\mathcal{A}$ is compact, requiring V to be continuous on a neighborhood of $\mathcal{A}$ is enough.

Corollary 1 shows that when maximal solutions from ξ converge to $\mathcal{A}$, then an upper bound on the cost $\mathcal{J}(\xi)$ (which is solution independent) is given by $V(\xi)$. On the one hand, when q_d and V are positive definite with respect to $\mathcal{A}$, (that is generally the case, e.g., in classical optimal control with $\mathcal{A}$ being the origin), (3) implies for any complete solution ϕ that $V \circ \phi$ approaches zero[2]. On the other hand, for maximal solutions that are not complete, finite-time convergence to $\mathcal{A}$ is needed. For this reason, next we provide a sufficient condition for finite-time convergence to a closed set $\mathcal{A}$ for (1). Such a condition is largely inspired by [9, Theorem 3.9].

Theorem 1: Let $\xi \in D$ and let $\mathcal{A}$ be closed. If there exists $W : \mathbb{R}^n \to \mathbb{R}$ positive definite with respect to $\mathcal{A}$ on $D \cup G(D)$ and such that for each $\phi \in \mathcal{S}(\xi)$ there exists $c > 0$ satisfying

$$(i) \qquad \inf_{\phi \in \mathcal{S}(\xi)} \sup \operatorname{dom} \phi > \left\lceil \frac{W(\xi)}{c} \right\rceil$$

$$(ii) \qquad \sup_{g \in G(x)} W(g) - W(x) \leq -\min\{c, W(x)\} \quad \forall x \in D$$

then, for each $\phi \in \mathcal{S}(\xi)$, there exists $j^\star_\phi \in \mathbb{N}_0$ such that $j \in \operatorname{dom} \phi$ with $j \geq j^\star_\phi$ implies $\phi(j) \in \mathcal{A}$. $\square$

C. Exact cost evaluation

In this section, our main objective is to obtain the exact value of the cost without explicitly computing it. To that end, next, under further assumptions on the system data and a stronger condition than (3), we provide a way to determine the exact value of $\mathcal{J}(\xi)$ for a given initial condition $\xi \in D$.

Corollary 2: Let $\mathcal{A} \subset \mathbb{R}^n$ be closed, $\xi \in D$, $q_d : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $G(x)$ be compact for each $x \in D$. Assume that there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ continuous on $G(D)$ such that

$$\max_{g \in G(x)} V(g) - V(x) + q_d(x) = 0 \qquad \forall x \in D \tag{7}$$

Furthermore, assume that for any solution ϕ_0 to $\mathcal{D}_0$ from ξ, $V \circ \phi_0$ is bounded. Pick any solution ϕ to

$$x^+ \in \operatorname*{arg\,max}_{g \in G(x)} V(g) \quad x \in D \tag{8}$$

with $\phi(0) = \xi$ and let ϕ_0 be any solution to $\mathcal{D}_0$ from ξ. Then, one has that J_{ϕ_0} and J_ϕ are finite and in particular

$$\begin{aligned} J_{\phi_0} + \limsup_{j \to \sup \operatorname{dom} \phi_0} V(\phi_0(j)) \leq \\ J_\phi + \limsup_{j \to \sup \operatorname{dom} \phi} V(\phi(j)) = V(\xi) \end{aligned} \tag{9}$$

Moreover, if V is uniformly continuous on a neighborhood of $\mathcal{A}$, $V(\mathcal{A}) = \{0\}$, and there exists a maximal solution ϕ to (8) with $\phi(0) = \xi$ such that

$$\lim_{j \to \sup \operatorname{dom} \phi} |\phi(j)|_\mathcal{A} = 0$$

then, one has

$$\mathcal{J}(\xi) = V(\xi) \tag{10}$$

$\square$

D. Applications and Examples

In this section we present some special cases of our results, which are useful in practical applications, along with examples.

First, notice that whenever G is a single-valued map, i.e., for $x \in D$, $G(x) = \{g(x)\}$ with $g : \mathbb{R}^n \to \mathbb{R}^n$, then for each $x \in D$, (7) reduces to $V(g(x)) - V(x) + q_d(x) = 0$. In particular, for the linear-quadratic case, we get the following result.

