On Robustness of Pre-Asymptotic Stability to Delayed Jumps in Hybrid Systems

Berk Altın and Ricardo G. Sanfelice

***Abstract*— We show that pre-asymptotic stability of a compact set for a hybrid system is semiglobally and practically robust in the presence of delayed jumps under mild conditions on the data. More precisely, when the delay-free system has a pre-asymptotically stable compact set, it is shown that for small enough delays, solutions of the delayed system converge to a neighborhood of a set of interest related to the aforementioned compact set. Unlike prior work, this notion of practical stability also holds for time-varying delays in the presence of Zeno solutions. Simulation results of a state estimator with intermittent and delayed information validate the findings.**

I. Introduction

For hybrid systems in the framework of [1], where a hybrid system is described by a combination of constrained differential and difference inclusions, we study the effects of delays on stability properties. The modeling approach adopted in [1] encapsulates a diverse set of related frameworks such as hybrid automata, impulsive differential equations, and switching systems, and emphasizes robustness of asymptotic stability to external perturbations under standard regularity conditions. The objective of this paper is to extend these results by scrutinizing the effects of a class of delays.

The present work is motivated largely by cyber-physical systems with delay phenomena arising from communication constraints and computational limitations, and focuses on the robustness properties of asymptotically stable hybrid dynamical systems in the presence of delays on events, or jumps. As an example, consider a continuous-time control system with state $x_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}}$ and input $u \in \mathbb{R}^{n_{\mathrm{c}}}$, evolving according to the vector field $f : \mathbb{R}^{n_{\mathrm{p}}} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{p}}}$. Then, given a state-feedback law $\kappa : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{p}}}$ and a sampling period $T_{\mathrm{s}} > 0$, a sample-and-hold implementation of κ can be modeled as a hybrid system by treating u as a state variable and introducing the sampling timer $\tau_{\mathrm{s}} \in [0, T_{\mathrm{s}}]$. In particular, the state $x = \begin{bmatrix} x_{\mathrm{p}}^\top & \tau_{\mathrm{s}}^\top & u^\top \end{bmatrix}^\top$ of the sampled-data control system evolves according to the continuous dynamics

$$\dot{x} = F(x) := \begin{bmatrix} f(x_{\mathrm{p}}, u) \\ -1 \\ 0 \end{bmatrix} \quad x \in C := \mathbb{R}^{n_{\mathrm{p}}} \times [0, T_{\mathrm{s}}] \times \mathbb{R}^{n_{\mathrm{c}}}, \tag{1}$$

and the discrete dynamics

$$x^+ = G(x) := \begin{bmatrix} x_{\mathrm{p}} \\ T_{\mathrm{s}} \\ \kappa(x_{\mathrm{p}}) \end{bmatrix} \quad x \in D := \mathbb{R}^{n_{\mathrm{p}}} \times \{0\} \times \mathbb{R}^{n_{\mathrm{c}}}, \tag{2}$$

describing the following principles: When $\tau_{\mathrm{s}} \in [0, T_{\mathrm{s}}]$, the timer variable counts down at a constant rate of 1 until it reaches zero, at which point it resets to T_{s}. In the intersample period when τ_{s} flows, the plant state evolves according to the differential equation $\dot{x}_{\mathrm{p}} = f(x_{\mathrm{p}}, u)$, while u is kept constant. When the timer τ_{s} resets to T_{s}, the input u is updated to $\kappa(x_{\mathrm{p}})$.

Under appropriate assumptions, robustness results in [1] help show that the system represented by (1) and (2), can not only tolerate disturbances, noise, and uncertainties on f, but also time-varying uncertainties on the sampling times. However, in practice, there is a strictly positive amount of time between the sampling timer update event $\tau_{\mathrm{s}}^+ = T_{\mathrm{s}}$ and the control update event $u^+ = \kappa(x_{\mathrm{p}})$ due to computational limitations. Due to the piecewise constant evolution of u, as in [2], the sampled-data control system with delays on the update of u can be described by a higher-order hybrid system. We generalize this approach to analyze the effects of delayed jumps on hybrid systems by invoking results from [1].

Although there have been a number of works studying delays in the hybrid systems setting (for example, [3], [4], [5]), existing literature fails to capture the generality associated with the hybrid inclusions formalism [1], and concentrate on specific hybrid models (e.g. impulsive systems and/or switching systems). With the observation that many (hybrid) controllers are designed and analyzed in a delay-free setting, our primary aim is to establish inherent (or nominal) robustness of hybrid inclusions against delayed jumps. While the valuable work extending hybrid inclusions to the delayed case in recent articles [6], [7] presents an opportunity to study the effects of delay in a more general sense, the sufficient conditions for robustness in the *hybrid systems with memory* framework imposes requirements on the system data with respect to the integrated set distance on the underlying infinite-dimensional space, which can be hard to check. On the contrary, the conditions we impose in this paper are commonly used to certify robustness with respect to a very general class of perturbations. The significance of our main result, though semiglobal and practical, is brought out by the fact that it holds for the case of time-varying delays, without any dwell-time restrictions.

The remainder of the paper is organized as follows. Section II introduces the necessary background on hybrid

*This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484, Grant no. ECS-1710621, and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, by the Air Force Research Laboratory under Grant no. FA9453-16-1-0053, and by CITRIS and the Banatao Institute at the University of California.

