Robust Stability of Hybrid Limit Cycles With Multiple Jumps in Hybrid Dynamical Systems

Xuyang Lou, Yuchun Li and Ricardo G. Sanfelice

***Abstract*—For a broad class of hybrid dynamical systems, we establish results for robust asymptotic stability of hybrid limit cycles with multiple jumps per period. Hybrid systems are given in terms of differential and difference equations with set constraints. Hybrid limit cycles are given by compact sets defined by periodic solutions that flow and jump. Under mild assumptions, we show that asymptotic stability of such hybrid limit cycles is not only equivalent to asymptotic stability of a fixed point of the associated Poincaré map but also robust to perturbations. Specifically, robustness to generic perturbations, which capture state noise and unmodeled dynamics, and to inflations of the flow and jump sets are established in terms of $\mathcal{KL}$ bounds. A two-gene network with binary hysteresis is presented to illustrate the notions and results throughout the paper.**

***Index Terms*—Hybrid limit cycle, Poincaré map, hybrid systems, stability, robustness.**

I. Introduction

In recent years, the study of limit cycles in hybrid systems has received substantial attention. One reason is the existence of hybrid limit cycles in many engineering applications, such as walking robots [1], genetic regulatory networks [2], mechanical systems [3], neuroscience [4], among others. Stability of hybrid limit cycles is often a fundamental requirement for their practical value in applications. The literature shows a variety of results for the study of limit cycles in several classes of hybrid systems [1], [2], [5]-[9]. In [1], the method of Poincaré sections is employed to establish existence and asymptotic stability of periodic orbits in impulsive systems emerging in bipedal walking. The approach consists of building a map that describes the evolution of the state of the system right before impulses (or jumps), which, in the setting of [1], occur at points belonging to a surface. By collapsing the flow dynamics into such a return map – resembling the so-called Poincaré map – the properties of the limit cycle can be studied using the theory of discrete-time systems. This approach leads to results in [5] highlighting properties of periodic orbits in general impulsive systems and the design of stabilizing controllers for walking robots. More precisely, [5] shows that periodic orbits for such systems that are within an invariant manifold (for which an explicit construction is provided) implies the existence of local coordinates in which the Poincaré map has a block upper triangular structure. An extension of the Poincaré map method was proposed in [6] for the analysis of limit cycles in left-continuous hybrid impulsive dynamical systems, which, as a difference to the models with state-triggered jumps in [1], have variables that exhibit jumps at pre-established time instances. For a similar class of hybrid systems, [7] presents differential conditions in terms of linear matrix inequalities for orbital stability within a contraction framework. Motivated by applications in power systems, a trajectory sensitivity approach was proposed in [8] to derive sufficient conditions for stability of limit cycles in switched systems with differential-algebraic constraints. In such models, the jumps occur due to the model switches or to the reinitialization of the variables needed to keep them within the so-called consistency spaces generated by the algebraic constraints [10]. More recently, in [9], the existence and stability of limit cycles in reset control systems, which are a specific class of hybrid systems, are investigated using techniques that rely on the linearization of the Poincaré map about its fixed point.

Besides our preliminary results in [11], [12], results for the study of robustness of limit cycles in hybrid systems are currently missing from the literature, being perhaps the main reason that a robust stability theory for such systems has only been recently developed in [13]. In fact, all of the aforementioned results about limit cycles are formulated for hybrid systems operating in nominal/noise-free conditions. The development of tools that characterize the robustness properties to perturbations of stable hybrid limit cycles is very challenging and demands a modeling framework that properly handles time and the complex combination of continuous and discrete dynamics.

In this paper, we establish sufficient conditions for guaranteeing (local and global) asymptotic stability of hybrid limit cycles. The constructions proposed to certify asymptotic stability of hybrid limit cycles are exploited to guarantee that such property is robust to perturbations. A result on robustness to generic perturbations, which allows for state noise and unmodeled dynamics, is proposed in terms of $\mathcal{KL}$ bounds. The satisfaction of the so-called hybrid basic conditions in [13] is shown to be a crucial property in guaranteeing robustness to such wide range of perturbations. Furthermore, due to the particular structure of the sets on which flows and jumps occur, we propose a result that guarantees robustness to inflations of those sets, as a function of a parameter. While this work investigates robust stability of a hybrid limit cycle with multiple jumps per period, the situation where a hybrid limit cycle only contains one jump per period has been studied in [14] (see also [11], [15]), where other sufficient conditions for stability, necessary conditions for existence of hybrid limit

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cycles, and numerous examples can be found.

The organization of the paper is as follows. Section II presents a motivational example. Section III presents needed preliminaries about hybrid systems. Section IV presents the definition of hybrid limit cycle, stability notions, and hybrid Poincaré map. In addition, with the hybrid limit cycle definition, sufficient conditions for stability of hybrid limit cycles are established. Section V highlights issues with perturbations and provides results on general robustness of stability to perturbations. In addition, a two-gene network with binary hysteresis, which exhibits periodic solutions with four jumps per period, illustrates our results throughout the paper.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geqslant 0} := [0, +\infty)$. $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \cdots\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of h at x in the direction of f is denoted by $L_f h(x) := \langle \nabla h(x), f(x)\rangle$. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$ when $\mathcal{A}$ is closed; $\overline{\mathcal{A}}$ (respectively, $\overline{\text{co}}\ \mathcal{A}$) denotes its closure (respectively, closed convex hull). $\mathbb{B}$ denotes a closed unit ball in Euclidean space (of appropriate dimension). Given $\delta > 0$ and $x \in \mathbb{R}^n$, $x + \delta\mathbb{B}$ denotes a closed ball centered at x with radius δ. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing; it belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if, in addition, is unbounded. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each $t \geqslant 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s, t) = 0$ and, for each $s \geqslant 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.

II. Motivational Example

Consider the genetic regulatory network with two genes, a and b, each encoding proteins A and B, respectively, proposed in [2]. The dynamics of such genetic network are given by

$$\begin{cases} \dot z_1 = k_1 s_-(z_2, \theta_2) - \gamma_1 z_1 \\ \dot z_2 = k_2 s_+(z_1, \theta_1) - \gamma_2 z_2 \end{cases} \tag{1}$$

where $z_1 \geqslant 0$ and $z_2 \geqslant 0$ represent the concentration of protein A and of protein B, respectively. The constants θ_1 and θ_2 are the thresholds associated with concentrations of proteins A and B, respectively. In this model, gene a and its protein A are expressed at a growth rate $k_1 > 0$ when z_2 is above the threshold θ_2. Similarly, gene b and its protein B are expressed at a growth rate $k_2 > 0$ when z_1 is above the threshold θ_1. Degradations of both proteins are assumed to be proportional to their own concentrations, a mechanism that is captured by $-\gamma_1 z_1$ and $-\gamma_2 z_2$, respectively. The positive constants γ_1 and γ_2 represent the degradation rates of protein A and B, respectively. The step functions s_+ and s_- are defined as

$$s_+(z_i, \theta) = \begin{cases} 1 & \text{if } z_i \geqslant \theta \\ 0 & \text{if } z_i < \theta \end{cases}, \quad s_-(z_i, \theta) = 1 - s_+(z_i, \theta), \tag{2}$$

where $i = 1, 2$, $s_+(z_i, \theta)$ represents the logic for gene expression when the protein concentration exceeds a threshold θ, while $s_-(z_i, \theta)$ represents the logic for gene inhibition.