Proposition 2: Let $\mathcal{A} = \{0\}$, $G(x) = \{Ax\}$ for each $x \in D$ with $A \in \mathbb{R}^{n \times n}$, $\xi \in D$, and $x \mapsto q_d(x) := x^\mathsf{T} Q x$, where $Q \in \mathbb{S}^n_+$. If A is Schur, then there exists $V : \mathbb{R}^n \to \mathbb{R}$ continuous and positive definite with respect to $\mathcal{A}$ on $G(D) \cup D$ such that

$$V(Ax) - V(x) + q_d(x) = 0 \qquad \forall x \in D$$

$\square$

An example within the setting considered in the above result is presented next.

Example 1: Consider system (1) defined by the following data: $D := \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$ and $G(x) = \{Ax\}$ for $x \in D$, where $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{4} \end{pmatrix}$. It is easy to check that for each $k \in \mathbb{N}$, one has

$$A^k = \begin{pmatrix} \frac{1}{2^k} & 4\left(\frac{1}{2^k} - \frac{1}{4^k}\right) \\ 0 & \frac{1}{4^k} \end{pmatrix} \tag{11}$$

At this stage, notice that one may explicitly characterize the set of initial conditions $\mathcal{O}$ that lead to complete maximal solutions. In particular, one has that

$$\mathcal{O} = \{x \in \mathbb{R}^2 : |A^k x| \preceq \mathbf{1}_2 \quad \forall k \in \mathbb{N}\}$$

Moreover, from (11), it follows that for each $x \in \mathbb{R}^2$ such that $|x| \preceq \mathbf{1}_2$ and $|Ax| \preceq \mathbf{1}_2$, for any $k \in \mathbb{N}_{\geq 2}$ one has $|A^k x| \preceq \mathbf{1}_2$. Therefore, $\mathcal{O}$ reduces to $\{x \in \mathbb{R}^2 : |x| \preceq \mathbf{1}_2, |Ax| \preceq \mathbf{1}_2\}$, which is the polyhedral set represented in Fig. 1. Now, define $x \mapsto q_d(x) := x^\mathsf{T} x$ and $x \mapsto V(x) := x^\mathsf{T} P x$, with

$$P = \begin{pmatrix} \frac{4}{3} & \frac{16}{21} \\ \frac{16}{21} & \frac{304}{105} \end{pmatrix}$$

Then, one has that for each $x \in \mathbb{R}^2$, $V(Ax) - V(x) + q_d(x) = 0$. To show the application of our results, we consider the following two solutions to (1):

$$\phi^{(1)}(j) = A^j \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} =: A^j \xi_1 \qquad \operatorname{dom} \phi^{(1)} = \mathbb{N}$$

$$\phi^{(2)}(j) = \begin{cases} \begin{pmatrix} -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix} =: \xi_2 & \text{if } j = 0 \\ \begin{pmatrix} -\frac{9}{8} \\ -\frac{3}{16} \end{pmatrix} & \text{if } j = 1 \end{cases} \qquad \operatorname{dom} \phi^{(2)} = \{0, 1\}$$

[1]As an alternative to uniform continuity, to ensure that $V \circ \phi$ converges to zero when ϕ approaches $\mathcal{A}$, one can assume that V is positive definite with respect to $\mathcal{A}$ and upper bounded by a continuous function positive definite with respect to $\mathcal{A}$. This is done in Corollary 3.

[2]Completeness of maximal solutions is guaranteed if and only if $G(D) \subset D$.

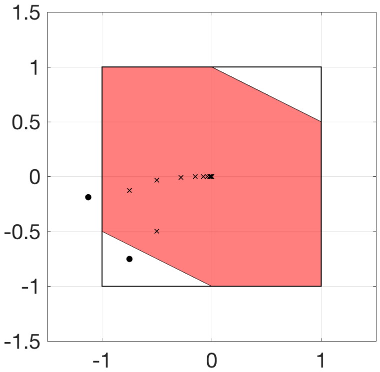

Fig. 1: The set $\mathcal{O}$ (red), the boundary of the set D (black), $\phi^{(1)}$ (crosses), and $\phi^{(2)}$ (circles) in Example 1.