Berk Altın and Ricardo G. Sanfelice are with the Department of Computer Engineering, University of California, Santa Cruz, CA 95064, USA; berkaltin@ucsc.edu, ricardo@ucsc.edu.

systems in the inclusions framework. Section III presents a higher-order model meant to capture the behavior of hybrid systems experiencing delays on jumps, parametrized by the maximum length of delays, and establishes basic properties of this model. Robustness of pre-asymptotic stability of the higher-order model in the presence of delays is shown in Section IV, in the semiglobal practical sense. This property is illustrated with a numerical example in Section V. Concluding remarks are given in Section VI. Due to space constraints, proofs of the technical results, among other content, are not included and will be published in another venue.

II. Background

We denote by $\mathbb{B}$ the closed unit ball at the origin in Euclidean space of appropriate dimension. We use $\mathbb{R}$ to represent real numbers, $\mathbb{R}_{\geq 0}$ its nonnegative and $\mathbb{R}_{>0}$ its positive subsets. The set of nonnegative integers is denoted $\mathbb{N}$. The Euclidean norm (2 norm) is denoted $|.|$. For a pair of sets S_1 and S_2, $S_1 \subset S_2$ indicates that S_1 is a subset of S_2, not necessarily proper. The distance of a vector $x \in \mathbb{R}^n$ to a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x-a|$. Let $S \subset \mathbb{R}^n$. The notation $S + \delta\mathbb{B}$ indicates the set of all $x \in \mathbb{R}^n$ such that $|x - s| \leq \delta$ for some $s \in S$. The closure and convex hull of S are denoted $\operatorname{cl} S$ and $\operatorname{con} S$, respectively. The domain of a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the set $\operatorname{dom} H = \{x \in \mathbb{R}^n : H(x) \neq \varnothing\}$. We denote by $\pi_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ the standard projection onto $\mathbb{R}^{n_i}$ such that $\pi_i(x) = x_i$, $i \in \{1, 2\}$. Similarly, we denote by $\pi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the standard projection onto $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ so that $\pi(x_1, x_2, y) = (x_1, x_2)$ for any $(x_1, x_2, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m$. Given $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, a strictly increasing continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ if $\alpha(0) = 0$. Similarly, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{s \to 0} \beta(s, t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$, and $\lim_{t \to \infty} \beta(s, t) = 0$ for all $s \in \mathbb{R}_{\geq 0}$. The zero vector in $\mathbb{R}^n$ is denoted $\{0_n\}$, or simply 0 when appropriate. Given vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $(x, y) = \begin{bmatrix} x^\top & y^\top \end{bmatrix}^\top$. Finally, given $a, b \in \mathbb{N}$, the set $\{a, a+1, \ldots, b\}$ is defined to be the singleton $\{a\}$ if $a = b$, and the empty set if $a > b$. Similarly, sequences and sets of the form $\{x_{k_i}\}_{i=a}^{b}$ and $\cup_{i=a}^{b} \{x_{k_i}\}$ are defined to be empty if $a > b$.

This paper considers hybrid systems in the framework introduced in [1], uniquely identified by the 4-tuple (C, F, D, G), called the data of the hybrid system. Hence, a hybrid system $\mathcal{H}$ is defined as $\mathcal{H} := (C, F, D, G)$, and described in the following form:

$$\mathcal{H} \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{3}$$

The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is referred to as the *flow map*, and it describes the continuous evolution of the state $x \in \mathbb{R}^n$ on the *flow set* $C \subset \mathbb{R}^n$. Similarly, the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the *jump map*, which describes the discrete evolution of x on the *jump set* $D \subset \mathbb{R}^n$. Solutions of $\mathcal{H}$ are parametrized by the pair $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the ordinary time keeping track of the continuous evolution (flows), and j is the jump time/index keeping track of the number of jumps. The domain $\operatorname{dom} \chi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ of a solution χ to $\mathcal{H}$ is a *hybrid time domain*, which means that for all $(t^*, j^*) \in \operatorname{dom} \chi$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{j^*+1}$ with $t_0 = 0$ such that

$$\operatorname{dom} \chi \cap ([0, t^*] \times \{0, 1, \ldots, j^*\}) = \bigcup_{j=0}^{j^*} [t_j, t_{j+1}] \times \{j\}.$$

A solution is called bounded if its range is bounded, maximal if its domain cannot be extended, and complete if its domain is unbounded. It is called eventually continuous if $J := \sup\{j \in \mathbb{N} : \exists (t, j) \in \operatorname{dom} \chi\}$ is finite and there exist at least two points belonging to the set $\operatorname{dom} \chi \cap (\mathbb{R}_{\geq 0} \times \{J\})$. The notation $\mathcal{S}_{\mathcal{H}}(S)$ indicates the set of all maximal solutions χ to $\mathcal{H}$ originating from S (i.e., $\chi(0, 0) \in S$). The set of all maximal solutions, i.e., $\mathcal{S}_{\mathcal{H}}(\mathbb{R}^n)$, is simply denoted $\mathcal{S}_{\mathcal{H}}$. Given a solution $\chi : \operatorname{dom} \chi \to \mathbb{R}^n$ to $\mathcal{H}$, we denote by $\operatorname{card}(\chi, t)$ the cardinality of the set $\operatorname{dom} \chi \cap (\{t\} \times \mathbb{N})$. Further details on the solution concept can be found in [1].

