# A Hybrid Predictive Control Approach to Trajectory Tracking for a Fully Actuated Biped

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*Abstract*— **We model a three-link fully actuated biped as a hybrid system and propose a prediction-based control algorithm for global tracking of reference trajectories. The proposed control strategy consists of a reference system that generates the desired periodic gait, a virtual system that generates a suitable reference trajectory using prediction, and a tracking control law that steers the biped to the virtual trajectory. The proposed algorithms achieves, in finite time, tracking in two steps. We present mathematical properties that define the main elements in the hybrid predictive controller for achieving convergence to the reference within the first two steps. The results are validated through numerical simulations.**

## I. Introduction

Mechanical systems with impacts have trajectories with intervals of continuous flow and instants where discrete changes occurs. These systems are classified by their intertwined continuous and discrete dynamics, which can be difficult to model using classical methods due to this complex behavior. Controller design for such systems is also challenging due to discrete jumps at unknown times, for which conventional control approaches are not applicable. Modeling systems with impacts as hybrid systems provides a way to describe the non-smooth behavior, and allows for the implementation of hybrid controllers. Hybrid controllers have the advantage of being able to perform discrete tasks while supplying a continuous input.

A biped is typically modeled as a mechanical system with impacts that occur at the end of each walking step. During each step, one leg is planted while the other is swinging forward towards the next impact. Both feet are briefly in contact with the surface during the transition into the next step. There are numerous biped models available in literature [2],[3],[12],[14],[16], and control strategies based on trajectory tracking methods [10],[16]. These trajectory tracking control strategies rely on either pre-computed trajectories [12] or trajectories computed on the fly [11], [15]. In this paper, we pursue a novel approach to trajectory tracking control for fully actuated bipeds that uses an additional "virtual trajectory" generated using prediction of the dynamics to steer the biped state to a given reference. The reference is generated based on the desired periodic gait defining a *hybrid limit cycle*. The virtual trajectory is generated by a hybrid algorithm with state variables that are reset to values that, according to the prediction of the future hybrid trajectories of the biped, guarantee tracking of the given reference trajectory, in finite time. The prediction of the hybrid trajectories of the biped relies on properties of solutions established analytically. With the proposed control strategy, the controller is able to steer the biped to track reference trajectories by relying only on the measurements of the limb angles and velocities. We show that convergence is achieved after the first two steps have occurred, regardless of the initial conditions of the system of the biped. The modeling and control design techniques we introduce in this paper can be extended to other complex systems, such as bipeds with more joints and variable walking characteristics.

The remainder of this paper is organized as follows. In Section II we introduce a model of a three-link biped using the hybrid inclusions framework presented in [1]. In Section III we propose methods to generate trajectories for a given periodic gate, and a hybrid control algorithm to track the generated trajectories in finite time. Simulations are provided in Section IV to support our claims. Due to space constraints, some details and proofs are not included, but will be published elsewhere.

## II. Hybrid System Model of a Biped

The three-link biped in Figure 1 is modeled as a hybrid system to describe the continuous and discrete dynamics of the system. The movement of the legs and torso during each step is described by the continuous dynamics of the model, while the discrete dynamics describe the instantaneous change that occurs upon the impact at the end of each step. At all times, one of the legs is the planted leg while the other is the swing leg, and they switch roles upon each step.

### A. State Variables, Inputs, and Parameters

To define a mathematical model of the fully actuated biped system, we introduce the state $x$, the input $u$, and parameters $\gamma$ as follows. The state component vector $x$ is comprised of the angle vector $\theta$, which contains the planted leg angle $\theta_p$, the swing leg angle $\theta_s$, and torso angle $\theta_t$; the velocity vector $\omega$, which contains the planted leg angular velocity $\omega_p$, the swing leg angular velocity $\omega_s$, and the torso angular velocity $\omega_t$. The input $u$ is the input torque, where $u_p$ is the torque applied on the planted leg from the ankle, $u_s$ is the torque applied on the swing leg from the hip, and $u_t$ is the torque applied on the torso from the hip. The vector of parameters

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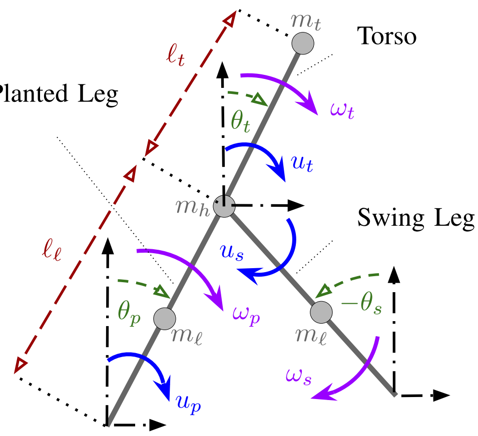


Fig. 1. Diagram of the state, input, and parameters of the biped model. The step angle and torso angle are measured from the same locations as $\theta_p$ and $\theta_t$, respectfully.