both solutions are displayed in Fig. 1. Obviously, $\phi^{(1)}$ is complete and, A being Schur, one has $\lim_{j\to\infty}\phi^{(1)}(j)=0$. Therefore, according to Corollary 2, one has

$$\mathcal{J}(\xi_1)=\lim_{j\to\infty}\sum_{i=0}^{j-1}q_d(\phi^{(1)}(i))=\sum_{i=1}^{\infty}|A^{i-1}\xi_1|^2=V(\xi_1)$$

and this is confirmed by Fig. 2, where the evolution of the function $j\mapsto\widetilde{J}_{\phi^{(1)}}(j):=\sum_{i=0}^{j-1}q_d(\phi^{(1)}(i))$ is reported. Concerning $\phi^{(2)}$, again from Corollary 2, one has $J_{\phi^{(2)}}+V(A\xi_2)=V(\xi_2)$, which, due to $\mathcal{J}(\xi)=J_{\phi^{(2)}}$, implies $\mathcal{J}(\xi)=V(\xi_2)-V(A\xi_2)$, that is $\mathcal{J}(\xi)=|\xi_2|^2$.

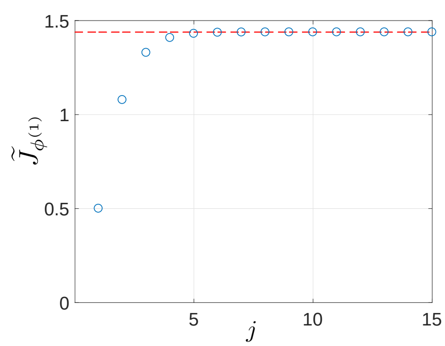

Fig. 2: The evolution of the function $j\mapsto\widetilde{J}_{\phi^{(1)}}(j)$ in Example 1.

A result similar to Proposition 2 holds for the case of linear difference inclusions. Such a result is given next.

Proposition 3: Let $\mathcal{A}=\{0\}$, $G(x)=\{A_1x,\ A_2x,\ \ldots,A_\nu x\}$ for each $x\in D$, with $A_1,A_2,\ldots,A_\nu\in\mathbb{R}^{n\times n}$, $\xi\in D$, and $x\mapsto q_d(x):=x^\mathsf{T}Qx$, where $Q\in\mathbb{S}^n_+$. Assume that there exists $P\in\mathbb{S}^n_+$ such that

$$\begin{aligned}\operatorname{diag}\{&A_1^\mathsf{T}PA_1-P-Q,A_2^\mathsf{T}PA_2-P-Q,\ldots\\&A_\nu^\mathsf{T}PA_\nu-P-Q\}\leq 0\end{aligned}\tag{12}$$

Then, there exists $V\colon\mathbb{R}^n\to\mathbb{R}$ continuous and positive definite with respect to $\mathcal{A}$ on $G(D)\cup D$ such that

$$\max_{i\in\{1,2,\ldots,\nu\}}V(A_ix)-V(x)+q_d(x)=0\qquad\forall x\in D$$

□

A necessary condition for (12) to be feasible for some $P\in\mathbb{S}^n_+$ is that the matrices $\{A_1,A_2,\ldots,A_\mu\}$ are simultaneously asymptotically stable; cf. [3].

Next, we showcase the applicability of the proposed result in the case of nonlinear dynamics.

Example 2: Let $\alpha,\beta,p,q\in\mathbb{R}_{>0}$, consider system (1) defined by the following data [10, Example 4.11]: $D:=[0,p]\times[0,q]$ and for all $x\in D$, $G(x)=\{g(x)\}$, where $x\mapsto g(x):=\left(\frac{\alpha x_2}{1+x_1^2},\frac{\beta x_1}{1+x_2^2}\right)$. Pick $x\mapsto q_d(x):=x^\mathsf{T}\left(\begin{smallmatrix}q_{11}&0\\0&q_{22}\end{smallmatrix}\right)x:=x^\mathsf{T}Qx$ with $q_{11},q_{22}\in\mathbb{R}_{>0}$. Define $x\mapsto V(x)=x^\mathsf{T}Px$, where $P\in\mathbb{S}^n_+$. Then, we want to compute P such that for each $x\in D$, $V(g(x))-V(x)\leq -q_d(x)$, that is