To incorporate binary hysteresis in the interaction between gene a and gene b, two discrete logic variables, q_1 and q_2, are incorporated. The dynamics of the logic variables depend on the thresholds, θ_1 and θ_2, respectively. The constants θ_1 and θ_2 inferred from biological data satisfy $0 < \theta_1 < \theta_1^{\max}$ and $0 < \theta_2 < \theta_2^{\max}$, where $\theta_1^{\max}$ and $\theta_2^{\max}$ are the maximal values of the concentration of protein A and of protein B, respectively.

The discrete dynamics of the hybrid system are described as follows. When $q_i = 0$ and $z_i = \theta_i + r_i$, the state q_i is updated to 1, i.e., $q_i^+ = 1$, where r_i, $i = 1, 2$, are positive constants defining the hysteresis width. When $q_i = 1$ and $z_i = \theta_i - r_i$, the state q_i is updated to 0, i.e., $q_i^+ = 0$, where $i = 1, 2$. Note that at jumps, the continuous states z_1 and z_2 do not change, i.e., $z_1^+ = z_1$ and $z_2^+ = z_2$. We can express the conditions for continuous and discrete behavior in a compact form using the following functions:

$$\eta_i(z_i, q_i) := (2q_i - 1)(-z_i + \theta_i + (1 - 2q_i)r_i), i \in \{1, 2\}$$

Then, the condition for continuous evolution is given by

$$\eta_1(z_1, q_1) \leqslant 0 \text{ and } \eta_2(z_2, q_2) \leqslant 0,$$

and the condition for discrete evolution is given by

$$\eta_1(z_1, q_1) = 0 \text{ or } \eta_2(z_2, q_2) = 0.$$

Parameters of the model include positive constants k_1, k_2, γ_1, γ_2, θ_1, θ_2, r_1, and r_2 satisfying $\theta_1 + r_1 < \theta_1^{\max}$, $\theta_2 + r_2 < \theta_2^{\max}$, $\theta_1 - r_1 > 0$, and $\theta_2 - r_2 > 0$.

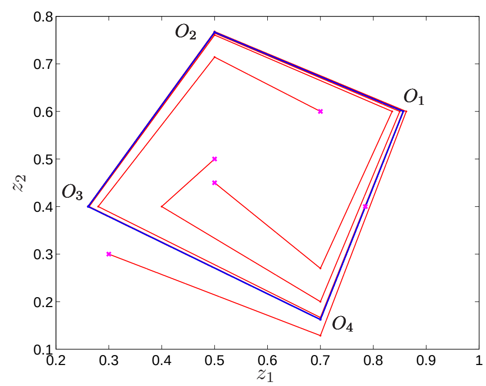

Fig. 1. Phase plot of solutions to the genetic network in (1) and (2) (projection to (z_1, z_2) plane). The point O_1 is given by $(z_1, z_2) = (0.85, 0.60)$, the point O_2 is $(z_1, z_2) = (0.50, 0.77)$, the point O_3 is $(z_1, z_2) = (0.26, 0.40)$, and the point O_4 is $(z_1, z_2) = (0.70, 0.16)$.

A simulation to the system with parameters $\theta_1 = 0.6$, $\theta_2 = 0.5$, $\gamma_1 = \gamma_2 = 1$, $k_1 = k_2 = 1$, and $r_1 = r_2 = 0.1$ is depicted in Fig. 1. The trajectory (blue line) in Fig. 1 shows a limit cycle $\mathcal{O}$ defined by the solution to the hybrid genetic network system with initial condition $(z_1, z_2, q_1, q_2) = (0.79, 0.40, 1, 0)$ that jumps at the points O_i, $i = 1, 2, 3, 4$, and flows in between points. As suggested from the simulation in Fig. 1, the limit cycle $\mathcal{O}$ is asymptotically stable for the system (more rigorous analysis is performed at a later section). A more detailed discussion of this example can be found in [2].

Motivated by the example, our interest is in developing analysis tools that can be applied to such systems so as to determine the stability and robustness properties of hybrid limit cycles with multiple jumps in a period, which are missing tools in the literature of hybrid limit cycles.

III. Preliminaries on Hybrid Systems

We consider hybrid systems $\mathcal{H}$ as in [13], given by

$$\mathcal{H}\begin{cases} \dot{x} &= f(x) \qquad x\in C \\ x^{+} &= g(x) \qquad x\in D \end{cases}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $\dot{x}$ denotes its derivative with respect to time, and x^+ denotes its value after a jump. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ (respectively, $g : \mathbb{R}^n \to \mathbb{R}^n$) is a single-valued map describing the continuous evolution (respectively, the discrete jumps) while $C \subset \mathbb{R}^n$ (respectively, $D \subset \mathbb{R}^n$) is the set on which the flow map f is effective (respectively, from which jumps can occur). The data of a hybrid system $\mathcal{H}$ is given by $\mathcal{H} = (C, f, D, g)$. A solution to $\mathcal{H}$ is parameterized by ordinary time t and a counter j for jumps. It is given by a hybrid arc $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ that satisfies the dynamics of $\mathcal{H}$; see [13] for more details. A solution ϕ to $\mathcal{H}$ is said to be complete if $\operatorname{dom}\phi$ is unbounded. It is said to be maximal if it is not a truncated version of another solution. The set of maximal solutions to $\mathcal{H}$ from the set K is denoted as

$$\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi \text{ is a maximal solution to } \mathcal{H} \text{ with } \phi(0,0)\in K\}.$$

We define $t \mapsto \phi^f(t, x_0)$ as a solution of the flow dynamics

$$\dot{x} = f(x) \quad x \in C$$

from $x_0 \in \overline{C}$. A hybrid system $\mathcal{H}$ is said to be well-posed if it satisfies the *hybrid basic conditions* [13, Assumption 6.5]. For more details about this hybrid systems framework, we refer the readers to [13].

IV. Hybrid Limit Cycles and Basic Properties

A. Definitions

In this work, we consider a class of flow periodic solutions defined as follows.

Definition 1: (flow periodic solution) A complete solution ϕ^* to $\mathcal{H}$ is *flow periodic with period T^* and N^* jumps in each period* if $T^* \in (0,\infty)$ and $N^* \in \mathbb{N}\setminus\{0\}$ are the smallest numbers such that $\phi^*(t+T^*, j+N^*) = \phi^*(t,j)$ for all $(t,j) \in \operatorname{dom}\phi^*$.

