Model Predictive Control under Intermittent Measurements due to Computational Constraints: Feasibility, Stability, and Robustness

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Abstract— **We propose a continuous-time model predictive control (MPC) strategy in the presence of intermittent sampling due to limited computational power. Using hybrid systems tools, the proposed scheme explicitly models the (not necessarily periodic) computation events associated with prediction and optimization. When the terminal cost is a control Lyapunov function and the implicit MPC control law in the proposed setting is continuous, the closed-loop system can tolerate disturbances, unmodeled dynamics, and measurement noise, as well as errors due to asynchronous actuation and sporadic data losses. These findings apply to a wide range of linear systems, and are particularly important for the target application area of cyber-physical systems, where real-time safety constraints require robustness in the presence of computational limitations.**

I. Introduction

Cyber-physical systems (CPSs) are characterized by complex phenomena arising from the interaction between physical mechanisms and the computational platforms controlling them. For such systems, optimization-based control strategies like model predictive control (MPC) [1] offer the ability to simultaneously address hard and soft constraints, which may emerge from safety concerns. When the system under consideration is complex and a closed-form control law may prove difficult to find, MPC has the added benefit of providing an implicit control law derived from its online, receding horizon implementation, with measurement updates triggering the solution of an optimal control problem [2].

Despite these advantages, it is well known that MPC algorithms might be nonrobust with respect to uncertainties [3]. As such, with an eye toward the CPS control problem, the objective of this paper is to propose a nominally robust MPC scheme in continuous time under intermittent sampling, where the time between consecutive events may be nonconstant, and is not necessarily small. Our approach is similar to some of the existing techniques (for example, [4]) and requires the terminal cost of the optimal control problem to be a control Lyapunov function (CLF) on the compact terminal constraint set. Along the lines of some of the work in the MPC literature, which have used regularity properties of the control law and/or the value function to show inherent robustness (see [2], [5] and references therein), we show that when the implicit feedback control law derived from the optimal control problem is continuous, the closed-loop system is semiglobally practically robust with respect to a broad class of perturbations such as timer errors and data dropouts, which applies to a wide range of linear systems. The continuity assumption is utilized to certify intrinsic robustness of MPC with respect to uncertainties that have not been explored as much in the literature. Although it could be seen as a strong requirement (as MPC can produce discontinuous controls, especially when continuous stabilizing controllers do not exist [6]), it can be guaranteed in certain simple cases [1], and checked in a systematic, albeit sometimes laborious, way [7]. On the other hand, while there are approaches to the MPC problem which take uncertainties explicitly into account [2], an immediate drawback is the added complexity to the optimal control problem.

Since MPC implementations are related to sampled-data control systems, our analysis treats the closed loop as a hybrid system by relying on the hybrid inclusions framework of [8]. As the class of uncertainties considered for the robustness analysis of hybrid systems includes perturbations to the constraint sets, this approach enables us to consider robustness of MPC under temporal uncertainties; these include asynchronous actuation, intermittent sampling (since sampling times are uncertain and not known in advance), and timer errors as possible sources of uncertainties. In addition, while delays are not accounted for in this work, it is possible to show robustness with respect to measurement/computation delays [9] without delay compensation [10]. Note that while network errors (delays, dropouts, uncertainties in the re-optimization time) have been considered before [11], such approaches have been restricted to the discrete-time case, as it requires these temporal uncertainties to be integer multiples of the sampling period. As a final note, we foresee the analysis pursued in this paper as a first step towards predictive control of hybrid systems, which should find significant use in systems with hybrid dynamics, for example, mechanical systems with impacts and power electronics.

The remainder of the paper is organized as follows. Section II introduces preliminaries on hybrid systems theory pertinent to our analysis. Section III presents the class of continuous-time systems and the corresponding optimal control problem considered, and introduces the hybrid systems model of the resulting closed loop. Asymptotic stability is discussed in Section IV. In Section V, it is shown that closed-loop stability is robust with respect to a large class of perturbations. The application of the robustness result to linear systems is discussed in Section VI. Concluding remarks are

*This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484, Grant no. ECS-1710621, and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, by the Air Force Research Laboratory under Grant no. FA9453-16-1-0053, and by CITRIS and the Banatao Institute at the University of California.

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given in Section VII. Because of space constraints, proofs of the technical results are not included, and will be published elsewhere.

Notation: We denote by $\mathbb{B}$ the closed unit ball in Euclidean space of appropriate dimension. We use $\mathbb{R}$ to represent real numbers, $\mathbb{R}_{\geq 0}$ its nonnegative and $\mathbb{R}_{>0}$ its positive subsets. The set of nonnegative integers is denoted $\mathbb{N}$. The Euclidean norm (2 norm) is denoted $|.|$. For a pair of sets S_1, S_2, the notation $S_1 \subset S_2$ indicates that S_1 is a subset of S_2, not necessarily proper. The distance of a vector $x \in \mathbb{R}^n$ to a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ is $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$. We use the typical definitions of class-$\mathcal{K}_\infty$ and class-$\mathcal{KL}$ functions as outlined in [8]. The identity function is denoted Id. The empty set is denoted $\varnothing$. For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $(x, y) = \begin{bmatrix} x^\top & y^\top \end{bmatrix}^\top$. Finally, given any $x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the notation $x \succcurlyeq 0$ means $x_i \geq 0$ for all $i \in \{1, 2, \ldots, n\}$.

