Robust Hybrid Global Asymptotic Stabilization of Rigid Body Dynamics using Dual Quaternions

Bharani P. Malladi* and Eric A. Butcher,†
University of Arizona, Tucson, AZ, 85721, USA

Ricardo G. Sanfelice‡
University of California at Santa Cruz, Santa Cruz, CA, 95064, USA

A hybrid feedback control scheme is proposed for stabilization of rigid body dynamics (pose and velocities) using unit dual quaternions, in which the dual quaternions and velocities are used for feedback. It is well-known that rigid body attitude control is subject to topological constraints which often result in discontinuous control to avoid the unwinding phenomenon. In contrast, the hybrid scheme allows the controlled system to be robust in the presence of uncertainties, which would otherwise cause chattering about the point of discontinuous control while also ensuring acceptable closed-loop response characteristics. The stability of the closed-loop system is guaranteed through a Lyapunov analysis and the use of invariance principles for hybrid systems. Simulation results for a rigid body model are presented to illustrate the performance of the proposed hybrid dual quaternion feedback control scheme.

I. Introduction

Rigid body control is often separated into two individual problems: rotational control and translational control. However, for many practical applications that include robotics, computer graphics,[1,2] unmanned air vehicle control and spacecraft proximity operations[3–5] to name a few, these translational and rotational dynamics are often coupled. Hence, some recent research on controlling rigid body dynamics utilizes the Lie group SE(3) for the configuration space (pose) of the rigid body and its tangent bundle TSE(3) for the state space which includes velocities.[6] It is a well-known fact that global asymptotic stabilization of rigid body attitude is subject to topological constraints.[7,8] Similar topological obstructions are also present in rigid body pose control when represented on $SE(3)$.[9] Hybrid feedback control[10] can overcome such topological obstructions and provide robust global solutions for the rigid body attitude stabilization problem.[8] Dual numbers introduced by Clifford[11] and later generalized in,[12] are often used to parametrize the members of $SE(3)$. In,[3] a continuous controller for rigid body pose stabilization was presented. Results associated with the kinematic sub-problem of rigid body motion using hybrid hysteresis based UDQs are presented in,[9] while an improved version using a bimodal approach to reduce higher average settling time or energy consumption is presented in.[13]

In this paper we adapt the hysteresis-based switching strategy of rigid body attitude presented in[7,8] to the Unit Dual Quaternion (UDQ) parameterization of rigid body pose. Specifically, a complete solution for rigid body kinematic and kinetic control is presented using a hybrid hysteresis-based switching strategy. Following the results in,[3] the rigid body dynamics are modeled using a state space consisting of the unit dual quaternions to parameterize the pose (postition and attitude) along with the angular and translational velocities and a discrete logic variable as in.[7,8] We also discuss the robustness of the suggested hybrid algorithm to uncertainties.

*Ph.D candidate, Department of Aerospace and Mechanical Engineering, 1130 N Mountain Ave, AIAA student member. Email: malladi@email.arizona.edu.

†Professor, Department of Aerospace and Mechanical Engineering, 1130 N Mountain Ave, AIAA Member. Email: ebutcher@email.arizona.edu.

‡Associate Professor, Department of Computer Engineering, 1156 High Street MS:SOE3. Email: ricardo@ucsc.edu

The remainder of the paper is organized as follows. In Section II, the notation used throughout the paper is defined and the needed background material on hybrid systems is provided. The problem of interest is formalized in Section IV.A. A general hybrid feedback control solution is presented in Section IV.B, where an energy-based controller is characterized for full state state measurements. Robustness of the proposed algorithms to uncertainties is discussed in Section IV.C.1. Numerical simulations for a rigid body example are given in Section V. Proofs of the results will be published elsewhere.

II. Preliminaries

II.A. Notation

The following notation and definitions are used throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}$ denotes the real numbers. $\mathbb{Z}$ denotes the integers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. An identity element $\mathbf{1} = (1, \mathbf{0}_{3\times 1})$ and $\mathbf{0} = (0, \mathbf{0}_{3\times 1})$. $\mathbb{B}$ denotes the open unit ball in a Euclidean space. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$. The equivalent notation $[x^\top \ y^\top]^\top$ and (x, y) is used for vectors. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \searrow 0} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$.

II.B. Well-posed hybrid systems

Hybrid systems are dynamical systems with both continuous and discrete dynamics. In this paper, we consider the framework for hybrid systems in[10, 14], where a hybrid system $\mathcal{H}$ is defined by the following objects:

- A map $f : \mathbb{R}^n \to \mathbb{R}^n$ called the *flow map*.
- A map $g : \mathbb{R}^n \to \mathbb{R}^n$ called the *jump map*.
- A set $C \subset \mathbb{R}^n$ called the *flow set*.
- A set $D \subset \mathbb{R}^n$ called the *jump set*.

The flow map f defines the continuous dynamics on the flow set C, while the jump map g defines the discrete dynamics on the jump set D. These objects are referred to as the data of the hybrid system $\mathcal{H}$. Additionally, it also permits explicit modeling of perturbations in the system dynamics, a feature that is very useful for robust stability analysis of dynamical systems; see[14] for more details. A solution ϕ to $\mathcal{H}$ is parametrized by pairs (t, j), where t is the ordinary time component and j is a discrete parameter that keeps track of the number of jumps; see [10, Definition 2.6]. A solution ϕ to $\mathcal{H}$ is said to be *nontrivial* if $\operatorname{dom}\phi$ contains at least one point different from $(0, 0)$, *complete* if $\operatorname{dom}\phi$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}\phi$ onto $\mathbb{R}_{\geq 0}$ is bounded and *maximal* if it cannot be extended, i.e., it is not a truncated version of another solution. The set $\mathcal{S}_{\mathcal{H}}(\xi)$ denotes the set of all maximal solutions to $\mathcal{H}$ from ξ. In this paper, we employ the following asymptotic stability notion of a set for closed-loop hybrid systems.