$$g(x)^\mathsf{T}Pg(x)-x^\mathsf{T}Px\leq -q_d(x)\qquad\forall x\in D\tag{13}$$

To this end, first notice that by defining

$$\widehat{G}(x):=\begin{pmatrix}0&\frac{\alpha}{1+x_1^2}\\\frac{\beta}{1+x_2^2}&0\end{pmatrix}\qquad\forall x\in D$$

for each $x\in D$, one has $g(x)=\widehat{G}(x)x$. Therefore, (13) can be rewritten as

$$x^\mathsf{T}\Delta(x)x\leq -q_d(x)\qquad\forall x\in D\tag{14}$$

with $\Delta(x):=\widehat{G}(x)^\mathsf{T}P\widehat{G}(x)-P$ for all $x\in D$. A sufficient condition for the above inequality to be satisfied is

$$\Delta(x)+Q\leq 0\qquad\forall x\in D\tag{15}$$

In particular, by denoting

$$P=\begin{pmatrix}p_{11}&p_{12}\\\bullet&p_{22}\end{pmatrix}\tag{16a}$$

one gets for each $x\in D$

$$\Delta(x)+Q=\begin{pmatrix}\frac{\beta^2p_{22}}{(x_2^2+1)^2}-p_{11}+q_{11}&\frac{\alpha\beta p_{12}}{(x_1^2+1)(x_2^2+1)}-p_{12}\\\bullet&\frac{\alpha^2p_{11}}{(x_1^2+1)^2}-p_{22}+q_{22}\end{pmatrix}$$

At this stage, if one enforces $p_{12}=0$, then (15) holds if

$$\left.\begin{matrix}\frac{\beta^2p_{22}}{(x_2^2+1)^2}-p_{11}+q_{11}\\\frac{\alpha^2p_{11}}{(x_1^2+1)^2}-p_{22}+q_{22}\end{matrix}\right\}\quad\forall x\in D\tag{16b}$$

which can be fulfilled by picking $p_{11},p_{22}\in\mathbb{R}_{>0}$ such that

$$\begin{aligned}&p_{11}\geq\max_{x\in D}\frac{\beta^2p_{22}}{(x_2^2+1)^2}+q_{11}=\beta^2p_{22}+q_{11}\\&p_{11}\leq\min_{x\in D}\frac{1}{\alpha^2}(p_{22}-q_{22})\left(x_1^2+1\right)^2\end{aligned}\tag{16c}$$

It can be easily checked that if $\alpha\beta<1$, then (16c) can be always fulfilled. Moreover, notice that due to q_d positive definite with respect to $\mathcal{A}$, the satisfaction of (13) ensures that maximal solutions to (1) converge to $\mathcal{A}$; see [6].

Now we show that under certain assumptions on $\alpha,\beta,p,$ and q, maximal solutions to (1) are complete. To this end, it suffices to select α, β, p, and q such that $G(D) \subset D$. Denote $x \mapsto (g_1(x), g_1(x)) := \left(\frac{\alpha x_2}{1+x_1^2}, \frac{\beta x_1}{1+x_2^2}\right)$ and observe that $g_1(D) \times g_2(D) = [0, \alpha q] \times [0, \beta p]$. Moreover, since $g(D) \subset g_1(D) \times g_2(D)$, it follows that if $\alpha q \leq p$ and $\beta p \leq q$, then $G(D) \subset D$.

For this example, we select $\alpha = \frac{1}{2}, \beta = 1, p = q = 5, q_{11} = q_{22} = 1$, which allows to enforce $G(D) \subset D$ and (16c). In particular, since as mentioned above maximal solutions to (1) converge to $\mathcal{A}$, from Corollary 1 one has that for each initial condition $\xi \in D$, $\mathcal{J}(\xi) \leq \xi^{\mathsf{T}} P \xi$, where $P \in \mathbb{S}_+^n$ is any matrix that satisfies (14). As mentioned earlier, to fulfill (14), P can be selected as in (16). In particular, one may operate a convenient selection of P so to minimize the conservatism in the upper bound on the cost $\mathcal{J}$ for each $x \in D$. This objective can be achieved by selecting $P = \operatorname{diag}\{p_{11}, p_{22}\}$ with $p_{11}, p_{22} \in \mathbb{R}_{>0}$ being the solution to the following optimization problem