II. Preliminaries on Hybrid Systems

A hybrid system $\mathcal{H}$ is defined by a quadruple (C, F, D, G), called the *data* of $\mathcal{H}$, and can be represented in the following form:

$$\mathcal{H} \begin{cases} \dot{z} \in F(z) & z \in C \\ z^+ \in G(z) & z \in D. \end{cases} \tag{1}$$

The state of $\mathcal{H}$ is z and takes values in $\mathbb{R}^n$. The set-valued mapping $F : C \rightrightarrows \mathbb{R}^n$ is the *flow map* describing the continuous evolution (*flows*) of the state z on the *flow set* $C \subset \mathbb{R}^n$. Similarly, the set-valued mapping $G : D \rightrightarrows \mathbb{R}^n$ is the *jump map* describing the discrete evolution (*jumps*) of the state z on the *jump set* $D \subset \mathbb{R}^n$. Solutions of $\mathcal{H}$ are parametrized by the scalar pair $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the ordinary time keeping track of the flow time and j is the jump time/index keeping track of the number of jumps. The domain $\operatorname{dom} \zeta \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ of a solution ζ to $\mathcal{H}$ is a *hybrid time domain*; i.e., for all $(T', J') \in \operatorname{dom} \zeta$ there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that

$$\operatorname{dom} \zeta \cap ([0, T'] \times \{0, 1, \ldots, J'\}) = \cup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\}.$$

A solution is called bounded if its range is bounded. It is called maximal if its domain cannot be extended, and complete if its domain is unbounded. The set $\mathcal{S}_{\mathcal{H}}(S)$ is the set of all maximal solutions ζ of $\mathcal{H}$ originating from S, i.e., $\zeta(0, 0) \in S$, with $\mathcal{S}_{\mathcal{H}} := \mathcal{S}_{\mathcal{H}}(\mathbb{R}^n)$. See [8, Definition 2.6] for a rigorous definition of a solution to $\mathcal{H}$.

The hybrid system $\mathcal{H}$ is said to satisfy the *hybrid basic conditions* if the following hold–see [8] for definitions of outer semicontinuity and local boundedness:

(H1) The sets C and D are closed.
(H2) The flow map F is locally bounded and outer semicontinuous. Furthermore, for each $z \in C$, the set $F(z)$ is convex.
(H3) The jump map G is locally bounded and outer semicontinuous.

These mild regularity conditions ensure that $\mathcal{H}$ is *well-posed* [8, Theorem 6.30] in the sense that it has structurally good properties; e.g., robustness to perturbations and closeness of solutions over finite hybrid horizons.

Stability concepts for hybrid systems are defined for closed sets, but our results will only consider compact sets. These concepts do not require completeness of solutions, thus giving rise to the notion of *pre*-asymptotic stability.

Definition 2.1: For a hybrid system $\mathcal{H}$ given by (1), the closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be pre-asymptotically stable if 1) for all $\epsilon > 0$, there exists $\delta > 0$ such that each solution ζ to $\mathcal{H}$ with $|\zeta(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|\zeta(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \operatorname{dom} \zeta$, and 2) there exists $\mu > 0$ such that each solution ζ to $\mathcal{H}$ with $|\zeta(0,0)|_{\mathcal{A}} \leq \mu$ is bounded and satisfies $\lim_{t+j \to \infty} |\zeta(t,j)|_{\mathcal{A}} = 0$ when complete.

A pre-asymptotically stable set is said to be globally pre-asymptotically stable if the scalar μ in Definition 2.1 can be chosen arbitrarily large. The basin of pre-attraction $\mathcal{B}^{\mathrm{p}}_{\mathcal{A}}$ of $\mathcal{A}$ is the set of all $z \in \mathbb{R}^n$ such that every $\zeta \in \mathcal{S}_{\mathcal{H}}(z)$ is bounded, and if it is complete, $\lim_{t+j \to \infty} |\zeta(t,j)|_{\mathcal{A}} = 0$.

A *hybrid control system* $\mathcal{H}_u$ is defined similarly by a quadruple (C_u, F_u, D_u, G_u):

$$\mathcal{H}_u \begin{cases} \dot{z} \in F_u(z, u) & (z, u) \in C_u \\ z^+ \in G_u(z, u) & (z, u) \in D_u. \end{cases}$$

The solution concept for $\mathcal{H}$ in (1) can be extended to a *solution pair* (ζ, υ) for $\mathcal{H}_u$ in a similar fashion, with $\operatorname{dom}(\zeta, \upsilon) = \operatorname{dom} \zeta = \operatorname{dom} \upsilon$. During flows, the input υ is Lebesgue measurable and locally essentially bounded. The set of *all* solution pairs originating from S is denoted $\mathcal{S}_{\mathcal{H}_u}(S)$, with $\mathcal{S}_{\mathcal{H}_u} := \mathcal{S}_{\mathcal{H}_u}(\mathbb{R}^n)$.