Definition II.1 ((pre-)asymptotic stability) *Consider a hybrid system $\mathcal{H}$. A closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be*

- stable *for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}\phi$;*
- pre-attractive *for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \leq \delta$ is bounded and if it is complete then $\lim_{t+j \to \infty} |\phi(t, j)|_{\mathcal{A}} = 0$;*
- pre-asymptotically stable *if it is both stable and pre-attractive;*
- asymptotically stable *if it is pre-asymptotically stable and there exists $\delta > 0$ such that any maximal solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \leq \delta$ is complete.*

The set of all points in $\overline{C} \cup D$ from which all solutions are bounded and the solutions that are complete converge to $\mathcal{A}$ is called the basin of pre-attraction *of $\mathcal{A}$.*[a] △

II.C. Unit Dual Quaternions

1. $\mathbb{H}$ denotes a set of quaternions (not necessarily normalized), i.e., $\mathbb{H} := \{q : q = (\eta, \mu), \eta \in \mathbb{R}, \mu \in \mathbb{R}^3\}$, in which $\eta \in \mathbb{R}$ is the scalar part and $\mu \in \mathbb{R}^3$ is the vector part.

2. $\mathbb{H}_v$ denotes a set of quaternions with zero scalar part, i.e., $\mathbb{H}_v := \{q \in \mathbb{H} : \eta = 0\}$.

3. $\mathbb{H}_s$ denotes a set of quaternions with zero vector part, i.e., $\mathbb{H}_0 := \{q \in \mathbb{H} : \mu = \mathbf{0}_{3\times 1}\}$.

4. Given $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we define
$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

5. Given two vectors $x, y \in \mathbb{R}^3$, their cross product $x \times y = S(x)y = -y^\top S(x)$.

6. Given two quaternions $q_1, q_2 \in \mathbb{H}$, $q_1 = (\eta_1, \mu_1)$, $q_2 = (\eta_2, \mu_2)$, the following properties hold:
   - Conjugate: $q_1^* = (\eta_1, -\mu_1)$
   - Addition: $q_1 + q_2 = (\eta_1 + \eta_2, \mu_1 + \mu_2)$
   - Quaternion multiplication: $q_1 \otimes q_2 = (\eta_1\eta_2 - \mu_1^\top \mu_2, \eta_1\mu_2 + \eta_2\mu_1 + S(\mu_1)\mu_2)$
   - Dot product: $q_1 \cdot q_2 = \eta_1\eta_2 + \mu_1 \cdot \mu_2$.
   - Norm: $\|q_1\| = \sqrt{q_1 \cdot q_1}$.

7. $\mathcal{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$ denotes a n-dimensional sphere embedded in $\mathbb{R}^{n+1}$. In particular, $\mathcal{S}^3$ denotes the set of unit quaternions, which is often used to parameterize the Lie group SO(3) of rigid body attitude, where each unit quaternion is such that $\|q\|^2 = \eta^2 + \mu^\top\mu = 1$.

8. The set $\mathcal{S}^3$ has, under the quaternion multiplication, an identity element $\mathbf{1} = (1, \mathbf{0}_{3\times 1})$ and the inverse given by the quaternion conjugate q^*.

9. Given a matrix $M \in \mathbb{R}^{4\times 4}$, and a quaternion $q \in \mathbb{H}$,
$$Mq = (M_{11}\eta + M_{12}\mu, M_{21}\eta + M_{22}\mu) \in \mathbb{H}$$
where $M_{11} \in \mathbb{R}$, $M_{12} \in \mathbb{R}^{1\times 3}$, $M_{21} \in \mathbb{R}^{3\times 1}$, $M_{22} \in \mathbb{R}^{3\times 3}$ are entries of $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

10. The set of dual quaternions is given by
$$\underline{\mathbb{H}} := \{\overline{q} : \overline{q} = (\overline{\eta}, \overline{\mu}) = q_r + \epsilon q_t, \overline{\eta} \in \mathbb{R}, \overline{\mu} \in \mathbb{R}^3\}$$
where ϵ is called the dual unit which is nilpotent, i.e., $\epsilon \neq 0$, $\epsilon^2 = 0$.
    - $\overline{\eta} = \eta_r + \epsilon\eta_t \in \mathbb{R}$ is the dual scalar part.
    - $\overline{\mu} = \mu_r + \epsilon\mu_t \in \mathbb{R}^3$ is the dual vector part.
    - $q_r = (\eta_r, \mu_r) \in \mathcal{S}^3$ is the rotational part parameterizing attitude.
    - $q_t = (\eta_t, \mu_t) \in \mathbb{H}_v$ is the translational part.

[a]Note that by definition, the basin of pre-attraction contains a neighborhood of $\mathcal{A}$. In addition, points in $\mathbb{R}^n \setminus (C \cup D)$ always belong to the basin of pre-attraction since there are no solutions starting at such points, and therefore, there is nothing to be checked. Furthermore, if $\mathcal{A}$ is pre-asymptotically stable and every maximal solution is complete, then we say that $\mathcal{A}$ is asymptotically stable (without the prefix "pre").

11. $\underline{\mathbb{H}}_v$ denotes a set of dual quaternions with zero scalar part, i.e., $\underline{\mathbb{H}}_v := \{\overline{q} \in \underline{\mathbb{H}} : \overline{\eta} = 0\}$.
12. Given a dual quaternion $\overline{q} \in \underline{\mathbb{H}}$, the following properties hold:
    - Conjugate: $\overline{q}^* = q_r^* + \epsilon q_t^* = (\overline{\eta}, -\overline{\mu})$.
    - Swap: $\overline{q}^s = q_t + \epsilon q_r$.
13. $\underline{\mathcal{S}}^3$ denotes the set of unit dual quaternions, where each unit dual quaternion $\overline{q} \in \underline{\mathbb{H}}$ is such that $\|q_r\| = 1$ and $q_r \otimes q_t^* + q_t \otimes q_r^* = \mathbf{0}_{4\times 1}$.
14. Given any dual quaternions $\overline{q}_1, \overline{q}_2, \overline{q}_3 \in \underline{\mathcal{S}}^3$, the following properties hold:
    - Dual quaternion multiplication: $\overline{q}_1 \otimes \overline{q}_2 = q_{r_1} \otimes q_{r_2} + \epsilon(q_{r_1} \otimes q_{t_2} + q_{t_1} \otimes q_{r_2})$.
    - Dot product: $\overline{q}_1 \cdot \overline{q}_2 = \frac{1}{2}(\overline{q}_1^* \otimes \overline{q}_2 + \overline{q}_2^* \otimes \overline{q}_1) = \frac{1}{2}(\overline{q}_1 \otimes \overline{q}_2^* + \overline{q}_2 \otimes \overline{q}_1^*) = q_{r_1} \cdot q_{r_2} + \epsilon(q_{t_1} \cdot q_{r_2} + q_{r_1} \cdot q_{t_2})$.
    - Circle product: $\overline{q}_1 \circ \overline{q}_2 = q_{r_1} \cdot q_{r_2} + q_{t_1} \cdot q_{t_2}$.
    - $\overline{q}_1 \circ (\overline{q}_2 \otimes \overline{q}_3) = \overline{q}_2^s \circ (\overline{q}_1^s \otimes \overline{q}_3^*) = \overline{q}_3^s \circ (\overline{q}_2^* \otimes \overline{q}_1^s)$.
    - $M \star \overline{q} = (M_{11} q_r + M_{12} q_t) + \epsilon(M_{21} q_r + M_{22} q_t)$, $M_{i,j} \in \mathbb{R}^{4\times 4}$, $i, j \in \{1, 2\}$.
15. Unit Dual Quaternions (UDQs) are often used to parametrize the Lie group $SE(3)$ of rigid body pose.
16. The set $\underline{\mathcal{S}}^3$ has, under the dual quaternion multiplication, an identity element $\overline{\mathbf{1}} = \mathbf{1} + \epsilon\mathbf{0}$ and the inverse given by the dual quaternion conjugate $\overline{q}^*$.