The definition of a flow periodic solution ϕ^* with period $T^*>0$ and N^* jumps per period above implies that if $(t,j)\in \operatorname{dom}\phi^*$, then $(t+T^*, j+N^*) \in \operatorname{dom}\phi^*$. A flow periodic solution to $\mathcal{H}$ generates a hybrid limit cycle.

Definition 2: (hybrid limit cycle) A flow periodic solution ϕ^* with period $T^* \in (0,\infty)$ and $N^* \in \mathbb{N}\setminus\{0\}$ jumps in each period defines a *hybrid limit cycle* $\mathcal{O} = \{x \in \mathbb{R}^n : x = \phi^*(t,j), (t,j) \in \operatorname{dom}\phi^*\}$.[1]

Next, the example in Section II is revisited to illustrate the hybrid limit cycle notion in Definition 2.

Example 3: Consider the hybrid genetic network system in in Section II. On the region $M_{\mathrm{G}} := \big\{x := (z_1, z_2, q_1, q_2) \in \mathbb{R}^2_{\geqslant 0}\times\{0,1\}^2 : ([0,\theta_1+r_1]\times[0,\theta_2]\times\{0\}\times\{0\})\cup([\theta_1,\theta_1^{\max}]\times[0,\theta_2+r_2]\times\{1\}\times\{0\})\cup([\theta_1-r_1,\theta_1^{\max}]\times[\theta_2,\theta_2^{\max}]\times\{1\}\times\{1\})\cup([0,\theta_1]\times[\theta_2-r_2,\theta_2^{\max}]\times\{0\}\times\{1\})\big\}$ (later, the set M_{G} will be part of our analysis), it can be described as a hybrid system $\mathcal{H}_{\mathrm{G}}$ as follows:

$$\mathcal{H}_{\mathrm{G}} : \begin{cases} \dot{x} = f_{\mathrm{G}}(x) := \begin{bmatrix} k_1(1-q_2)-\gamma_1 z_1 \\ k_2 q_1 - \gamma_2 z_2 \\ 0 \\ 0 \end{bmatrix} & x\in \widetilde{C}_{\mathrm{G}}\cap M_{\mathrm{G}} \\ x^+ = g_{\mathrm{G}}(x) & x\in\widetilde{D}_{\mathrm{G}}\cap M_{\mathrm{G}} \end{cases} \tag{3}$$

where $\widetilde{C}_{\mathrm{G}} := \{x \in \mathbb{R}^2_{\geqslant 0}\times\{0,1\}^2 : \eta_1(z_1,q_1) \leqslant 0, \eta_2(z_2,q_2)\leqslant 0\}$, $\widetilde{D}_{\mathrm{G}} := \{x \in \mathbb{R}^2_{\geqslant 0}\times\{0,1\}^2 : \eta_1(z_1,q_1) = 0 \text{ or } \eta_2(z_2,q_2)=0\}$. The jump map g_{G} is given by

$$g_{\mathrm{G}}(x) := \begin{cases} g_1(x) & \text{if } \eta_1(z_1,q_1)=0, \eta_2(z_2,q_2)<0 \\ g_2(x) & \text{if } \eta_1(z_1,q_1)<0, \eta_2(z_2,q_2)=0 \end{cases} \tag{4}$$

for each $x \in \widetilde{D}_{\mathrm{G}}\cap M_{\mathrm{G}}$, where $g_1(x) := (z_1, z_2, 1-q_1, q_2), g_2(x) := (z_1,z_2,q_1,1-q_2)$. It follows from [2] that when the conditions

$$\theta_1 + r_1 < k_1/\gamma_1 < \theta_1^{\max},\ \theta_2 + r_2 < k_2/\gamma_2 < \theta_2^{\max} \tag{5}$$

hold, there exists a hybrid limit cycle $\mathcal{O}$ for the hybrid system $\mathcal{H}_{\mathrm{G}}$ with four jumps per period. It is given by $\mathcal{O} = \bigcup_{i=1}^4 S_i$, with[2]

$$S_1 = \left\{z\in\mathbb{R}^2 : z = \begin{bmatrix} \frac{k_1}{\gamma_1} - (\frac{k_1}{\gamma_1} - p_0(1))e^{-\gamma_1 s} \\ p_0(2)e^{-\gamma_2 s}\end{bmatrix}, s\in[0,t_1']\right\}\times\{(0,0)\}$$

$$S_2 = \left\{z\in\mathbb{R}^2 : z = \begin{bmatrix} \frac{k_1}{\gamma_1} - (\frac{k_1}{\gamma_1} - p_1(1))e^{-\gamma_1 s} \\ \frac{k_2}{\gamma_2} - (\frac{k_2}{\gamma_2} - p_1(2))e^{-\gamma_2 s}\end{bmatrix}, s\in[0,t_2']\right\}\times\{(1,0)\}$$

$$S_3 = \left\{z\in\mathbb{R}^2 : z = \begin{bmatrix} p_2(1)e^{-\gamma_1 s} \\ \frac{k_2}{\gamma_2} - (\frac{k_2}{\gamma_2} - p_2(2))e^{-\gamma_2 s}\end{bmatrix}, s\in[0,t_3']\right\}\times\{(1,1)\}$$

$$S_4 = \left\{z\in\mathbb{R}^2 : z = \begin{bmatrix} p_3(1)e^{-\gamma_1 s} \\ p_3(2)e^{-\gamma_2 s}\end{bmatrix}, s\in[0,t_4']\right\}\times\{(0,1)\}$$

where

$$\begin{aligned} t_1' &= \ln\left[\frac{\frac{k_1}{\gamma_1}-p_0(1)}{\frac{k_1}{\gamma_1}-(\theta_1+h_1)}\right]^{\frac{1}{\gamma_1}},\ t_2' = \ln\left[\frac{\frac{k_2}{\gamma_2}-p_1(2)}{\frac{k_2}{\gamma_2}-(\theta_2+h_2)}\right]^{\frac{1}{\gamma_2}}, \\ t_3' &= \ln\left[\frac{p_2(1)}{\theta_1-h_1}\right]^{\frac{1}{\gamma_1}},\ t_4' = \ln\left[\frac{p_3(2)}{\theta_2-h_2}\right]^{\frac{1}{\gamma_2}}. \end{aligned} \tag{6}$$

and $p_0, p_1, p_2, p_3 \in \mathbb{R}^2$ are the vertices of the set $\mathcal{O}$ (projected to $\mathbb{R}^2$); see [2, Proposition 3.3]. This system has a hybrid limit cycle with period $T^* = t_1' + t_2' + t_3' + t_4'$ and $N^* = 4$. △

B. Basic Properties of Hybrid Limit Cycles

In what follows, we focus on a class of hybrid systems that satisfies the following assumption.