$\gamma$ contains the constants that describe the characteristics of the system, namely, the leg length $\ell_\ell$, the torso length $\ell_t$, the leg mass $m_\ell$, the hip mass $m_h$, the torso mass $m_t$, the gravity constant $\varrho$, the step angle $\phi_s$, the torso angle $\phi_t$, and the walking speed $v$.

### *B. Hybrid Model of a Biped*

A complete hybrid system model of a biped, denoted $\mathcal{H}_P$ is defined as

$$\mathcal{H}_P \begin{cases} \dot{x} = \begin{bmatrix} \omega \\ \alpha(x,u) \end{bmatrix} =: F_P(x,u) & (x,u) \in C_P \times \mathbb{R}^3 \\ x^+ = \begin{bmatrix} T(\theta) \\ \Omega(x) \end{bmatrix} =: G_P(x) & x \in D_P \end{cases} \quad (1)$$

where $x$ is the state and $u$ is the input. The continuous dynamics of $x = (\theta, \omega)$ come from the Lagrangian method [2]. It follows that

$$\dot{\theta} = \omega \quad (2)$$

$$D_f(\theta)\dot{\omega} + C_f(\theta,\omega)\omega + G_f(\theta) = Bu \quad (3)$$

where $D_f, C_f$, are the Inertial and Coriolis matrices and $B$ is the actuator relationship matrix. Solving this expression for $\dot{\omega}$ yields the angular accelerations of each limb, given by

$$\alpha(x,u) = D_f(\theta)^{-1}\left(-C_f(\theta,\omega)\omega - G_f(\theta) + Bu\right) \quad (4)$$

Equations (2) and (3) are used to define the flow map $F_P$ in (1). Impacts or jumps occur when the planted leg has reached the step angle such that both feet are in contact with the ground. To determine if the biped has reached the end of a step, we define the function $h$ as

$$h(x) := \phi_s - \theta_p \qquad \forall x \in \mathcal{X}. \quad (5)$$

When $h(x) = 0$, the angle of the planted leg has reached the step angle. A step will occur if the change of $h$ is such that $\theta_p$ is approaching $\phi_s$, and $h$ is equal zero. Thus, the jump set $D_P$ of $\mathcal{H}_P$ in (1) is defined as

$$\begin{aligned} D_P &:= \{x \in \mathcal{X} \ : \ h(x) = 0, \langle \nabla h(x), F_P(x,u) \rangle \le 0 \ \} \\ &= \{x \in \mathcal{X} \ : \ h(x) = 0, \omega_p \ge 0 \ \}. \end{aligned} \quad (6)$$

Where $F_P(x,u)$ is the flow mapping defined later in (8). It follows from $\langle \nabla h(x), F_P(x,u) \rangle = -\omega_p$ that the condition $\langle \nabla h(x), F_P(x,u) \rangle \le 0$ holds when the planted leg approaches the step angle with a nonnegative angular velocity $\omega_p$.The flow set, $C_P$ in (1) is a subset of the state space containing all of the states where the biped is evolving continuously. It is given by

$$C_P := \{x \in \mathcal{X} \ : \ h(x) \ge 0 \ \}. \quad (7)$$

Points $x$ such that $h(x) = 0$ are included to close the flow set, which, in particular, makes $\mathcal{H}_P$ well-posed (see [1]).