III. Rigid Body Kinematics and Dynamics

III.A. Kinematics

An arbitrary rigid body configuration with respect to a fixed inertial frame is characterized by rotation $q_r \in \mathcal{S}^3$ followed by the translation $q_t \in \mathbb{H}_v$ using dual quaternions[15] as

$$\overline{q} = q_r + \epsilon q_t, \tag{1}$$

where given an angle $\theta \in \mathcal{S}^1$ and an axis $\hat{n} \in \mathcal{S}^2$, a unit quaternion

$$q_r = \begin{bmatrix} \eta_r \\ \mu_r \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{n} \end{bmatrix}, \tag{2}$$

represents vector rotation from body frame to inertial frame. As noted in (1), given the relative position $r_d \in \mathbb{R}^3$ of the rigid body, with respect to the inertial frame and represented in the inertial frame, the translational part of the dual quaternion is given by

$$q_t = \frac{1}{2}\nu(r_d) \otimes q_r, \tag{3}$$

where $\nu(r_d) = (0, r_d)$. Therefore, from (1), (2), (3), the combined rotational and translational kinematics of a rigid body in UDQ representation[3,13] are

$$\dot{\overline{q}} = \frac{1}{2}\nu(\overline{\omega}_I) \otimes \overline{q}. \tag{4}$$

where $\nu(\overline{\omega}_I) = (0 + \epsilon 0, \overline{\omega}_I)$, $\overline{\omega}_I = \omega_I + \epsilon(\dot{r}_d + r_d \times \omega_I)$ is called the twist, $\omega_I \in \mathbb{R}^3$, $\dot{r}_d \in \mathbb{R}^3$ are the angular and translational velocities of the rigid body with respect to the inertial frame represented in the inertial frame, respectively. In many applications, it is a common practice to measure the angular and translational velocities of the rigid body in the body frame using onboard sensors. Hence, following the transformation presented in [15, Appendix A], the rigid body kinematics are

$$\dot{\overline{q}} = \tfrac{1}{2}\nu(\overline{\omega}_I) \otimes \overline{q} = \tfrac{1}{2}\overline{q} \otimes \nu(\overline{\omega}) = \tfrac{1}{2}\begin{bmatrix} -\overline{\mu}^\top \\ \overline{\eta} I + S(\overline{\mu}) \end{bmatrix} \overline{\omega} \qquad \overline{q} \in \underline{\mathcal{S}}^3, \tag{5}$$

where $\nu(\overline{\omega}) = (0 + \epsilon 0, \overline{\omega})$, $\overline{\omega} = \omega + \epsilon v \in \mathbb{R}^3$, $\omega, v \in \mathbb{R}^3$ are the angular and translational velocities or the rigid body with respect to the inertial frame represented in the body frame, respectively; $\overline{\eta} = \eta_r + \epsilon\eta_t \in \mathbb{R}$, $\overline{\mu} = \mu_r + \epsilon\mu_t \in \mathbb{R}^3$.

III.B. Rigid Body Kinetics

Let the dual inertia matrix be defined as the mass of the rigid body given by the dual inertia matrix

$$M = \begin{bmatrix} 1 & 0_{1\times3} & 0 & 0_{1\times3} \\ 0_{3\times1} & mI_{3\times3} & 0_{3\times1} & 0_{3\times3} \\ 0 & 0_{1\times3} & 1 & 0_{1\times3} \\ 0_{3\times1} & 0_{3\times3} & 0_{3\times1} & J \end{bmatrix} \in \mathbb{R}^{8\times8}, \tag{6}$$

where $m \in \mathbb{R}$ is the mass of the rigid body, $J \in \mathbb{R}^{3\times3}$ is the mass moment of inertia of the body about its center of mass written in the body frame, and I is the identity matrix. Following[3] the dynamics of the rigid body are

$$M \star \nu(\dot{\overline{\omega}}^s) = T - \nu(\overline{\omega}) \times (M \star \nu(\overline{\omega}^s)) \tag{7}$$

where $\nu(\overline{\omega}^s) = (0, (\overline{\omega}^s)) \in \underline{\mathbb{H}}_v$, $\overline{\omega}^s = v + \epsilon\omega \in \mathbb{R}^3$, $\overline{\omega} = \omega + \epsilon v \in \mathbb{R}^3$, $T = (0, F) + \epsilon(0, \tau) \in \underline{\mathbb{H}}_v$, $F \in \mathbb{R}^3$ represents control forces and $\tau \in \mathbb{R}^3$ represents control torques applied to the rigid body. Since the mass $m \in \mathbb{R}$ and the inertial matrix $J \in \mathbb{R}^3$ are considered a constant, the dual inertial matrix M in the above formulation is invertible and the inverse of M is M^{-1}. This formulation is advantageous over the formulation in,[16] as it does not include the operator $\frac{d}{d\epsilon}$ defined by $\frac{d}{d\epsilon}\overline{q} = \frac{d}{d\epsilon}(q_r + \epsilon q_t)$ (see[16] for more details).