Definition 2.1: For a hybrid system $\mathcal{H}$ given by (3), the closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that each solution χ of $\mathcal{H}$ with $|\chi(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|\chi(t, j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \operatorname{dom} \chi$. It is said to be pre-asymptotically stable if it is stable and there exists $\sigma > 0$ such that each solution χ of $\mathcal{H}$ with $|\chi(0, 0)|_{\mathcal{A}} \leq \sigma$ is bounded and satisfies $\lim_{t+j \to \infty} |\chi(t, j)|_{\mathcal{A}} = 0$ when complete.

Finally, we refer the readers to [1] for two concepts for set-valued mappings that play significant roles in our analysis, local boundedness and outer semicontinuity.

III. Modeling of Delayed Jumps in Hybrid Systems

This section details the construction of a hybrid system modeling delayed jumps for $\mathcal{H}$. The constructed system, denoted $\mathcal{H}'_T$, depends on the parameter $T \geq 0$ meant to capture the maximum length of delays. When $T = 0$, this construction can be viewed as a redundant, higher-order representation of the delay-free system $\mathcal{H}$. The following assumptions are used throughout the paper.

Assumption 3.1: The following are true for the data of $\mathcal{H}$:

(H1) The sets C and D are closed.

(H2) The flow map F is locally bounded and outer semicontinuous relative to C, and $C \subset \operatorname{dom} F$. Further, for each $x \in C$, the set $F(x)$ is convex.

(H3) The jump map G is locally bounded and outer semicontinuous relative to D, and $D \subset \operatorname{dom} G$.

Conditions (H1)-(H3) are a mild set of regularity assumptions on the data guaranteeing *well-posedness* [1, Definition 6.29, Theorem 6.30] of a hybrid system, called *hybrid basic conditions*. Well-posedness is exploited via the hybrid basic conditions to conclude robustness against delayed jumps. When F (respectively, G) is single-valued and continuous on C (respectively, D), condition (H2) (respectively, (H3)) is automatically satisfied.

To construct the high-dimensional model subject to delays on jumps, we introduce the decomposition of the plant state as $x := (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ for some $n_2 > 0$, where $n_1 + n_2 = n$, and x_2 represents those components of the state subject to delayed jumps. This formulation is motivated by cyber-physical systems, wherein measurements and/or control inputs might be subject to computational delays, while the physical plant state evolves delay-free.

Example 3.2: Consider the sampled-data control system discussed in Section I. Since the input $u \in \mathbb{R}^{n_c}$ is subject to delays, while the plant state $x_p \in \mathbb{R}^{n_p}$ and the sampling timer $\tau_s \in [0, T_s]$ evolve without any delays, the closed-loop state can be partitioned so that $x_1 = (x_p, \tau_s)$ and $x_2 = u$, with $n_1 = n_p + 1$ and $n_2 = n_c$. It is easy to show that the data of the hybrid system modeling the closed-loop system in (1) and (2) satisfies Assumption 3.1 when f and κ are continuous, as C and D are closed, and F and G are single-valued.

A. Higher-Order Modeling of Jump Delays

Let $G_i := \pi_i \circ G$ for each $i \in \{1, 2\}$, where $\circ$ denotes composition, and let $\widehat{G} : D \rightrightarrows \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ be such that $\widehat{G}(x) := G_1(x) \times \{x_2\} \times G_2(x)$. Furthemore, let $M := G_2(D)$. To ensure well-posedness of the delayed hybrid system $\mathcal{H}'_T$ to be introduced, as well as completeness of solutions for the specific case of $T = 0$, the following conditions will be enforced.

Assumption 3.3: The following conditions are true for the data (C, F, D, G) of $\mathcal{H}$:

(D1) The sets $\pi(\widehat{G}(D))$ and C satisfy $\pi(\widehat{G}(D)) \subset C$.

(D2) The set $M = G_2(D)$ is closed.

Condition (D1) is utilized to ensure that solutions can continue to flow until an active delay expires, by aid of an appropriate perturbation of the continuous dynamics, since it implies $G_1(x) \times \{x_2\} \subset C$ for all $x \in D$. It holds if $D \subset C$ and there exist sets $C_1 \subset \mathbb{R}^{n_1}$ and $C_2 \subset \mathbb{R}^{n_2}$ such that $G(D) \subset C = C_1 \times C_2$, as is the case with Example 3.2: for the sampled-data control system, we have $C = C_1 \times \mathbb{R}^{n_2}$ with $C_1 := \mathbb{R}^{n_1 - 1} \times [0, T_s]$, $D = \{x \in C : \tau_s = 0\}$, and $G(D) = \{x \in C : \tau_s = T_s\}$. Alternatively, it holds when $D \subset C$ and $n_1 = 0$, or simply when $C = \mathbb{R}^n$. The second condition of Assumption 3.3, (D2), is easily satisfied when D is compact and G is outer semicontinuous. While the jump set is not compact (as f is defined globally) for the sampled-data control system, it can satisfy (D2) in many cases; for instance, since $G_2(x) = \kappa(x_p)$ for all $x \in D$, when κ is linear, $G_2(D) = \kappa(\mathbb{R}^{n_p})$ is a linear subspace.