III. A Hybrid System Model of MPC with Intermittent Sampling and Prediction

We consider the continuous-time plant

$$\dot{x} = f_{\mathcal{P}}(x, u) \quad (x, u) \in X \times U, \tag{2}$$

where $f_{\mathcal{P}} : X \times U \to \mathbb{R}^n$, $X \subset \mathbb{R}^n$, and $U \subset \mathbb{R}^p$. Since we rely on certain hybrid systems results to establish desired properties, we define $\mathcal{P}_u := (X \times U, f_{\mathcal{P}}, \varnothing, \mathrm{Id})$ as the hybrid control system representation of (2).

We assume that the plant state x is measured at times $\{t_j\}_{j \in \mathbb{N}}$, where $t_0 \in [0, \bar{T}_{\mathrm{m}}]$ and

$$t_{j+1} - t_j \in [\underline{T}_{\mathrm{m}}, \bar{T}_{\mathrm{m}}] \quad \forall j \in \mathbb{N}$$

for some $\bar{T}_{\mathrm{m}} \in \mathbb{R}_{>0}$ and $\underline{T}_{\mathrm{m}} \in (0, \bar{T}_{\mathrm{m}}]$. We denote by x_{m} the *memory state*, which stores the measured value of the plant state x. This intermittent measurement scenario can be conveniently described by the hybrid dynamics [12]

$$\begin{cases} (\dot{x}_{\mathrm{m}}, \dot{\tau}_{\mathrm{m}}) = (0, -1) & \tau_{\mathrm{m}} \in [0, \bar{T}_{\mathrm{m}}] \\ (x_{\mathrm{m}}^+, \tau_{\mathrm{m}}^+) \in \{x\} \times [\underline{T}_{\mathrm{m}}, \bar{T}_{\mathrm{m}}] & \tau_{\mathrm{m}} = 0, \end{cases} \tag{3}$$

where τ_{m} is the timer state associated with x_{m}. The hybrid system (3) generates all possible sequences $\{t_j\}_{j \in \mathbb{N}}$ with the above properties.

Letting $z_u := (x, x_{\rm m}, \tau_{\rm m})$, and combining (2) and (3) results in the hybrid control system

$$\begin{cases} \dot{z}_u = (f_{\mathcal{P}}(x,u), 0, -1) & (z_u, u) \in C_{\rm ol} \\ z_u^+ \in \{x\} \times \{x\} \times [\underline{T}_{\rm m}, \bar{T}_{\rm m}] & (z_u, u) \in D_{\rm ol}, \end{cases} \quad (4)$$

where

$$C_{\rm ol} := X \times X \times [0, \bar{T}_{\rm m}] \times U,$$
$$D_{\rm ol} := X \times X \times \{0\} \times U.$$

Let $T \in [\bar{T}_{\rm m}, \infty)$ be the *prediction horizon* of the receding horizon scheme. The control objective is to render a compact subset of X asymptotically stable for (2) by a feedback law derived via an MPC scheme to be defined. Defining the hybrid system $\mathcal{P} := (X, F_{\mathcal{P}}, \varnothing, \mathrm{Id})$, where $F_{\mathcal{P}}(x) := f_{\mathcal{P}}(x, U)$, we enforce the following assumption on the data of (2), and the maximal solutions of $\mathcal{P}$ and $\mathcal{P}_u$.

Assumption 3.1: The hybrid systems $\mathcal{P}$ and $\mathcal{P}_u$ satisfy the following:

(P1) The flow map $f_{\mathcal{P}}$ is continuous.
(P2) The state constraint set $X \subset \mathbb{R}^n$ is closed.
(P3) The input constraint set $U \subset \mathbb{R}^p$ is compact.
(P4) For each $x \in X$, the set $F_{\mathcal{P}}(x)$ is convex.
(P5) Given $\chi \in \mathcal{S}_{\mathcal{P}}(X)$, χ does not have finite escape time.
(P6) Given any $x \in X$, for each $(\chi_i, \nu_i) \in \mathcal{S}_{\mathcal{P}_u}(x)$, where $i \in \{1,2\}$, if $\nu_1 \equiv \nu_2$ almost everywhere, then $\chi_1 \equiv \chi_2$.

A. The Optimal Control Problem during Flows

Given $t \in \mathbb{R}_{\geq 0}$ and $x \in X$, let us define the cost functional

$$\mathcal{J}_t(\chi, \nu; x) := \int_0^t \ell(\chi(s,0), \nu(s,0))\, ds + V_{\rm f}(\chi(t,0)) \quad (5)$$

and the constraints

$$\mathcal{C}_t(x) := \begin{cases} (\chi, \nu) \in \mathcal{S}_{\mathcal{P}_u}(x), \\ (t, 0) \in \mathrm{dom}(\chi, \nu), \\ \chi(t,0) \in X_{\rm f}, \end{cases} \quad (6)$$

where the set $X_{\rm f} \subset X$ is the terminal constraint set. The function $\ell : X \times U \to \mathbb{R}_{\geq 0}$ is called the *stage cost*, and the function $V_{\rm f} : X \to \mathbb{R}_{\geq 0}$ is called the *terminal cost*. For a given $t \in \mathbb{R}_{\geq 0}$ and $x \in X$, a pair (χ, ν) is said to be a *feasible pair* if it satisfies the constraints (6). In addition, for a given t, a point $x \in X$ is said to be a *feasible initial condition* if there exists a feasible pair (χ, ν) with $\chi(0,0) = x$. We denote by $X_t \subset X$ the set of all feasible initial conditions, called the *feasible set*.