$$F_P(x,u) := \begin{bmatrix} \omega \\ \alpha(x,u) \end{bmatrix} \quad (8)$$

The changes of $x$ at jumps are defined by the jump map $G_P$ in (1). Following [2], when a step occurs, $G_P$ swaps the leg angles and velocities so that the swing leg becomes the planted leg, and the planted leg becomes the swing leg. A transformation matrix $T$ is defined to swap angle and velocity variables accordingly. Thus the angles after a step are mapped according to

$$\theta^+ = T(\theta) \quad (9)$$

After a step, the angular velocities are determined by a contact model that requires the full five degrees of the robot (instead of three degrees during the swing phase), which is achieved by including the Cartesian coordinates of the planted leg [2]. This model produces expressions for the angular velocities after an impact at $x$ occurs, which are

$$\Omega(x) = \begin{bmatrix} \Omega_p(x) \\ \Omega_s(x) \\ \Omega_t(x) \end{bmatrix} \quad (10)$$

where $\Omega_p, \Omega_s$, and $\Omega_t$ are the angular velocities of the planted leg, swing leg, and torso, respectively. Equations (9) and (10) lead to the jump map in (1).

## III. Hybrid Feedback Control for Finite Time Tracking

In this section, we propose a hybrid control strategy that achieves convergence of the state of the biped $\mathcal{H}_P$ to a reference trajectory of the desired periodic gait in two steps. The proposed controller consists of three components: a reference system $\mathcal{H}_r$, a virtual system $\mathcal{H}_z$, and the physical system $\mathcal{H}_P$, which is the biped itself given in Section 1. Our hybrid predictive control strategy guarantees that the virtual system $\mathcal{H}_z$ tracks the reference trajectory generated by $\mathcal{H}_r$, and that the plant $\mathcal{H}_P$ tracks the virtual trajectory generated by $\mathcal{H}_z$, in finite time.

### A. Basic Properties of Solutions to $\mathcal{H}_P$

The reference system $\mathcal{H}_r$ generates periodic solutions for $\mathcal{H}_P$ to converge to. Using prediction, the virtual system computes the trajectory that will cause the second step (or impact) to occur simultaneously with that of the reference. To design these reference trajectories, we determine solutions that describe the evolution of the system over the course of a single step. We refer to these solutions as "single-step" solutions. The input torques that produce an acceleration $a$ for a specified state $x$ are determined by a function (according to (4)) $\mu$, defined as

$$\mu(x,a) = B^{-1}\left(D_f(\theta)a + C_f(\theta,\omega)\omega + G_f(\theta)\right) \quad (11)$$

The following lemma demonstrates the process in which a "single-step" solution is defined for a given set of initial and final conditions within the state space, and a nonzero step time.

*Lemma 3.1: Given $x_i \in C_P$, $x_f \in D_P$, and $t_+ \in \mathbb{R}_{>0}$, where*

$$x_i = \begin{bmatrix}\theta_i\\ \omega_i\end{bmatrix} \quad x_f = \begin{bmatrix}\theta_f\\ \omega_f\end{bmatrix}$$

*there exist constants $B_0, B_1$ given by*

$$\begin{bmatrix}B_0\\ B_1\end{bmatrix} = \beta(x_i,x_f,t_+) := \begin{bmatrix}\beta_0(x_i,x_f,t_+)\\ \beta_1(x_i,x_f,t_+)\end{bmatrix} := \begin{bmatrix}\frac{-2}{t_+^2}\left(3\left(\theta_i-\theta_f\right)+\omega_f t_+ + 2\omega_i t_+\right)\\ \frac{1}{t_+^3}\left(12\left(\theta_f-\theta_i-\omega_i t_+\right)-6t_+\left(\omega_f-\omega_i\right)\right)\end{bmatrix} \quad (12)$$

*such that the functions*

$$t \mapsto \phi^f(t) = \begin{bmatrix}\theta(t)\\ \omega(t)\end{bmatrix}, \quad t\mapsto \mu^f(t) = \mu(\phi^f(t),a(t)) \quad (13)$$

*define the state trajectory and control input of a "single step" solution $(\phi^f,\mu^f)$ to the flow equation of $\mathcal{H}_P$ given by $\dot{x} = F_P(x,u), (x,u)\in C_P\times\mathbb{R}^3$: for each $t\in[0,t_+]$, where $\mu$ is given in (11), $\phi^f$ and $\mu^f$ in (13) are defined by*

$$\theta(t) = \theta_i + \omega_i t + \frac{1}{2}B_0 t^2 - \frac{1}{6}B_1 t^3 \quad (14)$$

$$\omega(t) = \omega_i + B_0 t - \frac{1}{2}B_1 t^2 \quad (15)$$

$$a(t) = B_0 - B_1 t \quad (16)$$

Given a set of parameters $\gamma$, the following lemma demonstrates the methodology for computing the initial and final states of the hybrid limit cycle.