Next, given any $T > 0$, and a pair of class-$\mathcal{K}$ functions $\alpha_C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\alpha_F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we define the set C_T and set-valued mapping $F_T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightrightarrows \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ as follows:

$$C_T := \{x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (x + \alpha_C(T)\mathbb{B}) \cap C \neq \varnothing\} = C + \alpha_C(T)\mathbb{B},$$

$$F_T(x) := \mathrm{cl}(\mathrm{con}(F((x + \alpha_F(T)\mathbb{B}) \cap C))) + \alpha_F(T)\mathbb{B} \quad \forall x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

The set C_T (respectively, the mapping F_T) is an outer perturbation of the set C (respectively, the mapping F), and will allow solutions subject to delayed jumps to flow during the delay period. The functions α_C and α_F can be chosen to ensure that $\mathrm{dom}\, F_T \supset C_T$, and solutions to $\mathcal{H}'_T$ originating from certain subsets of C can flow for T units of time for small enough T. To make our construction precise, we let $z := (x, \mu, \tau) \in \mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}$, and introduce the jump map

$$G'_T(z) := \begin{cases} \widehat{G}(x) \times [0, T] & z \in D \times M \times \{-1\} \\ (x_1, \mu, \mu, -1) & z \in C_T \times M \times \{0\}, \end{cases} \tag{4}$$

along with the flow map

$$F'_T(z) := F_T(x) \times \{0_{n_2}\} \times \{-\min\{\tau + 1, 1\}\} \quad \forall z \in C'_T, \tag{5}$$

where the flow set is given as

$$C'_T := C_T \times M \times (\{-1\} \cup [0, T]). \tag{6}$$

Letting

$$D'_T := (D \times M \times \{-1\}) \cup (C_T \times M \times \{0\}), \tag{7}$$

the delayed system is denoted $\mathcal{H}'_T := (C'_T, F'_T, D'_T, G'_T)$. The hybrid system $\mathcal{H}'_T$ with the state $z = (x, \mu, \tau)$ is subject to the following:

- When x belongs to the jump set D and there is no active delay, i.e., $\tau = -1$, a jump records the delay-free "post-jump" value of x_2, which belongs to the set $G_2(x)$, in the memory state μ, and activates the delay timer.
- When the delay timer expires, i.e., $\tau = 0$, a jump deactivates the delay dynamics and updates the value of x_2 to μ.

For notational simplicity, from hereinafter, the case of $T = 0$ for $\mathcal{H}'_T$ will be denoted $\mathcal{H}' := (C', F', D', G')$, where $C' := C'_0$, $F' :\equiv F'_0$, $D' := D'_0$, and $G' :\equiv G'_0$. Note that $\mathcal{H}'_T$ is a perturbation of the delay-free system $\mathcal{H}'$, where the perturbation is derived from the definition of a generic perturbed hybrid system [1, Definition 6.27]. Given α_C and α_F, the perturbed system has the important property that for any $T_1, T_2 \in \mathbb{R}_{\geq 0}$ such that $T_1 \leq T_2$, every solution ζ of $\mathcal{H}'_{T_1}$ is a solution of $\mathcal{H}'_{T_2}$. Since our objective is to certify robustness against delayed jumps, this perturbation is somewhat specific in the sense that it is derived from a perturbation on only the flow dynamics of $\mathcal{H}$.

It is worth noting that the augmented jump map introduced in (4) encodes a "sequential" execution of jumps. That is, it does not allow solutions to jump due to the plant state x reaching D during an active delay, i.e., when $\tau \in [0, T]$. The modeling decision here is justified for small enough T when the ordinary time interval between jumps is uniformly lower bounded by a positive constant over the set of maximal solutions $\mathcal{S}_{\mathcal{H}}$. This fact is obvious for the sampled-data control system discussed in Example 3.2 since jumps are separated by T_s units of time, but also holds semiglobally for the generic hybrid system $\mathcal{H}$ under Assumption 3.1 provided it has a pre-asymptotically stable compact set, and $G(D)\cap D$ is empty, i.e., consecutive jumps are not allowed. If the set $G(D)\cap D$ is nonempty, the model can be justified by the implicit assumption that $\mathcal{H}$ arises from the interconnection of two hybrid (control) systems.

The following proposition shows that $\mathcal{H}'$ is a higher-order representation of $\mathcal{H}$, thereby justifying our study of $\mathcal{H}'$ (and by extension $\mathcal{H}'_T$, as it is a perturbation of $\mathcal{H}'$) in order to assess the robustness of the original hybrid system $\mathcal{H}$ against delays. Essentially, it states that for every solution χ to $\mathcal{H}$, there exists a solution ζ to $\mathcal{H}'$ with initial condition $(\chi(0,0), \mu, -1)$ for some μ satisfying the following:

- It flows when χ flows, and jumps twice each time χ jumps.
- During flows at ordinary time t, the plant state and the timer state components of ζ equal $\chi(t,j)$ and -1, respectively, where j is the unique integer so that $(t,j),(s,j)\in\operatorname{dom}\chi$ for some $s\neq t$.
- Before a pair of consecutive jumps by ζ occurring at ordinary time t, corresponding to the $(j+1)$-th jump by χ, the plant state and the timer state components of ζ equal $\chi(t,j)$ (the "pre-jump" state) and -1, respectively. Similarly, after a pair of consecutive jumps by ζ occurring at ordinary time t corresponding to the $(j+1)$-th jump by χ, the plant state and the timer state components of ζ equal $\chi(t,j+1)$ (the "post-jump" state) and -1, respectively.