Problem 3.2: Given $t \geq 0$, $x \in X_t$, and the hybrid control system $\mathcal{P}_u = (X \times U, f_{\mathcal{P}}, \varnothing, \mathrm{Id})$ in (2), minimize the cost functional $\mathcal{J}_t$ in (5) subject to the constraints $\mathcal{C}_t$ in (6).

When a solution (χ^*, ν^*) to Problem 3.2 exists, the minimum

$$\mathcal{J}_t^*(x) := \mathcal{J}_t(\chi^*, \nu^*; x)$$

of the cost functional $\mathcal{J}_t$ is called the *value function*. Some of the forthcoming results will use the following assumptions on Problem 3.2:

Assumption 3.3: For any $x \in X_T$, a solution to Problem 3.2 exists.

Assumption 3.3 simply states that for any feasible point, a minimizing optimal solution pair satisfying the constraints exist.

Assumption 3.4: Given a compact set $\mathcal{A} \subset X$, the cost functional (5), the terminal constraint set $X_{\rm f}$, and the hybrid system $\mathcal{P}_u$ satisfy the following:

(O1) The stage cost ℓ is continuous, $\ell(\mathcal{A}, U) = 0$, and there exists a class-$\mathcal{K}_\infty$ function $\underline{\alpha}$ such that

$$\ell(x,u) \geq \underline{\alpha}(|x|_{\mathcal{A}}) \quad \forall (x,u) \in X \times U.$$

(O2) The terminal cost $V_{\rm f}$ is continuous and positive definite with respect to the set $\mathcal{A}$.
(O3) The terminal constraint set $X_{\rm f} \subset X$ is compact, and $\mathcal{A} \subset X_{\rm f}$.
(O4) There exists a function $\kappa : [0,T] \times X_T \to U$ that is continuous, such that for any $x \in X_T$, there exists a feasible pair (χ^*, ν^*) with $\chi^*(0,0) = x$ that minimizes $\mathcal{J}_T$, where $\nu^*(t,0) := \kappa(t,x)$ for all $t \in [0,T]$.
(O5) There exists a Lebesgue measurable function $u_{\rm CLF} : X_{\rm f} \to U$ such that for all $x \in X_{\rm f}$

$$\langle \nabla V_{\rm f}(x), f_{\mathcal{P}}(x, u_{\rm CLF}(x)) \rangle \leq -\ell(x, u_{\rm CLF}(x)),$$

and maximal solutions of $(X_{\rm f}, f_{\mathcal{P}} \circ (\mathrm{Id}, u_{\rm CLF}), \varnothing, \mathrm{Id})$ are complete.

Conditions (O1)-(O3) and (O5) of Assumption 3.4 are common in the MPC literature, where (O5) is the familiar CLF condition. On the other hand, under uniqueness of solutions corresponding to a given input signal, imposed via (P6) of Assumption 3.1, Condition (O4) declares that the optimal control law is continuous, with respect to both the initial condition and time. The function κ is used only for analysis purposes; for real-time implementation, the optimal control $t \mapsto \kappa(x_{\rm m}, t)$ is found numerically by solving Problem 3.2 for the initial condition $x_{\rm m}$ and horizon T each time the plant state x is sampled via the memory state $x_{\rm m}$.

It is important to note here that discrete-time MPC variants which require the CLF property as in Condition (O5) can be nonrobust because of terminal constraints [3]. Nevertheless, we consider terminal constraints and costs for a few reasons. First, in practice, if state constraints are not present, i.e., $X = \mathbb{R}^n$, by making sure that the terminal cost is sufficiently large, one can implicitly enforce terminal constraints on the system trajectories [1]. Second, for systems with state constraints, relaxing the hard terminal constraint can lead to other assumptions that might be hard to guarantee. Furthermore, since the analysis of this paper is a first step towards predictive control for hybrid inclusions, which are characterized by constraints given by the flow and jump sets, we preferred not to add the complexity associated with MPC schemes without terminal constraints/costs.

We also require the following condition that relates the terminal constraint, the prediction horizon and the dynamics of the plant.

Assumption 3.5: Let $\mathcal{P}_{\text{back}} := (X, -F_{\mathcal{P}}, \varnothing, \text{Id})$. Then, given $\chi \in \mathcal{S}_{\mathcal{P}_{\text{back}}}(X_{\text{f}})$, χ does not have finite escape time.