Assumption 4: For a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$, there exist a closed set $M \subset \mathbb{R}^n$ and N^* continuously differentiable functions $h_i : \mathbb{R}^n \to \mathbb{R}$ such that

1) the flow set can be written as $C = \bigcap_{i=1}^{N^*} C_i$, and the jump set can be written as $D = \bigcup_{i=1}^{N^*} D_i$, where $C_i = \{x \in \mathbb{R}^n : h_i(x) \geqslant 0\}$ and $D_i = \{x\in\mathbb{R}^n : h_i(x) = 0, L_f h_i(x) \leqslant 0\}$ for each $i \in \{1,2,\cdots,N^*\}$;
2) the flow map f is continuously differentiable on an open neighborhood of $M \cap C$, and the jump map g is continuous on $M \cap D$;
3) for each $i,k \in \{1,2,\cdots,N^*\}$, $L_f h_i(x) < 0$ for all $x \in M\cap D_i$ and $g(M\cap D_i)\cap(M\cap D) = \emptyset$, and $(M\cap D_i)\cap(M\cap D_k) = \emptyset$ for each $i\neq k$;
4) $\mathcal{H}_M = (M\cap C, f, M\cap D, g)$ has a flow periodic solution ϕ^* with period $T^* > 0$ and $N^* \in \mathbb{N}\setminus\{0\}$ jumps per period that defines a hybrid limit cycle $\mathcal{O}\subset M\cap(C\cup D)$.

[1] Alternatively, the hybrid limit cycle $\mathcal{O}$ can be written as $\{x \in \mathbb{R}^n : x = \phi^*(t,j), t\in[t_s, t_s+T^*], (t,j)\in\operatorname{dom}\phi^*\}$ for some $t_s \in \mathbb{R}_{\geqslant 0}$.

[2] $p_i(n)$ is the n-th component of p_i, $i \in \{0,1,2,3\}$.

Remark 5: Item 1) in Assumption 4 implies that flows occur when every h_i is nonnegative and jumps only occur at points in zero level sets of h_i. The continuity property of f in item 2) of Assumption 4 is further required for the existence of solutions to $\dot{x} = f(x)$ according to [13, Proposition 2.10]. Items 3) and 4) in Assumption 4 allow us to restrict the analysis of a hybrid system $\mathcal{H}$ to a region of the state space $M \subset \mathbb{R}^n$. As we will show later, the set M is appropriately chosen for each specific system such that it guarantees completeness of nominal solutions to $\mathcal{H}_M$ and the existence of flow periodic solutions. It can be shown that the hybrid limit cycle generated by periodic solutions as defined in Definition 2 is closed and bounded.

Next, Example 3 is revisited to illustrate Assumption 4.

Example 6: Consider the hybrid genetic network system $\mathcal{H}_{\mathrm{G}}$ in Example 3. On the region M_{G} and under the conditions in (5), the sets $\widetilde{C}_{\mathrm{G}}$ and $\widetilde{D}_{\mathrm{G}}$ are equivalent to $C_{\mathrm{G}} := \{x \in M_{\mathrm{G}} : h_i(x) \geqslant 0 \ \forall i \in \{1,2,3,4\}\}$, and $D_{\mathrm{G}} := \bigcup_{i=1}^{4} D_{\mathrm{G}_i}$, respectively, where

$$\begin{aligned}
D_{\mathrm{G}_i} &:= \{x \in M_{\mathrm{G}} : h_i(x) = 0, \Gamma_i(x) f_1(x) \leqslant 0\}, i \in \{1,2\}\\
D_{\mathrm{G}_i} &:= \{x \in M_{\mathrm{G}} : h_i(x) = 0, \Gamma_i(x) f_2(x) \leqslant 0\}, i \in \{3,4\}\\
f_1(x) &:= k_1(1-q_2) - \gamma_1 z_1, \quad f_2(x) := k_2 q_1 - \gamma_2 z_2,\\
\Gamma_1(x) &= -e^{\theta_1 + r_1 - z_1}, \quad \Gamma_2(x) = e^{-(\theta_1 - r_1 - z_1)},\\
\Gamma_3(x) &= -e^{\theta_2 + r_2 - z_2}, \quad \Gamma_4(x) = e^{-(\theta_2 - r_2 - z_2)},
\end{aligned}$$

and the functions $h_i\colon C_{\mathrm{G}} \cup D_{\mathrm{G}} \to \mathbb{R}$, $i \in \{1,2,3,4\}$, are defined as

$$\begin{aligned}
h_1(x) &= e^{\theta_1 + r_1 - z_1} + e^{q_1} - 2,\\
h_2(x) &= e^{-(\theta_1 - r_1 - z_1)} + e^{1-q_1} - 2,\\
h_3(x) &= e^{\theta_2 + r_2 - z_2} + e^{q_2} - 2,\\
h_4(x) &= e^{-(\theta_2 - r_2 - z_2)} + e^{1-q_2} - 2.
\end{aligned}$$

With the closed set M_{G} introduced in Example 3 and the sets C_{G} and D_{G} given above, the system $\mathcal{H}_{\mathrm{G}}$ can be rewritten as $\mathcal{H}_{\mathrm{G}_M} = (M_{\mathrm{G}} \cap C_{\mathrm{G}}, f_{\mathrm{G}}, M_{\mathrm{G}} \cap D_{\mathrm{G}}, g_{\mathrm{G}})$. Then, using the conditions in (5), we obtain that for all $x \in M_{\mathrm{G}} \cap D_{\mathrm{G}_i}$ and each $i \in \{1,2,3,4\}$, $L_{f_{\mathrm{floor}(\frac{i+1}{2})}} h_i(x) = \Gamma_i(x) f_{\mathrm{floor}(\frac{i+1}{2})} < 0$. By definition, the sets C_{G} and D_{G} are closed, f_{G} is continuous on $M_{\mathrm{G}} \cap C_{\mathrm{G}}$, f_{G} is continuously differentiable on a neighborhood of $M_{\mathrm{G}} \cap C_{\mathrm{G}}$, and g_{G} is continuous on $M_{\mathrm{G}} \cap D_{\mathrm{G}}$. Moreover, it can be verified that $g_{\mathrm{G}}(M_{\mathrm{G}} \cap D_{\mathrm{G}_i}) \cap (M_{\mathrm{G}} \cap D_{\mathrm{G}}) = \emptyset$ and $(M_{\mathrm{G}} \cap D_{\mathrm{G}_i}) \cap (M_{\mathrm{G}} \cap D_{\mathrm{G}_k}) = \emptyset$, for all $i, k \in \{1,2,3,4\}$, $i \neq k$. Therefore, Assumption 4 holds. △

Remark 7: In [1] and [6], the authors extend the Poincaré method to analyze the stability properties of periodic orbits in nonlinear systems with impulsive effects. In particular, the solutions to the systems considered therein are right-continuous over (not necessarily closed) intervals of flow. In particular, the models therein (as well as those in [7]) require $C \cap D = \emptyset$, which prevents the application of the robustness results in [13] due to the fact that the hybrid basic conditions would not hold. On the other hand, our results allow us to establish robustness properties of hybrid limit cycles as shown in Section V.