Proposition 3.4: Suppose condition (D1) of Assumption 3.3 holds. Given $\chi\in\mathcal{S}_{\mathcal{H}}$, let $\{t_j\}_{j=1}^{J}$ be such that $(t_j,j),(t_j,j-1)\in\operatorname{dom}\chi$ for all $j\in\{1,2,\dots,J\}\cap\mathbb{N}$, where $J:=\sup\{j\in\mathbb{N}:\exists(t,j)\in\operatorname{dom}\chi\}$, and define

$$E:=\{(t,j)\in\mathbb{R}_{\geq 0}\times\mathbb{N}:(t,j/2)\in\operatorname{dom}\chi\} \bigcup\left(\bigcup_{j=1}^{J}\{(t_j,2j-1)\}\right).$$

Then, E is a hybrid time domain. Furthermore, the function $\zeta:\operatorname{dom}\zeta\to\mathbb{R}^n\times\mathbb{R}^{n_2}\times\mathbb{R}$, where $\operatorname{dom}\zeta=E$, defined as $\zeta(t,0):=(\chi(t,0),\mu,-1)$ for all $(t,0)\in\operatorname{dom}\zeta$ for some $\mu\in M$,

$$\zeta(t,j):=(\chi(t,j/2),\pi_2(\chi(t_{j/2},j/2)),-1)$$

for all $(t,j)\in\operatorname{dom}\zeta$ with $j\geq 1$ and even, and

$$\zeta(t,j):=(\pi_1(\chi(t,(j+1)/2)),\pi_2(\chi(t,(j-1)/2)),\pi_2(\chi(t,(j+1)/2)),0)$$

for all $(t,j)\in\operatorname{dom}\zeta$ with j odd, is a maximal solution to $\mathcal{H}'$.

B. Basic Properties of the Delayed Hybrid System

Having shown $\mathcal{H}'$ as a higher-order representation of $\mathcal{H}$, we will now proceed to establish basic properties of the delayed system $\mathcal{H}'_T$ that will be used later for stability and robustness analysis. We begin with an analog of Proposition 3.4, which shows that every maximal solution of $\mathcal{H}'$ originating from the set $(\operatorname{cl}(C)\cup D)\times M\times\{-1\}$ flows, or jumps an even number of times. For each such solution, there exists a unique corresponding maximal solution of $\mathcal{H}$. This result plays a key role in establishing pre-asymptotic stability of an appropriately constructed set for $\mathcal{H}'$.

Lemma 3.5: Suppose condition (D1) of Assumption 3.3 holds. Let $\zeta\in\mathcal{S}_{\mathcal{H}'}((\operatorname{cl}(C)\cup D)\times M\times\{-1\})$. Then, for each $t\geq 0$, $\operatorname{card}(\zeta,t)$ is odd if it is finite and nonzero. Moreover, the set $E:=\{(t,j)\in\mathbb{R}_{\geq 0}\times\mathbb{N}:(t,2j)\in\operatorname{dom}\zeta\}$ is a hybrid time domain, and the function $\chi:\operatorname{dom}\chi\to\mathbb{R}^n$, where $\operatorname{dom}\chi=E$, defined as $\chi(t,j):=\pi(\zeta(t,2j))$ for all $(t,j)\in\operatorname{dom}\chi$, is a maximal solution to $\mathcal{H}$ in (3).

The following lemma shows that by insisting on the set M to be closed and carefully selecting the functions α_C,α_F, well-posedness of $\mathcal{H}$ can be extended to the augmented system $\mathcal{H}'_T$ under the hybrid basic conditions.

Lemma 3.6: For any $T\geq 0$, the augmented hybrid system $\mathcal{H}'_T$ satisfies the hybrid basic conditions (and is therefore well-posed) if $\alpha_C(T)\leq\alpha_F(T)$, and conditions (H1)-(H3) of Assumption 3.1 and condition (D2) of Assumption 3.3 hold.

Finally, we show that the stability properties of the hybrid system $\mathcal{H}$ are preserved under the state augmentation leading to the delay-free system $\mathcal{H}'$. Hence, we assume the existence of a pre-asymptotically stable set for $\mathcal{H}$.

Assumption 3.7: There exists a pre-asymptotically stable nonempty compact set $\mathcal{A}\subset\mathbb{R}^n$ for the hybrid system $\mathcal{H}$ in (3).

In preparation for this result, it is necessary to construct a high dimensional set embedding $\mathcal{A}$ into $\mathbb{R}^n\times\mathbb{R}^{n_2}\times\mathbb{R}$ in an appropriate manner, so that stability properties of $\mathcal{H}$ can extend to the delay-free hybrid system $\mathcal{H}'$. Specifically, we define this set as

$$\mathcal{A}':=\underbrace{(\mathcal{A}\times M'\times\{-1\})}_{\mathcal{A}'_{-1}}\cup\underbrace{(\widehat{G}(\mathcal{A}\cap D)\times\{0\})}_{\mathcal{A}'_0},$$

where $M'\subset\mathbb{R}^{n_2}$ is nonempty. Intuitively, it is easy to see that the set $\mathcal{A}'$ should be pre-asymptotically stable under reasonable conditions: solutions to $\mathcal{H}'$ originating from $\mathcal{A}'_{-1}$ should flow in $\mathcal{A}'_{-1}$ and jump onto $\mathcal{A}'_0$, while solutions originating from $\mathcal{A}'_0$ should jump onto $\mathcal{A}'_{-1}$ when M' contains $\pi_2(\mathcal{A})$.