*Lemma 3.2: Given the parameters $\gamma$ and defining the step time $t_+$ as*

$$t_+ = t_{step}(\gamma) := \frac{2\ell_\ell \sin\phi_s}{v}, \quad (17)$$

*there exist initial and final angular velocities $\omega_i,\omega_f$ of the initial and final limit cycle states, $x_i$ and $x_f$, respectively, which are determined by solving the system of equations obtained from (14) and (15) with $t = t_+$ and unknown variables $\omega_i,\omega_f$, where*

$$\theta_i = \begin{bmatrix}-\phi_s\\ \phi_s\\ \phi_t\end{bmatrix} \text{ and } \theta_f = \begin{bmatrix}\phi_s\\ -\phi_s\\ \phi_t\end{bmatrix},$$

*assign the initial state components of $x_i$ and final state components $x_f$, respectively. In particular, the unknowns $\omega_i$ and $\omega_f$ are given by*

$$\omega_i = \Omega(x_f) = \Omega\left(\begin{bmatrix}\theta_f\\ \omega_f\end{bmatrix}\right), \quad (18)$$

*where*

$$\omega_f = \omega_i + t_+ B_0 - \frac{1}{2}t_+^2 B_1 \quad (19)$$

*and*

$$B_0 = \beta_0\left(G_P\left(x_f\right), x_f, t_+\right)$$
$$B_1 = \beta_1\left(G_P\left(x_f\right), x_f, t_+\right),$$

*where $\Omega$ is defined in (10), $\beta_0$ and $\beta_1$ are defined in (12).*

Given the system parameters $\gamma$ and the calculated limb angular velocities in (18) and (19), we define the initial and final limit cycle states, denoted $x_i^*, x_f^*$, respectively, and determine the trajectory constants, denoted $B_0^*, B_1^*$, using (12), of the single-step solution that defines a limit cycle:

$$x_i^* = \begin{bmatrix}\theta_i^*\\ \omega_i^*\end{bmatrix}, x_f^* = \begin{bmatrix}\theta_f^*\\ \omega_f^*\end{bmatrix}, B^* = \begin{bmatrix}B_0^*\\ B_1^*\end{bmatrix} = \beta(x_i^*, x_f^*, t_{step}) \quad (20)$$

where

$$\theta_i^* = \begin{bmatrix}-\phi_s\\ \phi_s\\ \phi_t\end{bmatrix} \quad \theta_f^* = \begin{bmatrix}\phi_s\\ -\phi_s\\ \phi_t\end{bmatrix} \quad \omega_i^* = \begin{bmatrix}\omega_{pi}\\ \omega_{si}\\ \omega_{ti}\end{bmatrix} \quad \omega_f^* = \begin{bmatrix}\omega_{pf}\\ \omega_{sf}\\ \omega_{tf}\end{bmatrix}$$

These expressions for $x_i^*$ and $x_f^*$ can be verified using (14) and (15) to confirm that, from $x_i^*$, the resulting limb angles and velocities at time $t_{step}$ match those of $x_f^*$, from where the jump map $G_P$ leads to $x_i^* = G_P(x_f^*)$.

### B. Problem Statement and Control Strategy

With the above construction, we consider the following control problem:

***Problem*** $\star$: *Given a reference trajectory $r$ design an algorithm guaranteeing that every solution to $\mathcal{H}_P$ converges to the reference trajectory $r$.*

To solve it we propose the following hybrid control algorithm, which uses a reference system $\mathcal{H}_r$ and a virtual system $\mathcal{H}_z$ for prediction:

**Algorithm to Solve Problem** $\star$:

At each impact of the biped $\mathcal{H}_P$ that occurs at the end of each step:

**Step 1)** Predict the time to the next impact of the virtual system, denoted as $\Delta_\tau$, by computing the next two impacts of the reference system. Denote the two reference impact times by $t_1^r$ and $t_2^r$, respectively. If $t_1^r$ is less than half of the total step time $t_{step}$ then $\Delta_\tau = t_1^r$, otherwise $\Delta_\tau = t_2^r$. This time is called the *impact-to-track time*, and is computed by the mapping given in (27).