$$\begin{aligned} &\underset{p_{11}, p_{22}}{\text{minimize}} && \max_{\xi \in D} \xi^{\mathsf{T}} P \xi \\ &\text{subject to} && p_{11} \geq \beta^2 p_{22} + q_{11} \\ & && p_{22} \leq \frac{1}{\alpha^2}(p_{22} - q_{22}), p_{11}, p_{22} \in \mathbb{R}_{>0} \end{aligned}$$

which turns out to be equivalent to the following optimization problem

$$\begin{aligned} &\underset{p_{11}, p_{22}}{\text{minimize}} && \max\{p_{11}, p_{22}\} \\ &\text{subject to} && p_{11} \geq \beta^2 p_{22} + q_{11} \\ & && p_{11} \leq \frac{1}{\alpha^2}(p_{22} - q_{22}), p_{11}, p_{22} \in \mathbb{R}_{>0} \end{aligned}$$

The solution $(p_{11}^\star, p_{22}^\star)$ to the above optimization problem can be easily computed by inspection of the feasible set. In particular, such a solution corresponds to the unique point belonging to

$$\Big\{(p_{11}, p_{22}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} : p_{11} = \beta^2 p_{22} + q_{11}, \quad p_{11} = \frac{1}{\alpha^2}(p_{22} - q_{22})\Big\}$$

which yields $(p_{11}^\star, p_{22}^\star) = (\frac{8}{3}, \frac{5}{3})$.

Let ϕ be the unique solution to (1) from $\xi = (0.5693, 1.6093)$, which converges to $\mathcal{A}$. Fig. 3 depicts the evolution of the cost function $j \mapsto \widetilde{J}_\phi(j) := \sum_{i=0}^{j-1} q_d(\phi(i))$, along with some upper bounds on the cost $\mathcal{J}(\xi)$ obtained via Corollary 1 through different selections of the matrix P in (13). The picture shows the relevance of the proposed optimization in reducing the conservatism in the upper bound of $\mathcal{J}(\xi)$.

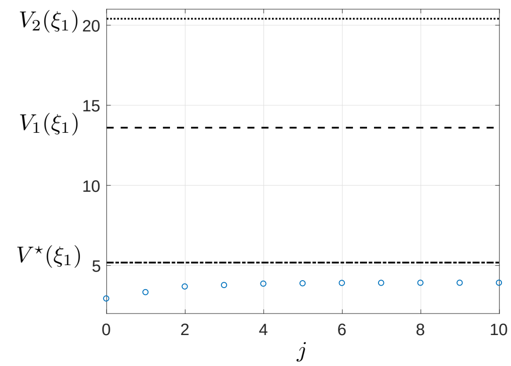

Fig. 3: The evolution of the function $j \mapsto \widetilde{J}_\phi(j)$ in Example 2 (blue bullets), along with different upper bounds on $\mathcal{J}(\xi)$ obtained for: $P = \operatorname{diag}\{p_{11}^\star, p_{22}^\star\}$ (dashed-dotted line), $P_1 = \operatorname{diag}\{10, 4\}$ (dashed line), and $P_2 = \operatorname{diag}\{15, 6\}$ (dotted line).

III. Towards Optimal Control with Asymptotic Terminal Constraints

Building on the results presented in the previous section, in this section we consider a particular optimal control problem for which a solution can be indirectly obtained by the use of a Lyapunov-like function, along with some additional conditions. The results contained in this section extend the ideas proposed in [1] to the case of constrained discrete-time systems. For simplicity, we assume the dynamics of the system being single valued. However, with the results in Section II-B, the extension to difference inclusions is straightforward.