Following [1], for a hybrid system $\mathcal{H}$, and for each $i \in \{1,2,\cdots,N^*\}$, the *time-to-impact function with respect to* D_i is defined by $T_{D_i}\colon C \cup D \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where[3]

$$T_{D_i}(x) := \inf\{t \geqslant 0 : \phi(t,j) \in D_i, \ \phi \in \mathcal{S}_{\mathcal{H}}(x)\} \tag{7}$$

for each $x \in C \cup D$.

Inspired by [1, Lemma 3], we have that for each $i \in \{1,2,\cdots,N^*\}$, the function $x \mapsto T_{D_i}(x)$ is continuous on a subset of $M \cap (C \cup D)$; see [12, Lemma 4.13]. Next, let us introduce the Poincaré map for hybrid systems. For each $i \in \{1,2,\cdots,N^*\}$, the hybrid Poincaré map $P_i : M \cap D_i \to M \cap D_i$ given by

$$\begin{aligned}
P_i(x) := \big\{\phi(T_{D_i}(g(x)), j) : \phi \in \mathcal{S}_{\mathcal{H}}(g(x)),\\
(T_{D_i}(g(x)), j) \in \operatorname{dom}\phi \ \big\} \quad \forall x \in M \cap D_i
\end{aligned} \tag{8}$$

is well-defined and continuous on $\mathcal{X}_i$ due to the continuity of T_{D_i} on $\mathcal{X}_i$ and well-posedness of $\mathcal{H}_M$. Note that for points x in the range of a hybrid limit cycle with N^* jumps, $P_i(x)$ is the value of the solution from x after N^* jumps; cf. [1]. The importance of the hybrid Poincaré map in (8) is that it allows one to determine the stability of hybrid limit cycles. Now, we define asymptotic stability using a hybrid Poincaré map. Below, P_i^k denotes k compositions of the Poincaré map P_i with itself.

Definition 8: For each $i \in \{1,2,\cdots,N^*\}$, a fixed point x^* of a Poincaré map $P_i : M \cap D_i \to M \cap D_i$ is said to be

- *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in M \cap D_i$, $|x - x^*| \leqslant \delta$ implies $|P_i^k(x) - x^*| \leqslant \epsilon$ for all integers $k \geqslant 0$;
- *globally attractive* with basin of attraction containing every point in $M \cap D_i$ if for all $x \in M \cap D_i$, $\lim_{k\to\infty} P_i^k(x) = x^*$;
- *globally asymptotically stable* if it is both stable and globally attractive with basin of attraction containing every point in $M \cap D_i$;[4]
- *locally attractive* if there exists $\mu > 0$ such that for all $x \in M \cap D_i$, $|x - x^*| \leqslant \mu$ implies $\lim_{k\to\infty} P_i^k(x) = x^*$;
- *locally asymptotically stable* if it is both stable and locally attractive.

C. Stability of Hybrid Limit Cycles

In this section, we present stability properties of hybrid limit cycles for $\mathcal{H}$. Following the stability notion introduced in [13, Definition 3.6], we employ the following notion for stability of hybrid limit cycles.

Definition 9: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a compact hybrid limit cycle $\mathcal{O}$. Then, the hybrid limit cycle $\mathcal{O}$ is said to be

- *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{O}} \leqslant \delta$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;

[3] In particular, when there does not exist $t \geqslant 0$ such that $\phi^f(t,x) \in D_i$, we have $\{t \geqslant 0 : \phi^f(t,x) \in D_i\} = \emptyset$ for each $i \in \{1,2,\cdots,N^*\}$, which gives $T_{D_i}(x) = \infty$.

[4] In this paper, our results employ the term "global" as in [13] and related references, which requires careful treatment. More precisely, in that reference, it is stated that, for a hybrid system $\mathcal{H} = (C, f, D, g)$ with state in $\mathbb{R}^n$, points outside $C \cup D$ belong to the basin of attraction, and that global asymptotic stability corresponds to the case when that basin is equal to $\mathbb{R}^n$, indicating that solutions from $C \cup D$ are required to converge to the asymptotically stable set; see [13, Definition 3.6].

- *globally attractive* for $\mathcal{H}$ if every maximal solution ϕ to $\mathcal{H}$ from $\bar{C}\cup D$ is complete and satisfies $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *globally asymptotically stable* for $\mathcal{H}$ if it is both stable and globally attractive;
- *locally attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that every maximal solution ϕ to $\mathcal{H}$ starting from $|\phi(0,0)|_{\mathcal{O}} \leqslant \mu$ is complete and satisfies $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} = 0$;
- *locally asymptotically stable* for $\mathcal{H}$ if it is both stable and locally attractive.

For each $i \in \{1,2,\cdots,N^*\}$, and for $x \in M \cap (C \cup D_i)$, define the "distance" function $d_i : M \cap (C \cup D_i) \to \mathbb{R}_{\geqslant 0}$ as

$$d_i(x) := \sup_{t\in[0,T_{D_i}(x)],\ (t,j)\in\operatorname{dom}\phi,\ \phi\in\mathcal{S}_{\mathcal{H}_M}(x)} |\phi(t,j)|_{\mathcal{O}}$$

when $0 \leqslant T_{D_i}(x) < \infty$, and

$$d_i(x) = \sup_{(t,j)\in\operatorname{dom}\phi,\ \phi\in\mathcal{S}_{\mathcal{H}_M}(x)} |\phi(t,j)|_{\mathcal{O}}$$

if $T_{D_i}(x) = \infty$. Note that d_i vanishes on $\mathcal{O}$. Denote the basin of attraction of the set $\mathcal{O}$ by $\mathcal{B}_{\mathcal{O}}$. Then, similar to [1, Lemma 4] but exploiting the hybrid basic conditions, we have that the functions d_i's are well-defined and continuous on $\mathcal{O}$; see [12, Lemma 4.15].

A relationship between stability of fixed points of Poincaré maps and stability of the corresponding hybrid limit cycles is established next.