**Step 2)** Predict the initial and final limit cycle states. Denoted as $x_i^*$ and $x_f^*$, these state values are the beginning and end states of the desired walking gait. [1]

**Step 3)** Using the state of $\mathcal{H}_P$ immediately after the step, and the final limit cycle state $x_f^*$, compute the trajectory coefficients $B_0^*, B_1^*$ required to induce a virtual trajectory that will arrive at the final limit cycle state $x_f^*$ at the impact-to-track time $\Delta_\tau$.

**Step 4)** Store the coefficients $B_0^*, B_1^*$ from Step 3 in the virtual system to generate a solution that will be tracked by $\mathcal{H}_P$. In between impacts, control $\mathcal{H}_P$ with a feedback law that tracks the state of the virtual system.

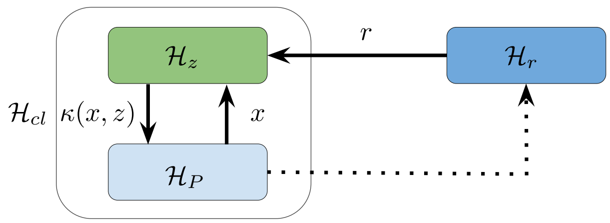


Fig. 2. A diagram illustrating the proposed control strategy. The full closed-loop hybrid system $\mathcal{H}_{cl}$ consists of the biped $\mathcal{H}_P$ and the virtual system $\mathcal{H}_z$, which performs tracking. The reference is generated externally, and is independent from $\mathcal{H}_{cl}$. The virtual system tracks the reference system $\mathcal{H}_r$ and the biped tracks the virtual system, resulting in the biped indirectly tracking the reference system.

### C. Generating the Reference Trajectory

We generate the reference trajectory by defining a system whose solution remains in a hybrid limit cycle, which is denoted $\mathcal{O}$ and defines the desired periodic walking gait that $\mathcal{H}_P$ will converge to. This is achieved by defining a hybrid system $\mathcal{H}_r$, given by a copy of $\mathcal{H}_P$ with two additional states used by the control law to produce the trajectory. The reference state $r$ is defined as

$$r = \begin{bmatrix} r_x \\ r_\tau \\ r_B \end{bmatrix} \tag{21}$$

where $r_x$ is the biped state, $r_B$ is the trajectory coefficient vector, and $r_\tau$ is the time elapsed since the beginning of the current step.

Then given a desired periodic walking gait specified by $x_i^*$, $x_f^*$ as the initial and final states, and coefficients $B_0^*, B_1^*$, leading to a hybrid limit cycle $\mathcal{O}$, any initial condition $r(0,0)$ in it leads to $r_x$ in $\mathcal{O}$. Then, $\mathcal{H}_r$ is defined as

$$\mathcal{H}_r \begin{cases} \dot{r} = \begin{bmatrix} F_P(r_x, \kappa_r(r)) \\ 1 \\ 0 \end{bmatrix} & (r_x, \kappa_r(r)) \in C_P \times \mathbb{R}^3 \\ r^+ = \begin{bmatrix} G_P(r_x) \\ 0 \\ \beta(x_0^*, x_f^*, t_{step}) \end{bmatrix} & r_x \in D_P \end{cases} \tag{22}$$

where $\kappa_r$ is the control law that produces the signal necessary to generate the trajectory that remains in $\mathcal{O}$, and is defined as

$$\kappa_r(r) = \mu(r_x, (r_{B_0} + r_{B_1} r_\tau)))$$

### D. Generating the Virtual Trajectory

The virtual system is essentially a copy of $\mathcal{H}_P$ with two additional states used by the control law to produce the virtual trajectory. It also uses the final limit cycle state $x_f^*$ to compute trajectories at each impact. The virtual state $z$ is defined as

$$z = \begin{bmatrix} z_x \\ z_\tau \\ z_B \end{bmatrix} \tag{23}$$

where $z_x$ is the virtual biped state, $z_\tau$ is the time elapsed since the beginning of the current step, and $z_B$ is the trajectory coefficient vector. The flows of $\mathcal{H}_z$ are given by

$$\left.\begin{aligned} \dot{z}_x &= F_P(z_x, \kappa_z(z)) \\ \dot{z}_\tau &= 1 \\ \dot{z}_B &= 0 \end{aligned}\right\} \quad (z_x, \kappa_z(z)) \in C_P \times \mathbb{R}^3 \tag{24}$$