B. The Receding Horizon Scheme under Intermittent Sampling

The receding horizon scheme is obtained by using the memory state x_{m} to recompute and apply an optimal control each time the plant state is sampled. After every jump, the measurement state x_{m} of the hybrid control system (4) gets updated so that $x_{\text{m}} = x$, and Problem 3.2 is solved for $x = x_{\text{m}}$ and $t = T$ in order to find the minimizer ν^*. The optimal control ν^* is then applied until the next sampling event, after which the process is repeated. By (O4), this means that the closed-loop system can be described by setting $u(t) = \kappa(t, x_{\text{m}})$ in (4). To obtain a time-invariant hybrid closed loop, we introduce the control timer state $\tau_{\text{c}} \in [0, T]$ keeping track of the continuous evolution of time during flows, described by the following hybrid control system with input τ_{m}:

$$\begin{cases} \dot{\tau}_{\text{c}} = 1 & \tau_{\text{m}} \in [0, \bar{T}_{\text{m}}] \\ \tau_{\text{c}}^+ = 0 & \tau_{\text{m}} = 0. \end{cases}$$

Letting $z := (x, x_{\text{m}}, \tau_{\text{m}}, \tau_{\text{c}})$ and replacing t with τ_{c} in the minimizer κ results in the closed-loop hybrid system

$$\mathcal{H}_{\text{cl}} \begin{cases} \dot{z} = \overbrace{(f_{\mathcal{P}}(x, \kappa(\tau_{\text{c}}, x_{\text{m}})), 0, -1, 1)}^{f_{\text{cl}}(z)} & z \in C_{\text{cl}} \\ z^+ \in \underbrace{\{x\} \times \{x\} \times [\underline{T}_{\text{m}}, \bar{T}_{\text{m}}] \times \{0\}}_{G_{\text{cl}}(z)} & z \in D_{\text{cl}}, \end{cases}$$

where $f_{\text{cl}} : C_{\text{cl}} \to \mathbb{R}^{2n+2}$, $G_{\text{cl}} : D_{\text{cl}} \to \mathbb{R}^{2n+2}$, and

$$C_{\text{cl}} := X \times X_T \times [0, \bar{T}_{\text{m}}] \times [0, T],$$
$$D_{\text{cl}} := X \times X_T \times \{0\} \times [0, T].$$

Under (P1) of Assumption 3.1 and (O4) of Assumption 3.4, the flow map f_{cl} is continuous. By definition, the jump map G_{cl} is locally bounded. Because X is closed by (P2) of Assumption 3.1, the flow and jump sets $C_{\text{cl}}, D_{\text{cl}}$ are closed if and only if X_T is closed. Closedness of X_T further guarantees outer semicontinuity of G_{cl} since the graph of G_{cl} is a topological embedding of C_{cl}. Moreover, compactness of the feasible set X_T can be established via backwards reachability analysis on the hybrid system $\mathcal{P}$ under Assumptions 3.1, 3.4, and 3.5. These lead to the following result:

Lemma 3.6: Suppose that Assumptions 3.1, 3.4, and 3.5 hold. Then, the closed-loop $\mathcal{H}_{\text{cl}}$ satisfies the hybrid basic conditions, and every maximal solution of $\mathcal{H}_{\text{cl}}$ is bounded or complete.

IV. Pre-Asymptotic Stability of the Closed-Loop System

We show pre-asymptotic stability of a compact set for the closed loop in two steps by relying on the hybrid *reduction principle* [8, Corollary 7.24]. Let us define the set

$$Z_{\text{sync}} := \{\zeta(t, j) : \zeta \in \mathcal{S}_{\mathcal{H}_{\text{cl}}}((C_{\text{cl}} \cup D_{\text{cl}}) \cap G_{\text{cl}}(D_{\text{cl}})), \quad (t, j) \in \operatorname{dom} \zeta\}.$$

The set Z_{sync} is related to the conventional time-varying analysis of MPC. Namely, it defines the reachable set of $\mathcal{H}_{\text{cl}}$ from points in $C_{\text{cl}} \cup D_{\text{cl}}$ corresponding to a synchronization between the plant state x and the control law κ assumed in (O4). That is, Z_{sync} is the reachable set of $\mathcal{H}_{\text{cl}}$ from points $z = (x, x_{\text{m}}, \tau_{\text{m}}, \tau_{\text{c}}) \in C_{\text{cl}} \cup D_{\text{cl}}$ such that $x = x_{\text{m}}$ for some $x_{\text{m}} \in X_T$, $\tau_{\text{m}} \in [0, \bar{T}_{\text{m}}]$, and $\tau_{\text{c}} = 0$.

Our goal is to show that the compact (due to compactness of Z_{sync}) set $\mathcal{A}_{\text{cl}} := (\mathcal{A} \times \mathcal{A} \times \mathbb{R} \times \mathbb{R}) \cap Z_{\text{sync}}$ is pre-asymptotically stable for the hybrid system $\mathcal{H}_{\text{cl}}$. In showing this result, the first step is to prove pre-asymptotic stability of Z_{sync} by "continuous dependence" of solutions on initial conditions [8, Proposition 6.14], which follows from the fact that $\mathcal{H}_{\text{cl}}$ satisfies the hybrid basic conditions under Assumptions 3.1, 3.4, and 3.5. In the second step, the value function $\mathcal{J}_T^*$ is utilized to construct a Lyapunov function to show that $\mathcal{A}_{\text{cl}}$ is globally pre-asymptotically stable for the reduced model $\mathcal{H}_{\text{cl}}^{\text{r}} := (C \cap Z_{\text{sync}}, f_{\text{cl}}, D \cap Z_{\text{sync}}, G_{\text{cl}})$; the properties of $\mathcal{J}_T^*$ required for this step (for example, positive definiteness and continuity) are shown as per standard proofs in the MPC literature. Finally, combination of the aforementioned results via the reduction principle allows us to conclude global pre-asymptotic stability of the closed-loop system.