Proposition 3.8: Suppose that conditions (H1) and (H3) of Assumption 3.1, and Assumptions 3.3 and 3.7 hold. Furthermore, assume that the set M' is closed and satisfies $M'\supset\pi_2(\mathcal{A})$. Then, the set $\mathcal{A}'$ is stable for the hybrid system $\mathcal{H}'$. If, in addition, either of the following conditions hold, then $\mathcal{A}'$ is pre-asymptotically stable for $\mathcal{H}'$:

(S1) There exists $\gamma>0$ such that if $\chi\in\mathcal{S}_{\mathcal{H}}(\mathcal{A}+\gamma\mathbb{B})$ is complete, either it is not eventually continuous, or there exist $(t_1,j_1),(t_2,j_2)\in\operatorname{dom}\chi$ so that $\chi(t_1,j_1)\in\mathcal{A}$ and $j_2>j_1$.

(S2) The sets M' and M satisfy $M'\supset M$.

The formal proof of this proposition is involved and relies on an upper semicontinuity-like property of the jump map G', along with a general stability result targeted towards a class of hybrid systems that are "stable in the absence of certain events". The main idea is to show, using the relationship between solutions of $\mathcal{H}'$ and $\mathcal{H}$ stated in Lemma 3.5, that given any $\zeta \in \mathcal{S}_{\mathcal{H}'}$, the solution converges to $\mathcal{A}'$ on the subset of its domain where $\tau = -1$, i.e., where "delays" are inactive. A straightforward way of satisfying the conditions on M' outlined in Proposition 3.8 to achieve pre-asymptotic stability is to take $M' = \pi_2(\mathcal{A}) \cup M$.

IV. Robustness against Delays in the Semiglobal Practical Sense

The goal of this section is to show that pre-asymptotic stability of the delay-free augmented hybrid system $\mathcal{H}'$ is robust in a semiglobal and practical sense, with respect to the perturbations described via the family of delayed hybrid systems $\mathcal{H}'_T$. This notion of robustness is called semiglobal practical robust $\mathcal{KL}$ pre-asymptotic stability [1, Definition 7.18]. It guarantees that for any compact subset of the basin of pre-attraction and any "acceptable" error level $\epsilon > 0$, there exists a maximum length of delay $T > 0$ for which pre-asymptotic stability is preserved, practically. To establish this result, it is necessary to note that when $\mathcal{H}'$ is well-posed and the set $\mathcal{A}'$ is compact, Proposition 3.8 can equivalently be stated in terms of uniform bounds given by class-$\mathcal{KL}$ functions, which follows via [1, Theorem 7.12]. In stating the results of this section, we rely on the basin of pre-attraction $\mathcal{B}^{\mathrm{p}}_{\mathcal{A}'}$ of $\mathcal{A}'$ for $\mathcal{H}'$, along with the continuous function $\omega : \mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\omega(x) := |x|_{\mathcal{A}'} / \left(|x|_{(\mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}) \setminus \mathcal{B}^{\mathrm{p}}_{\mathcal{A}'}} \right) \quad \forall x \in \mathcal{B}^{\mathrm{p}}_{\mathcal{A}'},$$

which is a proper indicator of $\mathcal{A}'$ on $\mathcal{B}^{\mathrm{p}}_{\mathcal{A}'}$ [1, Page 145].

Proposition 4.1: Suppose that Assumptions 3.1, 3.3, and 3.7 hold. Furthermore, assume that the set M' is compact and satisfies $M' \supset \pi_2(\mathcal{A})$, and at least one of conditions (S1)-(S2) hold. Then, the basin of pre-attraction $\mathcal{B}^{\mathrm{p}}_{\mathcal{A}'}$ of $\mathcal{A}'$ for $\mathcal{H}'$ is open, and there exists a class-$\mathcal{KL}$ function β such that for every $\zeta \in \mathcal{S}_{\mathcal{H}'}(\mathcal{B}^{\mathrm{p}}_{\mathcal{A}'})$

$$\omega(\zeta(t,j)) \leq \beta(\omega(\zeta(0,0)), t+j) \quad \forall (t,j) \in \operatorname{dom} \zeta. \tag{8}$$

Using Proposition 4.1, our main result can be stated as follows.

Theorem 4.2: Suppose that the conditions of Proposition 4.1 hold, and $\alpha_C(T) \leq \alpha_F(T)$ for all $T \in [0, T^*]$ for some $T^* \in \mathbb{R}_{\geq 0}$. Consider the class-$\mathcal{KL}$ function β of Proposition 4.1 satisfying (8) *and the basin of pre-attraction $\mathcal{B}^{\mathrm{p}}_{\mathcal{A}'}$ of $\mathcal{A}'$ for $\mathcal{H}'$. Then, for every*

1) compact set $K' \subset \mathcal{B}^{\mathrm{p}}_{\mathcal{A}'}$, and

2) scalar $\epsilon > 0$,

there exists $T \in (0, T^*)$ such that for every $\zeta \in \mathcal{S}_{\mathcal{H}'_T}(K')$

$$\omega(\zeta(t,j)) \leq \beta(\omega(\zeta(0,0)), t+j) + \epsilon \quad \forall (t,j) \in \operatorname{dom} \zeta. \tag{9}$$

Sketch of the Proof The proof of this result relies on the fact that for any $T \in (0, T^*]$, there exists an outer perturbation [1, Definition 6.27] of $\mathcal{H}'$, denoted $\mathcal{H}'_\rho$, such that $\mathcal{S}_{\mathcal{H}'_T} \subset \mathcal{S}_{\mathcal{H}'_\rho}$. This, coupled with the well-posedness of $\mathcal{H}'$ by Lemma 3.6, allows us to conclude the statement of the theorem by invoking Lemma 7.20 of [1]. ■

We remind the reader that since solutions of $\mathcal{H}'_{T_1}$ are solutions of $\mathcal{H}'_{T_2}$ as well when $0 \leq T_1 \leq T_2$, Theorem 4.2 indicates a positive upper bound on the length of time-varying delays that the system can tolerate for the delayed solutions originating from K' to converge to the ϵ-neighborhood of $\mathcal{A}'$, derived from (9). It is also worth pointing out that the hybrid system $\mathcal{H}'_T$ described in (4)-(7) is closely related to the notion of temporal regularization [8], and as a result of Theorem 4.2, well-posed hybrid systems with Zeno solutions can be temporally regularized in practice by the introduction of time delays, while maintaining practical stability.