where

$$\kappa_z(z) = \mu(z_x, (z_{B_0} + z_{B_1} z_\tau)) \tag{25}$$

The impacts of $\mathcal{H}_z$ occur when the biped state $x$ is in $D_P$, and are modeled by the set-valued discrete dynamics

$$\left.\begin{aligned} {z_x}^+ &= G_P(x) \\ {z_\tau}^+ &= 0 \\ {z_B}^+ &\in \kappa_B(G_P(x), r) \end{aligned}\right\} \quad x \in D_P \tag{26}$$

where $x$ is the state of the biped modeled by $\mathcal{H}_P$, $r$ is the state of the reference system $\mathcal{H}_r$, and $\kappa_B$ is a map that recomputes the trajectory coefficients using Steps 3 and 4 of the proposed algorithm, and according to the impact-to-track time in Step 1, $\Delta_\tau$ defined as

$$\Delta_\tau(r) := \left\{ \begin{array}{ll} t_{step} - r_\tau & \text{if } r_\tau < \frac{t_{step}}{2} \\ \{t_{step} - r_\tau, 2t_{step} - r_\tau\} & \text{if } r_\tau = \frac{t_{step}}{2} \\ 2t_{step} - r_\tau & \text{if } r_\tau > \frac{t_{step}}{2} \end{array} \right\} \tag{27}$$

This map predicts the impact-to-track time, as described in Step 1 of the proposed control algorithm in Section III-B, where $t_1^r = t_{step} - r_\tau$ and $t_2^r = 2t_{step} - r_\tau$. This is done to ensure that $\mathcal{H}_P$ does not attempt to converge too rapidly, and to ensure that the time to the next impact is nonzero. With the final limit cycle state $x_f^*$ from Lemma 3.2, the initial virtual biped state $z_x$, and the impact-to-track time in (27),

[1] This step does not need to be repeated if the biped characteristics do not change with time, but for the purpose of robustness, our algorithm performs this task recurrently.

we are able to compute coefficients that define the trajectory following Lemma 3.1. These coefficients are determined by (12) from Lemma 3.1. Then, we define $\kappa_B$ as

$$\kappa_B(x, r) = \beta(x, x_f^*, \Delta_\tau(r)) \tag{28}$$

which is set valued when $r_\tau = \frac{t_{step}}{2}$.

### E. Closed-Loop Hybrid System

The closed-loop system resulting from controlling $\mathcal{H}_P$ with the virtual system $\mathcal{H}_z$ can be written as the following hybrid system, which we denote by $\mathcal{H}_{cl}$:

$$\mathcal{H}_{cl}\begin{cases} \begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} F_P(x, \kappa(x,z)) \\ \begin{bmatrix} F_P(z_x, \kappa_z(z)) \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} & (x, \kappa(x,z)) \in C_P \\ \begin{bmatrix}x^+\\ z^+\end{bmatrix} \in \begin{bmatrix} G_P(x) \\ \begin{bmatrix} G_P(x) \\ 0 \\ \kappa_B(G_P(x), r) \end{bmatrix} \end{bmatrix} & x \in D_P \end{cases} \tag{29}$$

where $\kappa$ is the control law for $\mathcal{H}_P$, which is designed so that the trajectories of $\mathcal{H}_P$ track the trajectories of $\mathcal{H}_z$ (discussed in Section IV).

## IV. Main Results

In this section, we present our main results of the closed-loop system $\mathcal{H}_{cl}$. We show that the hybrid predictive control strategy ensures that $\mathcal{H}_P$ tracks the reference trajectories. Moreover, we numerically validate the hybrid model proposed in Section II and the hybrid controller proposed in Section III through simulations.

### A. Nominal Properties

Finite-time 0-tracking is defined as when the error between the biped state and the reference system trajectory is zero. The following result shows that the virtual system $\mathcal{H}_z$ is able to guarentee finite-time 0-tracking between the reference $\mathcal{H}_r$ and the plant $\mathcal{H}_P$.