IV. Hybrid Feedback Control and Stability

IV.A. Problem Description

Given the desired attitude of the rigid body with respect to the inertial frame $t \mapsto R_d(t) \in SO(3)$ and the desired position of the center of mass of the rigid body with respect to the inertial frame $t \mapsto r_d(t) \in \mathbb{R}^3$; the attitude and position errors are given by $R = R_d^\top R_m$ element of $SO(3)$ and $r = r_m - r_d \in \mathbb{R}^3$, respectively and the '$m$' subscript denotes the true attitude and position. Therefore, the objective is to design a controller to have $R_d = R_m$ and $r_d = r_m$ or to globally asymptotically stabilize $R = I$, $r = 0 \in \mathbb{R}^3$. In terms of dual quaternions, this reduces to the problem of designing a controller that globally asymptotically stabilizes $\overline{q} = (\overline{\eta}, \overline{\mu}) = \overline{q}_d{}^* \otimes \overline{q}_m = \overline{\mathbf{1}}$ for the rigid body kinematics in (5) along with $\overline{\omega} = \mathbf{0}_{3\times1} \in \mathbb{R}^3$ for the dynamics in (7).

Similar to the unit quaternion case (for attitude stabilization),[8] dual quaternions provide a dual cover for the elements in $SE(3)$, i.e., for every element in $SE(3)$, and for each r, there are exactly two UDQs $\pm\overline{q} = \pm(q_r + \epsilon q_t)$, such that $R = \mathcal{R}(q_r) = \mathcal{R}(-q_r)$, where given an angle $\theta \in \mathbb{R}$ and an axis $\hat{n} \in \mathcal{S}^2$, a unit quaternion q_r in (2) represents an element of $SO(3)$ by the map $\mathcal{R} : \mathcal{S}^3 \to SO(3)$ defined as $\mathcal{R}(q_r) = I + 2\eta_r S(\mu_r) + 2S^2(\mu_r)$. In addition, following the principle of transference presented in,[17] the characteristics of unit quaternion are completely inherited by dual quaternions. Therefore, the rigid body pose stabilization using dual quaternions can have disadvantages due to this non-uniqueness. Similar to the problem of rigid body attitude stabilization in $SO(3)$,[8] a continuous linear feedback law results in the 'unwinding' phenomenon. Alternatively, a discontinuous controller can be designed as in.[18,19]

Such a discontinuous controller would not be robust to small measurement noise as previously shown in the literature for both dual quaternion and quaternion cases.[7,8] Hence, we introduce a logic variable as in[7,8] to handle the topological obstruction of stabilizing a set on a manifold. We design a logic-based hybrid controller that steers the rigid body clockwise or counter-clockwise to take shortest route and reach the desired orientation and position while remaining robust to small perturbations. We assume that, measurements of dual quaternion $\overline{q}_m$ and dual velocity $\overline{\omega}_m$ are available and, hence, dual quaternion error $\overline{q}$ and dual velocity error $\overline{\omega}$ are available, respectively, in IV.C. Following the results in[8] we include a hybrid switching logic into the controller suggested in[20] to overcome topological issues with antipodal points.

IV.B. Hybrid Closed-Loop System

Given the rigid body kinematics and dynamics in (5), (7), to overcome the previously mentioned topological obstructions, and inspired from,[8] the following hybrid model is considered along with the logic variable $h \in \{-1, 1\} =: \mathcal{Q}$. The hybrid model of the rigid body kinematics and dynamics $\mathcal{H} = (C, f, D, g)$ has state

$\xi = (\overline{q}, \nu(\overline{\omega}^s), h) \in \underline{\mathcal{S}}^3 \times \underline{\mathbb{H}}_v \times \mathcal{Q} =: \mathcal{X}$, with flow and jump dynamics

$$\begin{array}{rclll} \dot{\xi} & = & f(\xi, T) & (\xi, T) & \in C \times \underline{\mathbb{H}}_v, \\ \xi^+ & = & g(\xi) & \xi & \in D, \end{array} \tag{8}$$

respectively, where maps $f : \mathcal{X} \times \underline{\mathbb{H}}_v \to \mathcal{X}$, $g : \mathcal{X} \to \mathcal{X}$ and the sets $C \subset \mathcal{X}$, $D \subset \mathcal{X}$ are

$$f(\xi, T) := \begin{bmatrix} \frac{1}{2}\overline{q} \otimes \nu(\overline{\omega}) \\ M^{-1} \star (T - \nu(\overline{\omega}) \times (M \star \nu(\overline{\omega}^s))) \\ 0 \end{bmatrix}, \qquad g(\xi) := \begin{bmatrix} \overline{q} \\ \nu(\overline{\omega}^s) \\ -h \end{bmatrix},$$

$C = \{\xi \in \mathcal{X} : h\eta_r \geq -\delta\}$, $D = \{\xi \in \mathcal{X} : h\eta_r \leq -\delta\}$, $\delta \in (0, 1)$. (Note that $\nu(\overline{\omega}^s) = (0, (\overline{\omega}^s)) \in \underline{\mathbb{H}}_v$ is considered as a state over $\overline{\omega}^s = v + \epsilon\omega \in \mathbb{R}^3$ to preserve the dimensionality of the system.) For this hybrid system, we design $T \in \underline{\mathbb{H}}_v$ to globally asymptotically stabilize the compact set

$$\mathcal{A} = \{\xi \in \mathcal{X} : \overline{q} = h\overline{\mathbf{1}}, \overline{\omega}^s = \mathbf{0}_{3\times1}\} \tag{9}$$

where $\overline{\mathbf{1}} = \mathbf{1} + \epsilon\mathbf{0}$.

IV.C. Dual quaternion and dual velocity feedback

We assume that the output of the system (8) is measured as $y = (\overline{q}_m, \overline{\omega}_m)$ and hence the error vector $(\overline{q}, \overline{\omega})$ is available for feedback. Therefore, considering the hybrid system $\mathcal{H}$ given in (8), our main result is as follows.