V. Application to State Estimation with Intermittent Information

Given a continuous linear time-invariant system subject to

$$\dot{x}_{\mathrm{p}} = A x_{\mathrm{p}} + B u \quad (x_{\mathrm{p}}, u) \in \mathbb{R}^{n_{\mathrm{p}}} \times \mathbb{R}^p, \tag{10}$$

where $x_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}}$ is the plant state, $u \in \mathbb{R}^p$ is the input, and A, B are real matrices of appropriate dimensions, suppose that the output $y_{\mathrm{p}} = Q x_{\mathrm{p}}$ for some real matrix Q is accessible only at a priori unknown times $\{t_j\}_{j\in\mathbb{N}}$. It is assumed that the sequence is strictly increasing, and there exist $\underline{T}_{\mathrm{s}}, \bar{T}_{\mathrm{s}} > 0$ so that $t_0 \in [0, \bar{T}_{\mathrm{s}}]$ and $\underline{T}_{\mathrm{s}} \leq t_{j+1} - t_j \leq \bar{T}_{\mathrm{s}}$ for all $j \in \mathbb{N}$. The observer problem in [9] is to design a real matrix L so that the state x_{o} of the impulsive system

$$\begin{aligned} \dot{x}_{\mathrm{o}} &= A x_{\mathrm{o}} + B u & t &\notin \{t_j\}_{j\in\mathbb{N}} \\ x_{\mathrm{o}}^+ &= x_{\mathrm{o}} + LQ(x_{\mathrm{p}} - x_{\mathrm{o}}) & t &\in \{t_j\}_{j\in\mathbb{N}} \end{aligned}$$

converges to the state x_{p} of (10). Defining the state estimation error $e := x_{\mathrm{p}} - x_{\mathrm{o}}$ and introducing a sampling timer τ_{s} as in Example 3.2, the problem can be restated as designing a real matrix L such that the set $[0, \bar{T}_{\mathrm{s}}] \times \{0_{n_{\mathrm{p}}}\}$ is asymptotically stable for the hybrid system

$$\begin{aligned} (\dot{\tau}_{\mathrm{s}}, \dot{e}) &= \overbrace{(-1, Ae)}^{F(\tau_{\mathrm{s}}, e)} & (\tau_{\mathrm{s}}, e) &\in \overbrace{[0, \bar{T}_{\mathrm{s}}] \times \mathbb{R}^{n_{\mathrm{p}}}}^{C} \\ (\tau_{\mathrm{s}}^+, e^+) &\in \underbrace{[\underline{T}_{\mathrm{s}}, \bar{T}_{\mathrm{s}}] \times \{(I - LQ)e\}}_{G(\tau_{\mathrm{s}}, e)} & (\tau_{\mathrm{s}}, e) &\in \underbrace{\{0\} \times \mathbb{R}^{n_{\mathrm{p}}}}_{D}. \end{aligned} \tag{11}$$

For the hybrid system described in (11), it is straightforward to check that Assumption 3.1 holds. To study the robustness of the system in the presence of jump delays when $\mathcal{A} = [0, \bar{T}_{\mathrm{s}}] \times \{0_{n_{\mathrm{p}}}\}$ is asymptotically stable, we suppose that the output estimation error Qe can be measured at the isolated time instances $\{t_j\}_{j\in\mathbb{N}}$ (i.e., the plant output y_{p} and the observer state x_{o} are accesible at time instances $\{t_j\}_{j\in\mathbb{N}}$), but the update given by the jump dynamics is delayed, according to the mechanism implemented by $\mathcal{H}'_T$. Hence, with $x = (x_1, x_2)$, we let $x_1 = \tau$ and $x_2 = e$, which imply $n_1 = 1$ and $n_2 = n_{\mathrm{p}}$. Then, for the data (C, F, D, G) given in (11), we have that

$$\widehat{G}(x) = [\underline{T}_{\mathrm{s}}, \bar{T}_{\mathrm{s}}] \times \{e\} \times \{(I - LQ)e\} \quad \forall x \in \{0\} \times \mathbb{R}^{n_2}. \tag{12}$$

Consequently, it follows that

$$\pi(\widehat{G}(D)) = [\underline{T}_s, \bar{T}_s] \times \mathbb{R}^{n_2} \subset C = [0, \bar{T}_s] \times \mathbb{R}^{n_2},$$

and the set $M = G_2(D)$ is closed as it is a linear subspace of $\mathbb{R}^{n_2}$. Thus, Assumption 3.3 is satisfied. Furthermore, no complete solution of the observer is eventually continuous, as the sampling events happen quasiperiodically, so condition (S1) holds, and global exponential stability of the set $\mathcal{A} = [0, \bar{T}_s] \times \{0_{n_2}\}$ can be guaranteed by a polytopic embedding technique [9, Section 4].