*Proposition 4.1: For each reference trajectory $r$ generated from $\mathcal{H}_r$, each initial condition of the virtual system $z_x(0,0) \in C_P$, $z_\tau(0,0) = 0$, $z_B(0,0) = \beta(z_x(0,0), x_f^*, t_1)$ for some initial step duration $t_1 \in (0, t_{step}]$ of $\mathcal{H}_z$ in (26), where $t_{step}$ is given in (17), each solution to $\mathcal{H}_z$ is bounded, and its $\theta$ and $\omega$ components finite-time 0-track the $\theta$ and $\omega$ components of the reference system $r$ after two impacts; that is, the difference between the $\theta$ and $\omega$ components of the virtual system and reference system are zero after the second impact occurs, and remain at zero for all future time.*

With the property guaranteed by Proposition 4.1, next we show that the trajectories of the plant $\mathcal{H}_P$ converge to those of the virtual system $\mathcal{H}_z$ in finite time. To this end, let $e_1 := \theta - z_{x_\theta}$ and $e_2 := \omega - z_{x_\omega}$, where $\theta, \omega$ and $z_{x_\theta}, z_{x_\omega}$ are the angle and velocity components of the biped with state $x$ and of the virtual biped with state $z_x$, respectively. Using (29), the resulting error dynamics are then given by

$$\dot{e}_1 = \dot{\theta} - \dot{z}_{x_\theta} = \omega - z_{x_\omega} = e_2$$
$$\dot{e}_2 = \dot{\omega} - \dot{z}_{x_\omega} = \alpha(x, u) - (z_{B_0} + z_{B_1} z_\tau)$$

where $\alpha$ is the acceleration of the plant $\mathcal{H}_P$ from (4), and $z_{B_0} + z_{B_1} z_\tau$ is the acceleration of the virtual system $\mathcal{H}_z$. The input $u$ is assigned to the feedback law $\kappa$, which is to be designed. A particular choice of $\kappa$ to accomplish tracking between $x$ and $z$ is given by

$$\kappa(x, z) = \mu(x, z_{B_0} + z_{B_1} z_\tau - k_1(\theta - z_{x_\theta}) - k_2(\omega - z_{x_\omega}))$$

where $k_1, k_2 \in \mathbb{R}^+$. Then, with the control input $u = \kappa(x, z)$, it follows that

$$\dot{e_1} = e_2 \qquad \dot{e_2} = -k_1 e_1 - k_2 e_2$$

*Theorem 4.2: There exist $k_1, k_2 \in \mathbb{R}^+$ such that for each initial condition $x(0,0)$, $z_x(0,0) \in C_P$, $z_\tau(0,0) = 0$, $z_B(0,0) = \beta(z_x(0,0), x_f^*, t_1)$ for some initial step duration $t_1 \in (0, t_{step}]$ of $\mathcal{H}_{cl}$ in (29), and each reference system trajectory $r$ generated from $\mathcal{H}_r$, the $\theta$ and $\omega$ components of the plant $x$ of $\mathcal{H}_{cl}$ finite-time 0-tracks the $\theta$ and $\omega$ components of the virtual system $z$ after one impact; that is, the $\theta$ and $\omega$ components of the plant converge to those of the virtual system after the first impact occurs, and remain identical for all future time.*

### B. Simulations

We developed a software package to validate the results presented in Section IV-A for numerous different initial conditions and parameters. The package[2] also computes the solutions presented in Section III as both numeric or symbolic expressions. To evaluate the performance of the complete hybrid system, we ran simulations with randomized initial conditions and parameters. Figures 3 and 4 depict a few of the simulation results. The solid blue line corresponds to the trajectory of the physical system, the green dashed line to the virtual system, and the dotted red line to the reference system. The simulation shown in Figure 3 depicts the behavior of the system during the first two steps, where the physical, virtual, and reference systems are initialized to different initial conditions. This simulation shows the physical system converging to the virtual system after the first step, and the physical and virtual system converging to the reference system during the second step, demonstrating the closed-loop systems ability to converge with the reference by the time the second step occurs. The simulatios shown in Figure 4 has randomized step angle perturbations that emulate an uneven walking surface to demonstrate the robustness guaranteed by our hybrid predictive controller. Perturbations that cause the step angle to be less than its nominal value can be related to the biped walking up an incline as the swing leg would impact earlier than expected, and vice versa, when the step angle is greater than its nominal value.