Theorem IV.1 *The hybrid feedback*

$$T = \kappa(\xi) := -k_p h(\overline{q}^* \otimes (h\overline{q}^s - \epsilon\mathbf{1})) - k_d\nu(\overline{\omega}^s) \in \underline{\mathbb{H}}_v, \tag{10}$$

where $k_p, k_d > 0$, dual number $\epsilon \neq 0$, $\epsilon^2 = 0$, renders the set $\mathcal{A}$ globally asymptotically stable for the hybrid system $\mathcal{H}$ in (8).

Sketch of proof : For the closed-loop hybrid system resulting from controlling the plant (8) with the controller (10), we first show that every complete solution to it converges to $\mathcal{A}$. For this purpose, we use the invariance principle for hybrid systems in[10] for which $\mathcal{H}$ has to satisfy the hybrid basic conditions, which is the case. Specifically, the sets C and D are closed subset of $\mathcal{X}$ and by definition, the maps $f : \mathcal{X} \times \underline{\mathbb{H}}_v \to \mathcal{X}$, $g : \mathcal{X} \to \mathcal{X}$ are continuous, respectively, and we conclude that the closed-loop system $\mathcal{H}$ is well-posed following [10, Assumption 6.5]. In addition, since $\mathcal{H}$ satisfies the hybrid basic conditions, following [10, Proposition 6.10], we can conclude every maximal solution to the hybrid system is complete.

Now to show the convergence of maximal solutions to $\mathcal{A}$, consider the function $V : \mathcal{X} \to \mathbb{R}$ defined as

$$V(\xi) = k_p(h\overline{q} - \overline{\mathbf{1}}) \circ (h\overline{q} - \overline{\mathbf{1}}) + \frac{1}{2}\nu(\overline{\omega}^s) \circ (M \star \nu(\overline{\omega}^s)), \tag{11}$$

where '$\circ$' operator for the UDQs is defined in Section II.C. Applying invariance principles for hybrid systems in [10, Theorem 8.2], every precompact (complete and bounded) solution to the hybrid system (8) converges to the largest weakly invariant set W inside

$$\{\xi \in \mathcal{X} : h\eta_r \geq -\delta, \overline{\omega} = \mathbf{0}_{3\times1}\} \cap V^{-1}(a)$$

for some $a \in \mathbb{R}$. Since $\mathcal{A}$ is compact, and the function $V : \mathcal{X} \to \mathbb{R}$ in (11) is positive definite relative to $\mathcal{A}$ and non-increasing along the solutions of $\mathcal{H}$, then the set W is contained in $\mathcal{A}$. Hence, the set $\mathcal{A}$ is asymptotically stable and global asymptotic stability is a consequence of well-posedness and the asymptotic stability property of $\mathcal{A}$; see [10, Chapter 7].

IV.C.1. Robustness of the Closed-loop System

To take our stability analysis close to the real world problems, let us consider that the plant (8) is affected by unmodeled dynamics given by $d_1 \in \mathcal{X}$, actuator error $d_2 \in \mathbb{R}^{4\times 1}$ and measurement error $d_3 \in \mathcal{X}$ as

$$\dot{\xi} = f(\xi + d_3, T + d_2) + d_1.$$

where $d_3 = (d_{3_q}, d_{3_\omega}, 0) \in \mathcal{X}$, $d_{3_q} \in \underline{\mathcal{S}}^3$, $d_{3_\omega} \in \mathbb{H}_v$. Denoting $\tilde{\xi} := \xi + d_3$, the closed-loop system $\mathcal{H}$ in (8) results in the perturbed closed-loop system $\mathcal{H}_d$ with dynamics

$$\begin{array}{rcll} \dot{\xi} & = & f_d(\tilde{\xi}, T + d_2) + d_1 & (\tilde{\xi}, T + d_2) + d_1 \in C_d \times \underline{\mathbb{H}}_v, \\ \xi^+ & = & g_d(\tilde{\xi}) & \tilde{\xi} \in D_d, \end{array} \tag{12}$$

where the maps $f_d : \mathcal{X} \times \underline{\mathbb{H}}_v \to \mathcal{X}$, $g_d : \mathcal{X} \to \mathcal{X}$ and the sets $C_d \subset \mathcal{X}$, $D_d \subset \mathcal{X}$, respectively, are

$$f_d(\tilde{\xi}, T + d_2) := \begin{bmatrix} \frac{1}{2}(\overline{q} + d_{3_q}) \otimes \nu(\overline{\omega} + d_{3_\omega}) \\ M^{-1} \star (T + d_2 - \nu(\overline{\omega} + d_{3_\omega}) \times (M \star \nu(\overline{\omega}^s + d_{3_\omega}))) \\ 0 \end{bmatrix}, \qquad g_d(\xi) := \begin{bmatrix} \overline{q} + d_{3_q} \\ \nu(\overline{\omega}^s + d_{3_\omega}) \\ -h \end{bmatrix}$$

$C_d = \{\tilde{\xi} \in \mathcal{X} : h\eta_r \geq -\delta\}$, $D_d = \{\tilde{\xi} \in \mathcal{X} : h\eta_r \leq -\delta\}$, where $\delta \in (0, 1)$.

Following the global asymptotic stability property of the set $\mathcal{A}$ for the closed-loop system $\mathcal{H}$ established in Theorem IV.1, we have the following result.

Theorem IV.2 *There exists $\beta \in$ class-$\mathcal{KL}$ such that, for each $\varepsilon > 0$ and each compact set $\mathcal{M} \subset \mathcal{X}$, there exists $d > 0$ such that for each measurable $d : \mathbb{R}_{\geq 0} \to \delta\mathbb{B}$ every solution ϕ to the hybrid system $\mathcal{H}$ with hybrid feedback controller* (10) *with initial condition $\phi(0, 0) \in \mathcal{M}$ satisfies*

$$|\phi(t, j)|_{\mathcal{A}} \leq \beta(|\phi(0, 0)|_{\mathcal{A}}, t + j) + \varepsilon \qquad \forall (t, j) \in \operatorname{dom} \phi.$$