Theorem 10: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4. Suppose every maximal solution to $\mathcal{H}_M = (M\cap C, f, M\cap D, g)$ is complete. Then, the following equivalences hold:

1) *for each $i \in \{1,2,\cdots,N^*\}$, $x_i^* \in M\cap D_i$ is a stable fixed point of the Poincaré map P_i in (8) if and only if the hybrid limit cycle $\mathcal{O}$ of $\mathcal{H}_M$ generated by a flow periodic solution ϕ^* with period T^* and N^* jumps in each period from $\phi^*(0,0) = x_i^*$ for each $i \in \{1,2,\cdots,N^*\}$ is stable for $\mathcal{H}_M$;*
2) *for each $i \in \{1,2,\cdots,N^*\}$, $x_i^* \in M\cap D_i$ is a globally asymptotically stable fixed point of the Poincaré map P_i if and only if the unique hybrid limit cycle $\mathcal{O}$ of $\mathcal{H}_M$ generated by a flow periodic solution ϕ^* with period T^* and N^* jumps in each period from $\phi^*(0,0) = x_i^*$ for each $i \in \{1,2,\cdots,N^*\}$ is globally asymptotically stable for $\mathcal{H}_M$ with basin of attraction[5] containing every point in $M\cap(C\cup D)$.*

Proof: We first prove the sufficiency of item $1)$. By Assumption 4, every maximal solution to $\mathcal{H}_M$ is unique via [13, Proposition 2.11]. Consider the hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution to $\mathcal{H}_M$ from x_i^* with $x_i^* \in M\cap D_i$ for each $i \in \{1,2,\cdots,N^*\}$. Since $\mathcal{O}$ is stable for $\mathcal{H}_M$, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution ϕ to $\mathcal{H}_M$, $|\phi(0,0)|_{\mathcal{O}} \leqslant \delta$ implies $|\phi(t,j)|_{\mathcal{O}} \leqslant \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$. Since ϕ is complete and $P_i^k(x_i^*) = \phi(T_{D_i}(g(x_i^*)), j)$ for some j, in particular, we have that $|P_i^k(x_i^*)|_{\mathcal{O}} \leqslant \varepsilon$ for each $k \in \mathbb{N}$. Therefore, $x_i^* \in M\cap D_i$ is a stable fixed point of the Poincaré map P_i.

Next, we prove the necessity of item $1)$ as in the proof of [1, Theorem 1]. Suppose that for each $i \in \{1,2,\cdots,N^*\}$, $x_i^* \in M\cap D_i$ is a stable point of P_i. Then, $P_i(x_i^*) = x_i^*$ due to the continuity of P_i in (8) and, for any $\bar{\epsilon} > 0$, there exists $\bar{\delta} > 0$ such that $\tilde{x} \in (x_i^* + \bar{\delta}\mathbb{B}) \cap (M\cap D_i)$ implies $P_i^k(\tilde{x}) \in (x_i^* + \bar{\epsilon}\mathbb{B}) \cap (M\cap D_i)\quad \forall k \in \mathbb{N}$. Moreover, by assumption, every maximal solution ϕ to $\mathcal{H}_M$ from $\tilde{x} \in (x_i^* + \bar{\delta}\mathbb{B}) \cap (M\cap D_i)$ is complete and unique. Since solutions are guaranteed to exist from $M\cap D_i$, there exists a complete solution ϕ from every such point $\tilde{x}$. Furthermore, the distance between ϕ and the hybrid limit cycle $\mathcal{O}$ satisfies[6]

$$\sup_{(t,j)\in\operatorname{dom}\phi} |\phi(t,j)|_{\mathcal{O}} \leqslant \max_{i\in\{1,2,\cdots,N^*\}} \sup_{x\in(x_i^*+\bar{\delta}\mathbb{B})\cap(M\cap D_i)} d_i\circ g(x).$$

Since the functions d_i's are well-defined and continuous on $\mathcal{O}$, d_i is continuous at x_i^*. Since $\mathcal{O}$ is transversal to $M\cap D_i$ (see [12, Lemma 4.9]), $\mathcal{O}\cap(M\cap D_i)$ is a singleton, $g(x_i^*) \in \mathcal{O}$, and g is continuous, we have that $d_i\circ g$ is continuous at x_i^*. Moreover, since $d_i\circ g(x_i^*) = 0$, it follows by continuity that given any $\epsilon > 0$, we can pick $\bar{\epsilon}$ and $\bar{\delta}$ such that $0 < \bar{\epsilon} < \epsilon$ and

$$\max_{i\in\{1,2,\cdots,N^*\}} \sup_{x\in(x_i^*+\bar{\delta}\mathbb{B})\cap(M\cap D_i)} d_i\circ g(x) \leqslant \epsilon.$$

Therefore, an open neighborhood of $\mathcal{O}$ given by $\mathcal{V} := \{x \in \mathbb{R}^n : \max_{i\in\{1,2,\cdots,N^*\}} d_i(x) \in [0,\epsilon)\}$ is such that any solution ϕ to $\mathcal{H}_M$ from $\phi(0,0) \in \mathcal{V}$ satisfies $|\phi(t,j)|_{\mathcal{O}} \leqslant \epsilon$ for all $(t,j) \in \operatorname{dom}\phi$. Thus, the necessity of item $1)$ follows immediately.

The stability part of item $2)$ follows similarly. Sufficiency of the global attractivity part in item $2)$ is proved as follows. Suppose the hybrid limit cycle $\mathcal{O}$ generated by a flow periodic solution to $\mathcal{H}_M$ from x_i^*, $i \in \{1,2,\cdots,N^*\}$ is globally attractive for $\mathcal{H}_M$ with basin of attraction containing every point in $M\cap(C\cup D)$. Then, given $\epsilon > 0$, for any solution ϕ to $\mathcal{H}_M$, there exists $\bar{T} > 0$ such that $|\phi(t,j)|_{\mathcal{O}} \leqslant \epsilon$ for all $(t,j) \in \operatorname{dom}\phi$ with $t \geqslant \bar{T}$. Note that ϕ is complete and $\operatorname{dom}\phi$ is unbounded in the t-direction as Assumption 4 prevents solutions from being Zeno via [16, Lemma 2.7]. It follows that $|P_i^k(x_i^*)|_{\mathcal{O}} \leqslant \epsilon$ for sufficiently large k. Therefore, x_i^* is a globally attractive fixed point of P_i.

Finally, we prove the necessity of the global attractivity part in item $2)$. For each $i \in \{1,2,\cdots,N^*\}$, assume that $x_i^* \in M\cap D_i$ is a globally attractive fixed point of P_i. Then, for any $\bar{\epsilon} > 0$, there exists $\bar{\delta} > 0$ such that,

$$\tilde{x} \in (x_i^* + \bar{\delta}\mathbb{B}) \cap (M\cap D_i)$$

implies $\lim_{k\to\infty} P_i^k(\tilde{x}) = x_i^*$. Moreover, following from Definition 8, it is implied that a maximal solution ϕ to $\mathcal{H}_M$ from x^* is complete. Then, by continuity of d_i and g,

$$\lim_{k\to\infty} d_i\circ g(P_i^k(\tilde{x})) = d_i\circ g(x_i^*) = 0,$$

from which it follows that

$$\begin{aligned}
&\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{O}} \\
&\quad\leqslant \lim_{k\to\infty} \max_{i\in\{1,2,\cdots,N^*\}} \sup_{\tilde{x}\in(x_i^*+\bar{\delta}\mathbb{B})\cap(M\cap D_i)} d_i\circ g(P_i^k(\tilde{x})) \\
&\quad\leqslant \max_{i\in\{1,2,\cdots,N^*\}} \sup_{\tilde{x}\in(x_i^*+\bar{\delta}\mathbb{B})\cap(M\cap D_i)} \lim_{k\to\infty} d_i\circ g(P_i^k(\tilde{x})) \\
&\quad\leqslant \max_{i\in\{1,2,\cdots,N^*\}} d_i\circ g(x_i^*) = 0. \qquad (9)
\end{aligned}$$

□

Remark 11: In [1], sufficient and necessary conditions for

[5] A global property for $\mathcal{H}_M$ implies a global property of the original system $\mathcal{H}$ only when M contains $C\cup D$.