We consider the simulation scenario in [9, Example 2], with the system matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^\top, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$
$$L = \begin{bmatrix} 0.7752 & 0.1812 & -0.1212 & -0.1741 & 0.2247 \end{bmatrix}^\top,$$

representing a mass-spring-damper interconnection with biased measurements, along with the sampling interval $[\underline{T}_s, \bar{T}_s] = [0.2, 3]$. A simplified higher-order model is used to study delayed updates, given by the data

$$\begin{aligned} \tilde{C}'_T &= C \times M \times (\{-1\} \cup [0, T]), \\ \tilde{F}'_T(z) &= F(x) \times \{0\} \times (-\min\{\tau + 1, 1\}), \\ \tilde{D}'_T &= (D \times M \times \{-1\}) \cup (C \times M \times \{0\}), \end{aligned}$$

and

$$\tilde{G}'_T(z) = \begin{cases} \widehat{G}(x) \times [0, T] & z \in D \times M \times \{-1\} \\ (x_1, \mu, \mu, -1) & z \in C \times M \times \{0\}, \end{cases}$$

where (C, F, D, G) is given in (11), $\widehat{G}$ is given in (12), and $M = \cup_{x_2 \in \mathbb{R}^{n_2}} (I - LQ)x_2$. For the set $\mathcal{A}'$, we let $M' = \pi_2(\mathcal{A}) = \{0_{n_2}\}$, which implies

$$\begin{aligned} \mathcal{A}' = ([0, \bar{T}_s] &\times \{0_{n_2}\} \times M \times \{-1\}) \\ &\cup ([\underline{T}_s, \bar{T}_s] \times \{0_{n_2}\} \times \{0_{n_2}\} \times \{0\}). \end{aligned}$$

Therefore, the estimation error e is expected to converge to the origin for small delays, in a practical sense.

The evolution of the norm of the estimation error e for a solution corresponding to the case $T = 0.15$, projected onto ordinary time t, can be observed in Figure 1. The simulation[1] is performed with a random initial condition and fixed delay; i.e., each time $z \in D \times M \times \{-1\}$, the state of the closed-loop is updated to a point $(y, 0.15)$, where $y \in \widehat{G}(x)$. Note that the sampling intervals are selected randomly. In spite of the large delay, which is close to the lower bound on the sampling interval, the error converges to a small neighborhood of the origin.

[1]Files for this simulation can be found at the following adress: https://github.com/HybridSystemsLab/NetworkEstimationRobustnessDelays.

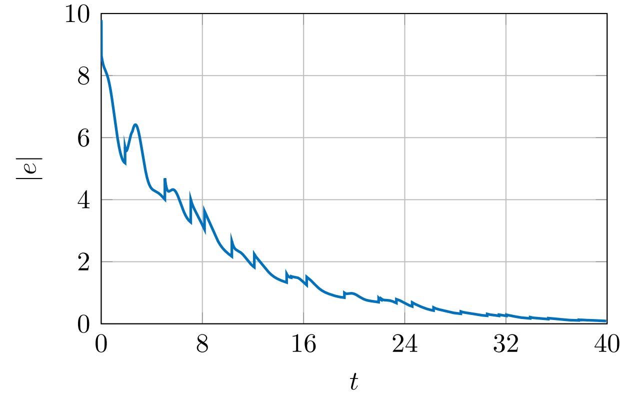

Fig. 1: Evolution of the error norm $|e|$ for a solution corresponding to $T = 0.15$, over ordinary time t.

VI. Conclusion

For hybrid systems experiencing delays in their jumps, we constructed a higher-order model that depends parametrically on the length T of delays. Under mild conditions imposed on the data of the delay-free system, we showed that the higher-order system preserves pre-asymptotic stability properties of the delay-free system for the case of $T = 0$. This fact was utilized to show semiglobal practical robustness in the presence of jump delays. The obtained robustness property was illustrated in an example about a hybrid state estimator with intermittent measurements, validating the theoretical findings. Future work will focus on sufficient conditions preserving pre-asymptotic stability in the presence of delayed jumps.

References

[1] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton, NJ: Princeton University Press, 2012.

[2] W. P. M. H. Heemels, A. R. Teel, N. van de Wouw, and D. Nešić, "Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance," *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1781–1796, Aug 2010.

[3] R. Yuan, Z. Jing, and L. Chen, "Uniform asymptotic stability of hybrid dynamical systems with delay," *IEEE Transactions on Automatic Control*, vol. 48, no. 2, pp. 344–348, Feb 2003.

[4] X. Liu and J. Shen, "Stability theory of hybrid dynamical systems with time delay," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 620–625, April 2006.

[5] Y. Sun, A. N. Michel, and G. Zhai, "Stability of discontinuous retarded functional differential equations with applications," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1090–1105, Aug 2005.

[6] J. Liu and A. R. Teel, "Lyapunov-based sufficient conditions for stability of hybrid systems with memory," *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 1057–1062, April 2016.

[7] ——, *Hybrid Dynamical Systems with Finite Memory*. Springer International Publishing, 2016, pp. 261–273. [Online]. Available: https://doi.org/10.1007/978-3-319-18072-4_13

[8] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry, "On the regularization of Zeno hybrid automata," *Systems & Control Letters*, vol. 38, no. 3, pp. 141–150, 1999. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0167691199000596

[9] F. Ferrante, F. Gouaisbaut, R. G. Sanfelice, and S. Tarbouriech, "State estimation of linear systems in the presence of sporadic measurements," *Automatica*, vol. 73, pp. 101 – 109, 2016. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109816302400