Theorem 4.1: Suppose that Assumptions 3.1, 3.4, and 3.5 hold. Then, the compact set $\mathcal{A}_{\text{cl}}$ is globally pre-asymptotically stable for $\mathcal{H}_{\text{cl}}$. Furthermore, for each initial condition from the set Z_{sync}, a solution exists, and maximal solutions are complete.

Theorem 4.1 shows that in addition to being stable in the conventional sense, the closed loop can tolerate small initialization errors in the timer associated with the controller, in addition to mismatch between the plant state and its measurement, as the initial condition could be such that $x \neq x_{\text{m}}$. An important consequence of this is an inherent ability to recover from sporadic losses in measurements. Note, however, since we consider nominal robustness properties, as opposed to other methods explicitly modeling perturbations, robust feasibility cannot be guaranteed without further information on the geometry of the sets X and U, and therefore that of X_T. It can be guaranteed in certain reasonable cases, such as when $\mathcal{A}$ is in the interior of $X_{\text{f}} = X_0 \supset X_T$, where X_0 is X_t evaluated at $t = 0$, as a result of the semiglobal practical robustness property discussed in the next section.

V. Robustness of the Closed-Loop System

As stated in Section II, hybrid systems satisfying conditions (H1)-(H3) portray desirable robustness properties. The objective of this section is to show the robustness of the closed loop $\mathcal{H}_{\text{cl}}$ to small perturbations. Given a function $\rho : \mathbb{R}^{2n+2} \to \mathbb{R}_{\geq 0}$, we will define $\mathcal{H}_{\text{cl}}^{\rho} := (C_{\text{cl}}^{\rho}, F_{\text{cl}}^{\rho}, D_{\text{cl}}^{\rho}, G_{\text{cl}}^{\rho})$ to be the ρ-perturbation [8, Definition 6.27] of the closed-loop system $\mathcal{H}_{\text{cl}}$.

A. Semiglobal Practical Robust Stability

Under the hybrid basic conditions, which guarantee well-posedness, Theorem 4.1 results in an equivalent characterization of the pre-asymptotic stability of the closed loop, see [8, Definition 7.10]), which follows via [8, Theorem 7.12].

Proposition 5.1: Suppose that Assumptions 3.1, 3.4, and 3.5 hold. Then, there exists a class-$\mathcal{KL}$ function β such that for every $\zeta \in \mathcal{S}_{\mathcal{H}_{\rm cl}}$

$$|\zeta(t,j)|_{\mathcal{A}_{\rm cl}} \leq \beta(|\zeta(0,0)|_{\mathcal{A}_{\rm cl}}, t+j) \quad \forall (t,j) \in \operatorname{dom}\zeta. \quad (7)$$

In addition, well-posedness of $\mathcal{H}_{\rm cl}$, coupled with Proposition 5.1, leads to a semiglobal practical robust stability [8, Definition 7.18] result by [8, Lemma 7.20].

Theorem 5.2: Suppose that Assumptions 3.1, 3.4, and 3.5 hold. Consider the class-$\mathcal{KL}$ function β of Proposition 5.1 satisfying (7)*. Then, for every*

1) *continuous function $\rho : \mathbb{R}^{2n+2} \to \mathbb{R}_{\geq 0}$ that is positive on $\mathbb{R}^{2n+2} \setminus \mathcal{A}_{\rm cl}$,*
2) *compact set $K \subset \mathbb{R}^{2n+2}$, and*
3) *scalar $\epsilon > 0$,*

there exists $\delta > 0$ such that for every $\zeta \in \mathcal{S}_{\mathcal{H}_{\rm cl}^{\delta\rho}}(K)$

$$|\zeta(t,j)|_{\mathcal{A}_{\rm cl}} \leq \beta(|\zeta(0,0)|_{\mathcal{A}_{\rm cl}}, t+j) + \epsilon \quad \forall (t,j) \in \operatorname{dom}\zeta.$$