[6] Given two functions $d : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$, the operator $\circ$ defines a function composition, i.e., $d\circ g(x) = d(g(x))$ for all $x \in \mathbb{R}^n$.

stability properties of periodic orbits in impulsive systems are established using properties of the fixed points of the corresponding Poincaré maps. Compared to [1], Theorem 10 enables the use of the Lyapunov stability tools in [13] to certify asymptotic stability of a fixed point without even computing the Poincaré map.

At times, one might be interested only on local asymptotic stability of the fixed point of the Poincaré map. Such case is handled by the following result.

Corollary 12: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4. Suppose every maximal solution to $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ is complete. Then, for each $i \in \{1, 2, \cdots, N^*\}$, $x_i^* \in M \cap D_i$ is a locally asymptotically stable fixed point of the Poincaré map P_i if and only if the unique hybrid limit cycle $\mathcal{O}$ of $\mathcal{H}_M$ generated by a flow periodic solution ϕ^* with period T^* and N^* jumps in each period from $\phi^*(0,0) = x_i^*$ for each $i \in \{1, 2, \cdots, N^*\}$ is locally asymptotically stable for $\mathcal{H}_M$.

The following example illustrates the sufficient condition in Corollary 12 by checking the eigenvalues of the Jacobian matrix of each Poincaré map at its fixed points.

Example 13: Consider the hybrid genetic network system $\mathcal{H}_{\mathrm{G}_M}$ introduced in Example 6. By [2, Proposition 3.1], every maximal solution to $\mathcal{H}_{\mathrm{G}_M}$ is complete. Therefore, the hybrid system $\mathcal{H}_{\mathrm{G}}$ on M_{G} satisfies Assumption 4 and has a flow periodic solution ϕ^* with period T^* given in Example 3 and four jumps per period, which defines a unique hybrid limit cycle $\mathcal{O} \subset M_{\mathrm{G}} \cap (C_{\mathrm{G}} \cup D_{\mathrm{G}})$. Now, for each $i \in \{1, 2, 3, 4\}$, let P_i be the Poincaré map for $\mathcal{H}_{\mathrm{G}_M}$ associated to the fixed point x_i^*. The sufficient condition in Corollary 12 can be verified by computing the eigenvalues of the Jacobian matrix of the Poincaré maps P_i at each fixed point x_i^*, $i \in \{1, 2, 3, 4\}$. Due to the linear form of the flow and jump maps, it is possible to obtain an analytic form of the Jacobian matrices of the Poincaré maps; see [2, Eq.(19)]. Here, we apply the shooting method in [8] to compute the Jacobian matrices based on approximate Poincaré maps numerically for parameters $\theta_1 = 0.6$, $\theta_2 = 0.5, \gamma_1 = \gamma_2 = 1$, $k_1 = k_2 = 1$, and $r_1 = r_2 = 0.1$. The four fixed points are obtained as

$$x_1^* \approx (0.70, 0.16, 0, 0) \in D_{\mathrm{G}_1}, x_2^* \approx (0.85, 0.60, 1, 0) \in D_{\mathrm{G}_2},$$
$$x_3^* \approx (0.50, 0.77, 1, 1) \in D_{\mathrm{G}_3}, x_4^* \approx (0.26, 0.40, 0, 1) \in D_{\mathrm{G}_4},$$

and the period time of the hybrid limit cycle is $T^* \approx 2.83$. It can be verified that the four eigenvalues of the Jacobian matrix are located inside the unit circle. Therefore, the hybrid limit cycle $\mathcal{O}$ of the hybrid genetic network system is locally asymptotically stable. The properties of the hybrid limit cycle $\mathcal{O}$ are illustrated numerically in Fig. 1 (blue line). △

V. Robustness of Hybrid Limit Cycles

A. Robustness to General Perturbations

In this section, we present results guaranteeing robustness to generic perturbations of asymptotically stable hybrid limit cycles. More precisely, we consider the perturbed continuous dynamics of the hybrid system $\mathcal{H}_M = (M \cap C, f, M \cap D, g)$ given by

$$\dot{x} = f(x + d_1) + d_2 \quad x + d_3 \in M \cap C$$

where d_1 corresponds to state noise, d_2 captures unmodeled dynamics and d_3 captures generic disturbances on the state such as measurement noise. Similarly, we consider the perturbed discrete dynamics

$$x^+ = g(x + d_1) + d_2 \quad x + d_4 \in M \cap D.$$

where d_4 captures generic disturbances on the state. The hybrid system $\mathcal{H}_M$ with such perturbations results in the perturbed hybrid system

$$\tilde{\mathcal{H}}_M \left\{ \begin{array}{rcll} \dot{x} & = & f(x+d_1)+d_2 & x+d_3 \in M \cap C \\ x^+ & = & g(x+d_1)+d_2 & x+d_4 \in M \cap D \end{array} \right. \tag{10}$$

The perturbations d_i $(i = 1, 2, 3, 4)$ might be state or hybrid time dependent, but assumed to have Euclidean norm bounded by $\bar{M}_i \geqslant 0$ $(i = 1, 2, 3, 4)$, and to be admissible, namely, $\operatorname{dom} d_i$ $(i = 1, 2, 3, 4)$ is a hybrid time domain and the function $t \mapsto d_i(t, j)$ is measurable on $\operatorname{dom} d_i \cap (\mathbb{R}_{\geqslant 0} \times \{j\})$ for each $j \in \mathbb{N}$.

The following result establishes that the stability of $\mathcal{O}$ for $\mathcal{H}_M$ is robust to a class of the perturbations defined above.

Theorem 14: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4. If $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}_M$ with basin of attraction $\mathcal{B}_{\mathcal{O}}$, then for every proper indicator ω of $\mathcal{O}$ on $\mathcal{B}_{\mathcal{O}}$ there exists $\tilde{\beta} \in \mathcal{KL}$ such that for every $\varepsilon > 0$ and every compact set $K \subset \mathcal{B}_{\mathcal{O}}$, there exist $\bar{M}_i > 0$, $i \in \{1, 2, 3, 4\}$, such that for any admissible perturbations d_i, $i \in \{1, 2, 3, 4\}$, with Euclidean norm bounded by $\bar{M}_i$, respectively, every solution $\tilde{\phi}$ to $\tilde{\mathcal{H}}_M$ with $\tilde{\phi}(0,0) \in K$ satisfies

$$\omega(\tilde{\phi}(t,j)) \leqslant \tilde{\beta}(\omega(\tilde{\phi}(0,0)), t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} \tilde{\phi}.$$