[2] https://github.com/HybridSystemsLab/HybridThreeLinkBiped

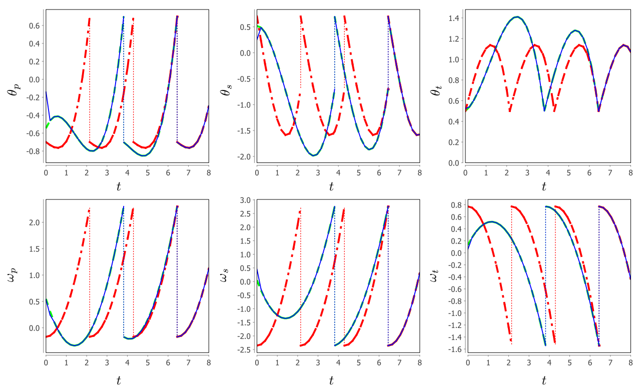

Fig. 3. Simulation results showing limb angles (rad) and velocities (rad/sec) with parameters: $\ell_\ell = 1, \ell_t = 1, m_t = 1, m_h = 1, m_\ell = 1, \phi_s = 0.7, v = 0.6, \phi_t = 0.5, k_1 = 2000, k_2 = 100$. The red dashed line indicates the reference system, the green dashed line indicates the virtual system, and the solid blue line indicates the physical system.

The step angle deviations caused by the perturbation angles result in unknown variations to the anticipated impact times, which requires the virtual system to adjust the trajectory of the following step to compensate. The simulations shown in Figure 4 shows the leg angles and velocities of the closed-loop system with randomized perturbations applied to the step angle, showing how the virtual system adjusts its trajectory to track the reference system when unknown disturbances are present.

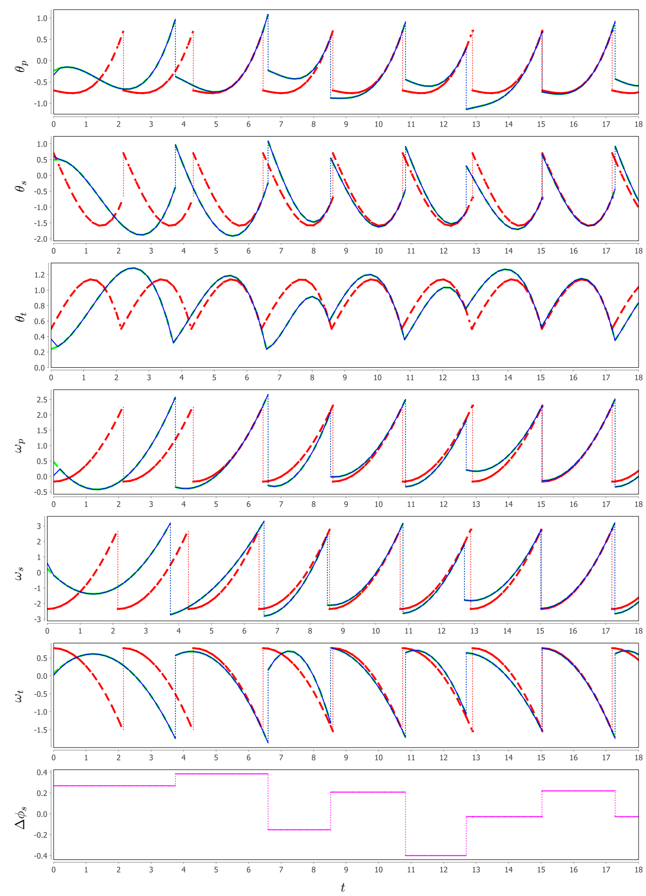

Fig. 4. Simulation results showing limb angles (rad) and velocities (rad/sec), and the perturbation in the step angle, denoted $\Delta\phi_s$, with parameters: $\ell_\ell = 1, \ell_t = 1, m_t = 1, m_h = 1, m_\ell = 1, \phi_s = 0.7, v = 0.7, \phi_t = 0.5, k_1 = 2000, k_2 = 100$. The red dashed line indicates the reference system, the green dashed line indicates the virtual system, and the solid blue line indicates the physical system.

## V. Conclusion

This paper presents a hybrid predictive control algorithm to drive a biped to tracking in finite time. The main results are validated in simulations. The results indicate that the control algorithm is capable of tracking even under perturbations. The modeling and proposed hybrid control strategy with prediction can be applied to other complex systems. Our potential next steps are to include additional joints, implement obstacle avoidance, accommodate gait variation, and extend the control strategy to the underactuated case.

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