B. Robustness to Disturbances, Unmodeled Dynamics, Measurement Noise, and Timer Errors

Given $\sigma := (d, \Delta, v, \theta_{\rm m}, \theta_{\rm c}, \omega_{\rm c}) \succcurlyeq 0$, consider the following perturbation of the closed-loop hybrid system $\mathcal{H}_{\rm cl}$:

$$\mathcal{H}_{\rm cl}^{\sigma} \begin{cases} \dot{z} \in F_{\rm cl}^{\sigma}(z) & z \in C_{\rm cl}^{\sigma} \\ z^+ \in G_{\rm cl}^{\sigma}(z) & z \in D_{\rm cl}^{\sigma}, \end{cases} \quad (8)$$

where

$$F_{\rm cl}^{\sigma}(z) := (f_{\mathcal{P}}(x, (\kappa(\tau_{\rm c}, x_{\rm m}) + d\mathbb{B}) \cap U) + \Delta\mathbb{B}, \\ 0, -1, 1 + \omega_{\rm c}\mathbb{B}) \quad \forall z \in C_{\rm cl}^{\sigma},$$

$$G_{\rm cl}^{\sigma}(z) := \{x\} \times \{x + v\mathbb{B}\} \times \left([\underline{T}_{\rm m}, \bar{T}_{\rm m}] + \theta_{\rm m}\mathbb{B}\right) \\ \times [0, \theta_{\rm c}] \quad \forall z \in C_{\rm cl}^{\sigma},$$

and

$$C_{\rm cl}^{\sigma} := X \times X_T \times [0, \bar{T}_{\rm m} + s_{\rm m}] \times [0, T], \\ D_{\rm cl}^{\sigma} := X \times X_T \times \{0\} \times [0, T].$$

In (8), d, Δ, v represent the magnitude of disturbances, bounded unmodeled dynamics, and measurement noise, respectively. The constant $\theta_{\rm m}$ introduces further uncertainty to the sampling period and ω_c allows us to consider time-varying perturbations to the rate of the flow/control timer $\tau_{\rm c}$. The scalar $\theta_{\rm c}$ acts to model asynchronous behavior between the sensor and the actuator.

With some abuse of notation, given any $\tilde{\rho} \geq 0$, define $\mathcal{H}_{\rm cl}^{\tilde{\rho}} := (C_{\rm cl}^{\tilde{\rho}}, F_{\rm cl}^{\tilde{\rho}}, D_{\rm cl}^{\tilde{\rho}}, G_{\rm cl}^{\tilde{\rho}})$, a ρ-perturbation to $\mathcal{H}_{\rm cl}$ of the form (8) with the constant perturbation function $\rho(z) = \tilde{\rho}$ for all $z \in \mathbb{R}^{2n+2}$. By simple algebraic manipulations, it is easy to see that $\mathcal{H}_{\rm cl}^{\tilde{\rho}}$ can be used to outer approximate the perturbed model $\mathcal{H}_{\rm cl}^{\sigma}$, with $\tilde{\rho}$ depending continuously on σ. This observation leads to the following result, certifying inherent robustness of the closed loop to input disturbances, bounded unmodeled dynamics, measurement noise, and timer errors.

Theorem 5.3: Suppose that Assumptions 3.1, 3.4, and 3.5 hold. Consider the class-$\mathcal{KL}$ function β of Proposition 5.1 satisfying (7)*. Then, for every*

1) *compact set $K \subset \mathbb{R}^{2n+2}$, and*
2) *scalar $\epsilon > 0$,*

there exists $\bar{\sigma} > 0$ such that if $\sigma \succcurlyeq 0$ and $|\sigma| \leq \bar{\sigma}$,

$$|\zeta(t,j)|_{\mathcal{A}_{\rm cl}} \leq \beta(|\zeta(0,0)|_{\mathcal{A}_{\rm cl}}, t+j) + \epsilon \quad \forall (t,j) \in \operatorname{dom}\zeta$$

for every $\zeta \in \mathcal{S}_{\mathcal{H}_{\rm cl}^{\sigma}}(K)$.

Remark 5.4: An analogue of Theorem 5.3 has been developed in [9] to certify robustness of general hybrid systems to delays in their jumps, which can be used to show robustness of the MPC scheme against delays due to sensor limitations or the high computational cost of solving Problem 3.2.

VI. Application to Linear Systems

The objective of this section is to show the applicability of Theorems 4.1 and 5.3 to linear systems.

A. On Stabilizable Linear Systems

As mentioned before, while the continuity assumption on optimal controls is restrictive, it can be guaranteed for certain special cases. Below, we show that Assumptions 3.1, 3.4, and 3.5 hold for a class of linear systems–see also [5] for similar results. The main idea is to choose the terminal cost matrix P as the solution of the algebraic Riccati equation, given in (9). This ensures that the state-feedback gain resulting from finite-horizon linear-quadratic regulation (LQR) is time-invariant, and equal to the gain derived from the infinite-horizon LQR problem.

Proposition 6.1: Suppose that the pair (A, B) is stabilizable, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$. Furthermore, suppose $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{p\times p}$ are positive definite. Let $P \in \mathbb{R}^{n\times n}$ be the unique positive definite solution of

$$PA + A^\top P + Q - PBR^{-1}B^\top P = 0. \quad (9)$$

Assume $f_{\mathcal{P}}(x, u) = Ax + Bu$ for all $(x, u) \in X \times U$, where

$$X = \{x \in \mathbb{R}^n : x^\top P x \leq \gamma\}, \\ U \supset \{u \in \mathbb{R}^p : \exists x \in X, u = -R^{-1}B^\top P x\},$$

for some $\gamma > 0$. If $X_{\rm f} = X$, U is compact, and

$$\ell(x, u) = x^\top Q x + u^\top R u \quad \forall (x, u) \in X \times U, \\ V_{\rm f}(x, u) = x^\top P x \quad \forall (x, u) \in X \times U,$$

then, $X_T = X$, and Assumptions 3.1, 3.4, and 3.5 hold with $\mathcal{A} = \{0\}$. Moreover,

$$\kappa(t, x) = -R^{-1}B^\top P \chi(t, 0) \quad \forall (t, x) \in [0, T] \times X,$$

where χ is the complete solution of $(\mathbb{R}^n, f'_{\mathcal{P}}, \varnothing, \mathrm{Id})$ satisfying $\chi(0,0) = x$, with $f'_{\mathcal{P}}(x) := Ax - BR^{-1}B^\top P x$ for all $x \in \mathbb{R}^n$.