A. Preliminaries on Controlled Constrained Difference Equations

We consider controlled constrained difference equations with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ of the form

$$\mathcal{D}\colon \quad x^+ = g(x, u) \qquad (x, u) \in D \tag{17}$$

where $x \in \mathbb{R}^n$, $D \subset \mathbb{R}^n \times \mathbb{R}^m$, and $g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Let $\mathcal{U}$ be the set of functions $u\colon \operatorname{dom} u \to \mathbb{R}^m$, with $\operatorname{dom} u = \mathbb{N}_0 \cap \{0, 1, \ldots, J\}$, for some $J \in \mathbb{N}_0 \cup \{\infty\}$. A solution pair (ϕ, u) to $\mathcal{D}$ is any $(\phi, u) \in \mathcal{X} \times \mathcal{U}$, for which $\operatorname{dom} \phi = \operatorname{dom} u$, $(\phi(0), u(0)) \in D$, and for all $j \in \operatorname{dom} \phi$ such that $j + 1 \in \operatorname{dom} \phi$

$$\begin{aligned} &(\phi(j), u(j)) \in D \\ &\phi(j+1) = g(\phi(j), u(j)) \end{aligned}$$

In particular, a solution pair (ϕ, u) to $\mathcal{D}$ is said to be *complete* if its domain $\operatorname{dom} \phi$ is unbounded and *maximal* if it is not the truncation of another solution pair.

Definition 1 (Set of maximal solution pairs): Given $\xi \in \mathbb{R}^n$, we denote by $\mathcal{S}_u(\xi)$ the set of maximal solution pairs $(\phi, u) \in \mathcal{X} \times \mathcal{U}$ to (17) such that $\phi(0) = \xi$.

Definition 2 (Set of maximal responses): Given $\xi \in \mathbb{R}^n$ and $u \in \mathcal{U}$, we denote the set of maximal responses by

$$\mathcal{R}(\xi, u) = \{\phi \in \mathcal{X} \colon (\phi, u) \in \mathcal{S}_u(\xi)\}$$

Definition 3 (Set of closed-loop maximal solutions): Given $\xi \in \mathbb{R}^n$ and a function $\kappa\colon \mathbb{R}^n \to \mathbb{R}^u$, we denote by $\mathcal{S}_\kappa(\xi)$ the set of maximal solutions ϕ to

$$x^+ = g(x, \kappa(x)) \quad (x, \kappa(x)) \in D \tag{18}$$

such that $\phi(0) = \xi$.

Let us define the projection of D onto $\mathbb{R}^n$ as $\Pi(D) = \{\xi \in \mathbb{R}^n \colon \exists u \in \mathbb{R}^m \text{ s.t. } (\xi, u) \in D\}$. Now, let $\mathcal{A} \subset \mathbb{R}^n$ be closed, consider the following sets:

$$\mathcal{X}_{\mathcal{A}} := \{x \in \mathcal{X} \colon \lim_{j \to \sup \operatorname{dom} x} |x(j)|_{\mathcal{A}} = 0\}$$

and $\mathcal{U}_{\mathcal{A}}(\xi) := \{u \in \mathcal{U} \colon \exists x \in \mathcal{R}(\xi, u) \cap \mathcal{X}_{\mathcal{A}}\}$. Essentially, $\mathcal{X}_{\mathcal{A}}$ is the set of sequences in $\mathcal{X}$ converging to $\mathcal{A}$, while $\mathcal{U}_{\mathcal{A}}(\xi)$, for each $\xi \in \mathbb{R}^n$, is the set of inputs such that the resulting response to (17) from ξ converges to $\mathcal{A}$. For each initial condition $\xi \in \mathbb{R}^n$ and $u \in \mathcal{U}_{\mathcal{A}}(\xi)$, consider the following cost[3]

$$\mathcal{J}_u(\xi, u) = \sum_{\phi \in \mathcal{R}(\xi,u), j=1}^{\sup \operatorname{dom} \phi} q_d(\phi(j-1), u(j-1)) \qquad (19)$$

where $q_d \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. Then, the following result can be established.

Theorem 2: Let $\mathcal{A} \subset \mathbb{R}^n$ be closed, $\xi \in \Pi(D)$, and $q_d \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Assume there exist functions $V \colon \mathbb{R}^n \to \mathbb{R}$ and $\kappa \colon \mathbb{R}^n \to \mathbb{R}^m$ such that V is uniformly continuous on a neighborhood of $\mathcal{A}$, $V(\mathcal{A}) = \{0\}$, and

$$\begin{array}{ll} V(g(x, \kappa(x))) - V(x) + q_d(x, \kappa(x)) = 0 & \forall (x, \kappa(x)) \in D \\ V(g(x, u)) - V(x) + q_d(x, u) \geq 0 & \forall (x, u) \in D \end{array} \qquad (20)$$