V. Simulations

To verify the ideas presented in this paper we apply the hybrid hysteresis based switching strategy to a rigid body model with mass $m = 1$kg, and inertia

$$J = \begin{bmatrix} 1 & 0.1 & 0.15 \\ 0.1 & 0.63 & 0.05 \\ 0.15 & 0.05 & 0.85 \end{bmatrix} \text{kg-m}^2$$

as in.[20] In the results presented below, each of the the plots show simulations of 'hybrid', 'discontinuous' and 'continuous', controllers. For the simulations labeled hybrid, the hysteresis half-width $\delta \in (0, 1)$ and $h(t, j) \in \{-1, 1\}$. When the hysteresis width $\delta = 0$, the controller reduces to discontinuous scheme where

$$h(t, j) := sgn(\eta_r) = \begin{cases} -1 & \eta_r < 0 \\ 1 & \eta_r \geq 0. \end{cases} \tag{13}$$

When $\delta > 1$, $h(t, j) = 1$ and a continuous controller exhibiting unwinding is implemented. To this end, simulations associated with various initial conditions are presented to show that the suggested hybrid controller (10) is insensitive to initial conditions and indeed robustly globally asymptotically stable. Next, simulations associated with full state feedback using controller (10), where the output of the system (8) is measured as $y = (\overline{q}_m, \overline{\omega}_m)$ (and hence the error vector $(\overline{q}, \overline{\omega})$ is available for feedback) are presented in Section V.A. Following the results in Section IV.C.1, for all the simulation results below, a zero-mean Gaussian noise with variance $(0.2)^2$ is added to the state $\eta_r \in \mathbb{R}$ (corresponding to the principal angle), and a variance $(0.2m)^2$ is added to $\mu_t \in \mathbb{R}^3$ (corresponding to the position) to account for measurement errors in configuration. In addition, a zero-mean Gaussian noise with variances $(0.02rad/sec)^2$ and $(0.02m/sec)^2$ are added to the angular velocity $\omega \in \mathbb{R}^3$ and translational velocity $v \in \mathbb{R}^3$ of the rigid body, respectively, to account for measurement errors in velocities. This additional noise in the states results in chattering behavior for the switching signal $sgn(\eta_r)$ for the discontinuous controller, while the hysteresis based hybrid logic is impervious to such noise as shown in Figures 1-3.

V.A. Dual quaternion and dual velocity feedback

V.A.1. Measurement noise

The response of the closed-loop rigid body dynamics with energy-based controller (10) when dual quaternion error and velocity errors $(\overline{q}, \overline{\omega})$ are available for feedback is presented in Figure 1. The simulations are performed with the initial condition set to $q_r(0,0) = (0, 0.4243, 0.5657, 0.7071)$, $r_p(0,0) = (0, 25, 25, 25)$, $\omega(0,0) = (0.2, 0.4, 0.6) rad/s$, $v(0,0) = (0.1, 0.2, 0.3) m/s$ and $h(0,0) = 1$. The energy-based controller has the gains $k_d = 0.5$, $k_p = 0.5$ and a hysteresis gap of $\delta = 0.1$. Figure 1 also shows a comparison between the

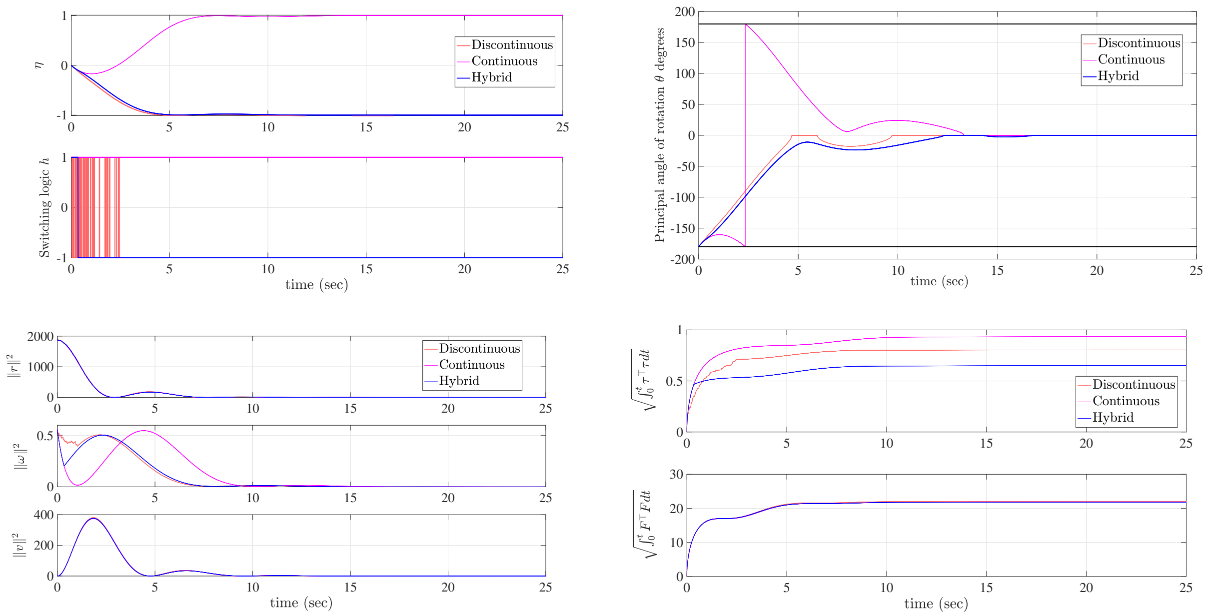

Figure 1. Closed-loop response of the continuous, discontinuous and hybrid controllers subjected to measurement noise with the switching logic $h(0,0) = 1$ and $\delta = 0.1$.

linear continuous controller with $h(t,j) = 1$, a discontinuous controller where the switching logic variable $h(t,j) := sgn(\eta_r)$ as in (13) and the hybrid controller with $h(t,j) \in \{-1, 1\}$ as in Section IV.C. Next, we consider a larger hysteresis width of $\delta = 0.4$ and repeat the the simulations with the same set of initial conditions and uncertainties as above. The hybrid controller now exhibits the same unwinding solution as the linear continuous controller due to the larger hysteresis gap.