B. Numerical Example

An example illustrating Proposition 6.1 is straightforward. Instead, while Proposition 6.1 does not cover the time-varying case, the following example shows that it is possible to numerically show that Assumptions 3.1, 3.4, and 3.5 hold, when the LQR feedback gain is nonconstant. Although this example is simple, it is helpful to show the robustness of MPC to various types of temporal uncertainties such as data dropouts and asynchronous actuation.

Let $n = 2$, $p = 1$, and consider the constrained unstable linear system defined by the flow map

$$f_{\mathcal{P}}(x, u) = \underbrace{\begin{bmatrix} 0.2 & 0.7 \\ 0.7 & 0.2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}}_{b} u,$$

where $x \in X = \mathbb{B}$, $u \in U = 6\mathbb{B}$. We assume that the measurements of x for this linear system occur every 0.5 to 2 seconds, i.e., $\underline{T}_{\rm m} = 0.5$ and $\bar{T}_{\rm m} = 2$. The horizon $T = 3 > 2$ is chosen for the MPC algorithm in order to stabilize the origin, hence $\mathcal{A} = \{0\}$. Then, by linearity, it is straightforward to check that Assumptions 3.1 and 3.5 hold. The stage cost is chosen to be $\ell(x, u) = |(x, u)|^2$. To satisfy (O5), a Lyapunov equation is solved for a positive definite matrix P, and the terminal cost and constraint set are chosen to be $V_{\rm f}(x) = x^\top P x$ and

$$X_{\rm f} = \{x \in \mathbb{R}^2 : V_{\rm f}(x) \leq 5\} \subset X,$$

respectively. The optimal solution pairs for the unconstrained linear system $(\mathbb{R}^2 \times \mathbb{R}, Ax + bu, \varnothing, \mathrm{Id})$ from the set $X = \mathbb{B}$ satisfy the constraints imposed by the original problem, which can be checked via simulations: the unconstrained problem is solved by LQR, and given $x \in X$, optimal pairs (χ^*, ν^*) with $\chi^*(0, 0) = x$ satisfy $\chi^*(t, 0) \in X$ and $\nu^*(t, 0) \in U$ for all $t \in [0, T]$, as well as the terminal constraint that $\chi^*(T, 0) \in X_{\rm f}$. Moreover, (O4) is satisfied since the optimal controls are given by a continuous time-varying feedback gain derived from the differential Riccati equation, and $X = X_T$.

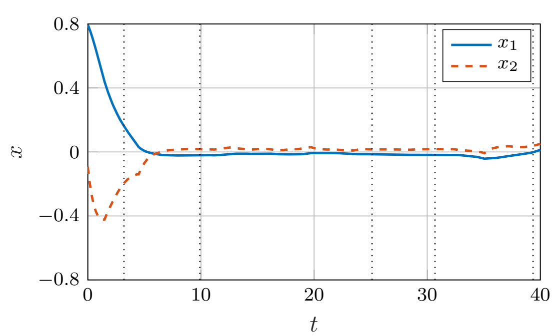

Fig. 1: Plant trajectory (projected to t) under perturbations. The vertical dotted lines indicate data dropouts.

A simulation[1] scenario is realized by introducing perturbations of the form given in (8), with the parameters $d = 0.1$, $\Delta = 0.05$, and $v = 0.01$, and $\theta_c = 0.025$. The parameter $\theta_{\rm m}$ is omitted since the large gap between the prediction horizon and maximum sampling period, $T - \bar{T}_{\rm m} = 1$, provides an immediate robustness window against this parameter. In lieu of this perturbation, up to 5 measurement events are randomly chosen to be subject to communication dropouts. Fig. 1 shows that despite these challenging uncertainties, the corresponding state trajectory converges to a small neighborhood of the origin, thereby verifying Theorems 4.1 and 5.3.

[1]Files for this simulation can be found at the following adress: https://github.com/HybridSystemsLab/IntermittentMPCPlanarSystem.

VII. Conclusion

We studied the robustness of a general class of continuous-time MPC algorithms in the presence of intermittent measurements, via hybrid systems tools. When the implicit MPC feedback law is continuous, the so-called hybrid basic conditions can be exploited to show that the system is robust with respect to a large class of perturbations, including temporal errors. Simulation results show that under the continuity assumption, the proposed method can also tolerate data dropouts and initialization errors. Future work will relax the assumptions by relying on continuity properties of the value function, and extend the results to hybrid dynamical systems.

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