Furthermore, assume that the unique $\phi_\kappa \in \mathcal{S}_\kappa(\xi)$ is such that

$$\lim_{j \to \sup \operatorname{dom} \phi_\kappa} |\phi_\kappa(j)|_{\mathcal{A}} = 0$$

Then, one gets

$$\min_{u \in \mathcal{U}_{\mathcal{A}}(\xi)} \mathcal{J}_u(\xi, u) = V(\xi)$$

where the above minimum exists. In particular, κ is the optimal control, in the sense that

$$\min_{u \in \mathcal{U}_{\mathcal{A}}(\xi)} \mathcal{J}_u(\xi, u) = \sum_{\phi_\kappa \in \mathcal{S}_\kappa(\xi), j=1}^{\sup \operatorname{dom} \phi_\kappa} q_d(\phi_\kappa(j-1), \kappa(\phi_\kappa(j-1)))$$

□

Remark 2: For each $x \in \mathbb{R}^n$, define the following set-valued map $\Pi_u(x, D) := \{u \in \mathbb{R}^m \colon (x, u) \in D\}$. Then, it follows that (20) is equivalent to

$$0 = \min_{u \in \Pi_u(x, D)} \left(V(g(x, u)) - V(x) + q_d(x, u)\right) \quad \forall x \in \Pi(D)$$

which is a state-and-input constrained version of the steady state discrete-time Hamilton-Jacobi-Bellman equation; see [2].

The applicability of Theorem 2 requires the feedback law κ to induce convergence towards the set $\mathcal{A}$. On the other hand, it is worthwhile to observe that when q_d is positive definite with respect to $\mathcal{A}$, one can add further assumptions on V so that (20) implies convergence of maximal solutions towards the set $\mathcal{A}$. This aspect is illustrated in the result given next.

Corollary 3: Let $\mathcal{A} \subset \mathbb{R}^n$ be closed, q_d be positive definite with respect to $\mathcal{A}$ on D. Assume that there exist functions $V \colon \mathbb{R}^n \to \mathbb{R}, \kappa \colon \mathbb{R}^n \to \mathbb{R}^m, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$, (20) holds, and

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(D) \cup g(D)$$

Furthermore, assume that the following conditions are satisfied

$$\begin{array}{ll} \kappa(x) \in \{u \in \mathbb{R}^m \colon (x, u) \in D\} & \forall x \in \Pi(D) \\ \kappa(x) \in \{u \in \mathbb{R}^m \colon (g(x, u), u) \in D\} & \forall x \in \Pi(D) \end{array}$$

Then, for each $\xi \in \Pi(D)$, $\min_{u \in \mathcal{U}_{\mathcal{A}}(\xi)} J(\xi, u) = V(\xi)$, where such a minimum exists. In particular, κ is the optimal control, in the sense that

$$\min_{u \in \mathcal{U}_{\mathcal{A}}(\xi)} \mathcal{J}_u(\xi, u) = \sum_{\phi_\kappa \in \mathcal{S}_\kappa(\xi), j=1}^{\sup \operatorname{dom} \phi_\kappa} q_d(\phi_\kappa(j-1), \kappa(\phi_\kappa(j-1)))$$

□

IV. Conclusion

In this paper we addressed cost evaluation and optimal control problems for constrained difference inclusions. The results are obtained via a Lyapunov-like approach and provide sufficient conditions to solve a meaningful optimal control problem for which completeness and boundedness of maximal solutions are not required. The approach we pursue sets this paper apart from the literature of model predictive control, where the main focus is to provide constructive design methods for optimal feedback.

Future research directions include the use of inverse optimality approaches to allow for the design of optimal control laws, as well as the extension of the proposed approach to discrete-time dynamical games in the spirit of [8]. Moreover, the extension to the case of hybrid dynamical systems in the framework [6], which is currently part of our work.

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[3]Notice that, due to g being single valued, for any $\xi \in \mathbb{R}^n$, given $u \in \mathcal{U}_{\mathcal{A}}(\xi)$, the set $\mathcal{R}(\xi, u)$ is a singleton. This implies that the cost J_u is univocally determined once u and ξ are given.