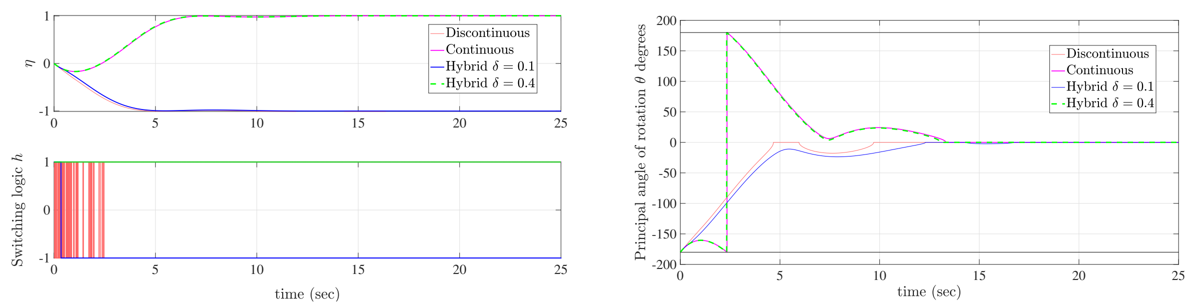

Figure 2. Unwinding in rigid body rotational and translational dynamics with the switching logic $h(0,0) = 1$ and $\delta = 0.4$.

In addition, Figure 3 presents the results with the initial condition set to $r_p(0,0) = (0, 2, 2, 1)$, $q_r(0,0) = (-0.4618, 0.1917, 0.7999, 0.3320)$, $\omega(0,0) = (-0.1, 0.2, -0.3) rad/s$, $v(0,0) = (0.1, -0.2, 0.3) m/s$ and $h(0,0) = 1$. The energy-based controller has the gains $k_d = 0.5$, $k_p = 0.5$ and a hysteresis gap of $\delta = 0.4$. Noise is added similar to the previous simulations. As discussed previously in,[8] there is a correlation between hysteresis width δ and the sensitivity of the controller (10) to noise and the control effort as shown in Figures 2- 3.

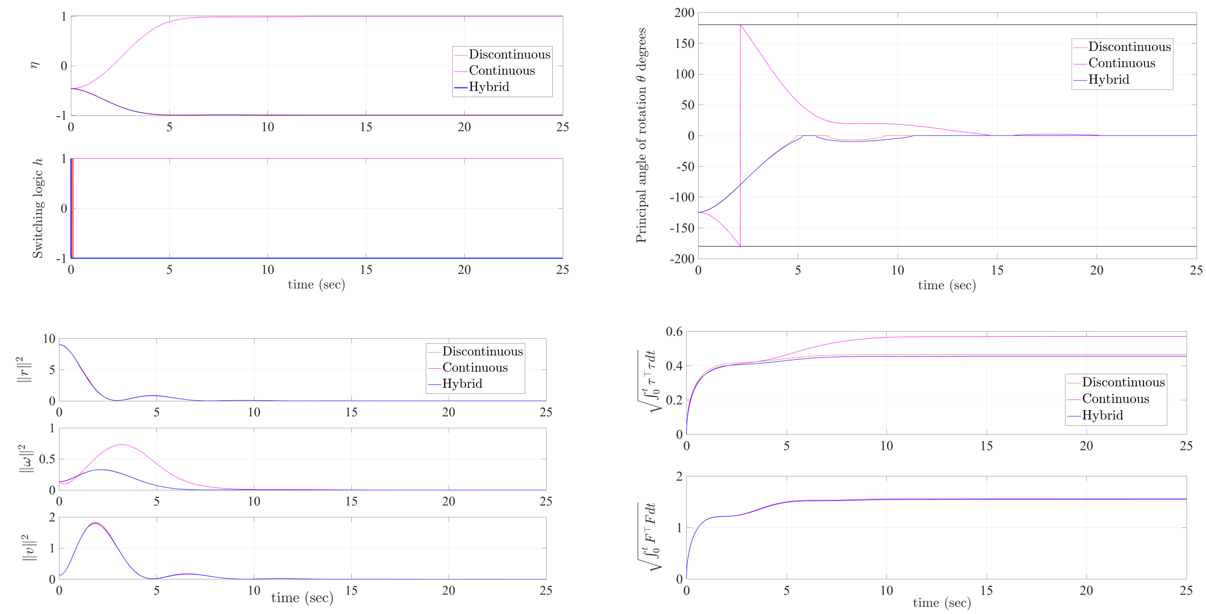

Figure 3. Rigid body rotational and translational dynamics with the switching logic $h(0,0) = 1$ and $\delta = 0.4$.

V.A.2. Measurement noise and actuator error

In addition to the measurement noise in section V.A.1, to include actuator errors, a zero-mean Gaussian process noise with variance $(0.2)^2$ and $(0.2m)^2$ is added to the input states η_r and μ_t, respectively. And a zero-mean Gaussian process noise with variance $(0.02rad/sec)^2$ and $(0.02m/sec)^2$ is added to the angular velocity ω and translational velocity v of the input states, respectively.

As discussed in Section IV.A and evident from the simulation results, the continuous linear feedback control exhibits unwinding, while the discontinuous feedback control with $h(t,j) := \text{sgn}(\eta_r)$ exhibits chattering in the presence of measurement noise and is not impervious to noise and requires additional control effort. The hybrid feedback control, however, is robust to the disturbances and achieves robust global asymptotic stability for the rigid body dynamics as shown in Figure 4.