Proof: We introduce the following perturbed hybrid system $\mathcal{H}_M^\rho$ with $\rho > 0$:

$$\mathcal{H}_M^\rho \left\{ \begin{array}{rcll} \dot{x} & \in & F_\rho(x) & x \in C_\rho \\ x^+ & \in & G_\rho(x) & x \in D_\rho \end{array} \right. \tag{11}$$

where

$$\begin{aligned} C_\rho &:= \{x \in \mathbb{R}^n : (x + \rho\mathbb{B}) \cap (M \cap C) \neq \emptyset\}, \\ F_\rho(x) &:= \overline{\mathrm{co}} f((x + \rho\mathbb{B}) \cap (M \cap C)) + \rho\mathbb{B}, \\ D_\rho &:= \{x \in \mathbb{R}^n : (x + \rho\mathbb{B}) \cap (M \cap D) \neq \emptyset\}, \\ G_\rho(x) &:= \{v \in \mathbb{R}^n : v \in \eta + \rho\mathbb{B}, \\ &\qquad \eta \in g((x + \rho\mathbb{B}) \cap (M \cap D))\}. \end{aligned}$$

Then, every solution to $\tilde{\mathcal{H}}_M$ with admissible perturbations d_i having Euclidean norm bounded by $\bar{M}_i$, $i \in \{1, 2, 3, 4\}$, is a solution to the hybrid system $\mathcal{H}_M^\rho$ with $\rho \geqslant \max\{\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4\}$, which corresponds to an outer perturbation of $\mathcal{H}_M$ and satisfies [17, (C1)-(C4)] (see [17, Example 5.3] for more details). Then, the claim follows by [17, Theorem 6.6] and the fact that every solution to $\tilde{\mathcal{H}}_M$ is a solution to (11). In fact, using [17, Theorem 6.6], for every proper indicator ω of $\mathcal{O}$ on $\mathcal{B}_{\mathcal{O}}$ there exists $\tilde{\beta} \in \mathcal{KL}$ such that for each compact set $K \subset \mathcal{B}_{\mathcal{O}}$ and each $\varepsilon > 0$, there exists $\rho^* > 0$ such that for each $\rho \in (0, \rho^*]$, every solution ϕ_ρ to (11) from K satisfies $\omega(\phi_\rho(t,j)) \leqslant \tilde{\beta}(\omega(\phi_\rho(0,0)), t+j) + \varepsilon$ for all $(t,j) \in \operatorname{dom} \phi_\rho$. The proof concludes using the relationship between the solutions to $\tilde{\mathcal{H}}_M$ and (11), and picking $\bar{M}_i$, $i \in \{1, 2, 3, 4\}$, such that $\max\{\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4\} \in (0, \rho^*]$. □

Through an application of [13, Lemma 7.19], it can be shown that the hybrid limit cycle is robustly $\mathcal{KL}$ asymptotically stable on $\mathcal{B}_{\mathcal{O}}$. In certain applications, a relationship between the maximum value ρ of the perturbation and the factor ε in the semiglobal and practical $\mathcal{KL}$ bound in (13) can be established numerically, as shown in the next example.

Example 15: Consider the hybrid system $\mathcal{H}_{\mathrm{G}_M}$ in Example 6. The admissible state perturbation and genetic perturbations considered are $d_1 = d_3 = d_4 = (\rho\sin(t), 0, 0, 0)$. The unmodeled dynamics considered is $d_2 = (0, \rho\cos(t), 0, 0)$. To validate Theorem 14, more simulations are performed to quantify the relationship between ρ^* (the maximal value of the perturbation parameter ρ) and ε (the desired level of closeness to $\mathcal{O}$). Given a compact set $K := [0.36, 0.44] \times [0.46, 0.54] \times \{0\} \times \{0\}$ and different desired region radiuses $\varepsilon = \{0.02, 0.04, 0.06, 0.2, 0.4, 0.6\}$, the simulation results are shown in Table I, which indicates that the relationship between ρ^* and ε can be approximated as $\rho^* \approx 0.5\varepsilon$. As it can be seen, the larger admissible convergence error the larger perturbation parameter ρ^* can be. These validate the result in Theorem 14. $\triangle$

TABLE I
SIMULATION RESULTS FOR DIFFERENCE VALUES OF ε

ρ^*	ε	ρ^*/ε
0.0103	0.02	0.5150
0.0204	0.04	0.5100
0.0305	0.06	0.5083
0.1019	0.20	0.5095
0.2000	0.40	0.5000
0.3021	0.60	0.5035

B. Robustness to Inflations of C and D

We consider the following specific parametric perturbation on h, in both the flow and jump sets, with $\epsilon > 0$ denoting the parameter: the perturbed flow set is an inflation of the original flow set while the condition $h(x) = 0$ in the jump set is replaced by $h(x) \in [-\epsilon, \epsilon]$. The resulting system is denoted as $\mathcal{H}_M^\epsilon$ and is given by

$$\mathcal{H}_M^\epsilon \begin{cases} \dot{x} &= f(x) \qquad x \in C_\epsilon \cap M \\ x^+ &= g(x) \qquad x \in D_\epsilon \cap M \end{cases} \tag{12}$$

where the flow set and the jump set are replaced by $C_\epsilon = \{x \in \mathbb{R}^n : h(x) \geqslant -\epsilon\}$ and $D_\epsilon = \{x \in \mathbb{R}^n : h(x) \in [-\epsilon, \epsilon], L_f h(x) \leqslant 0\}$, respectively, while the flow map and jump map are the same as for $\mathcal{H}_M$. We have the following result, whose proof follows from the proof of Theorem 14.

Theorem 16: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4. If $\mathcal{O}$ is an asymptotically stable compact set for $\mathcal{H}_M$ with basin of attraction $\mathcal{B}_{\mathcal{O}}$, then there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for every $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}_{\mathcal{O}}$, there exists $\bar{\epsilon} > 0$ such that for each $\epsilon \in (0, \bar{\epsilon}]$ every solution ϕ to $\mathcal{H}_M^\epsilon$ in (12) with $\phi(0,0) \in K$ satisfies

$$|\phi(t,j)|_{\mathcal{O}} \leqslant \tilde{\beta}(|\phi(0,0)|_{\mathcal{O}}, t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom}\phi. \tag{13}$$

Theorem 16 implies that the asymptotic stability property of the hybrid limit cycle $\mathcal{O}$ is robust to a parametric perturbation on h.

VI. CONCLUSION

Notions and tools for the analysis of hybrid limit cycles in hybrid dynamical systems are proposed. In addition to nominal results, the key novel contributions include conditions for robustness of asymptotically stable hybrid limit cycles with respect to perturbations and to inflations of flow and jump sets. The proposed results are applicable to the situation where a hybrid limit cycle may contain multiple jumps within a period. An example is included to aid the reading and illustrate the concepts and the methodology of applying the new results. Current research efforts include determining necessary conditions for the existence of hybrid limit cycles [14], and hybrid control design for asymptotic stabilization of such limit cycles as well as their robust implementation.

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