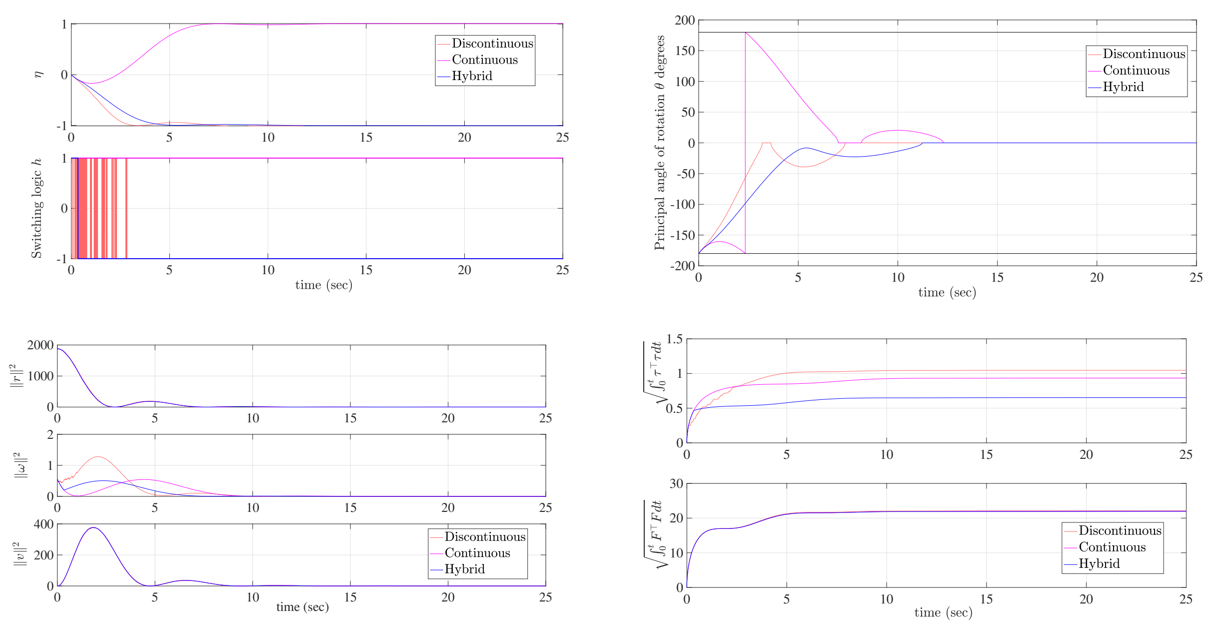

Figure 4. Closed-loop response of the continuous, discontinuous and hybrid controllers subjected to measurement noise and input error with the switching logic $h(0,0) = 1$ and $\delta = 0.1$.

As suggested in,[8] as well as the results in Sections IV.C, V, there is a trade-off between the choice of hysteresis width δ (which should be increased for greater amounts of noise to ensure robustness) and the control energy expended with the dual quaternion formulation.

VI. Conclusion

In this paper, a hybrid UDQ feedback control scheme was proposed for rigid body robust pose stabilization with full state of the system available for feedback. The stability of the closed-loop system was guaranteed through an energy-based Lyapunov function analysis using invariance principles for hybrid systems. We showed that the proposed control schemes can globally asymptotically stabilize the kinematics and kinetics and establish global asymptotic stability for a rigid body. In addition, the proposed hybrid scheme allows for the controlled system to be stable in the presence of small uncertainty, which would otherwise cause chattering about the point of discontinuous control.

Acknowledgments

Research by B. P. Malladi and E. A. Butcher is supported by the Dynamics, Control, and Systems Diagnostics Program of the National Science Foundation under Grant CMMI-1561836. Research by R. G. Sanfelice is partially supported by NSF Grants no. ECS-1150306 and CNS-1544396, and by AFOSR Grant no. FA9550-16-1-0015.

References

[1] J. R. Dooley and J. M. McCarthy. Spatial rigid body dynamics using dual quaternion components. In *Proc. IEEE International Conference on Robotics and Automation*, pages 90–95, 1991.

[2] R. M. Murray, Z. Li, and S. S. Sastry. *A mathematical introduction to robotic manipulation.* CRC press, 1994.

[3] N. Filipe and P. Tsiotras. Adaptive position and attitude-tracking controller for satellite proximity operations using dual quaternions. *Journal of Guidance, Control, and Dynamics*, 38(4):566–577, 2014.

[4] X. Wang, C. Yu, and Z. Lin. A dual quaternion solution to attitude and position control for rigid-body coordination. *IEEE Transactions on Robotics*, 28(5):1162–1170, 2012.

[5] M. Nazari, E. A. Butcher, T. Yucelen, and A. K. Sanyal. Decentralized consensus control of a rigid-body spacecraft formation with communication delay. *Journal of Guidance, Control, and Dynamics*, 39(4):838–851, 2016.

[6] A. Sarlette, S. Bonnabel, and R. Sepulchre. Coordinated motion design on lie groups. *IEEE Transactions on Automatic Control*, 55(5):1047–1058, 2010.

[7] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel. Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points. In *Proc. 25th American Control Conference*, pages 3352–3357, 2006.

[8] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel. Quaternion-based hybrid controller for robust global attitude tracking. *IEEE Transactions on Automatic Control*, 56(11):2555–2566, November 2011.

[9] H.T. M. Kussaba, L. F. C. Figueredo, J.Y. Ishihara, and B. V. Adorno. Hybrid kinematic control for rigid body pose stabilization using dual quaternions. *Journal of the Franklin Institute*, 354(7):2769–2787, 2017.

[10] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton University Press, New Jersey, 2012.

[11] W. K. Clifford. Preliminary sketch of bi-quaternions. In *Proc. London Mathematical Society*, pages 381–395, 1873.

[12] E. Study. *Geometrie der Dynamen.* Leipzig, 1901.

[13] P. P. M. Magro, H.T. M. Kussaba, L. F. C. Figueredo, and J.Y. Ishihara. Dual quaternion-based bimodal global control for robust rigid body pose kinematic stabilization. In *Proc. American Control Conference*, pages xx–xx, 2017.

[14] R. Goebel, R. G. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.

[15] Y. Wu, X. Hu, D. Hu, T. Li, and J. Lian. Strapdown inertial navigation system algorithms based on dual quaternions. *IEEE transactions on aerospace and electronic systems*, 41(1):110–132, 2005.

[16] J. Wang and Z. Sun. 6-dof robust adaptive terminal sliding mode control for spacecraft formation flying. *Acta Astronautica*, 73:76–87, 2012.

[17] AP. Kotelnikov. Screw calculus and some applications to geometry and mechanics. *Annals of the Imperial University of Kazan*, 24, 1895.

[18] D-P. Han, Q. Wei, and Z-X. Li. Kinematic control of free rigid bodies using dual quaternions. *International Journal of Automation and Computing*, 5(3):319–324–87, 2008.

[19] X. Wang and C. Yu. Unit dual quaternion-based feedback linearization tracking problem for attitude and position dynamics. *Systems & Control Letters*, 62(3):225–233, 2013.

[20] N. Filipe and P. Tsiotras. Simultaneous position and attitude control without linear and angular velocity feedback using dual quaternions. In *Proc. IEEE American Control Conference*, pages 4808–4813, 2013.