

POINTWISE ASYMPTOTIC STABILITY IN A HYBRID SYSTEM AND WELL-POSED BEHAVIOR BEYOND ZENO*

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Abstract. Hybrid dynamical systems, modeled by hybrid inclusions—a combination of differential equations or inclusions, of difference equations or inclusions, and of constraints on the resulting motions—are considered. Pointwise asymptotic stability, a property of a set of equilibria in a hybrid system where every equilibrium is Lyapunov stable and solutions from near the equilibria converge to some equilibrium, is studied. Sufficient conditions, relying on set-valued Lyapunov functions with strict or weak decrease, on invariance arguments, or on standard Lyapunov functions that also limit the lengths of solutions, are given. Structural properties of sets of solutions to a hybrid system, of reachable sets, and of limits of solutions are investigated in the presence of a pointwise asymptotically stable set of equilibria, and also under further uniform Zeno assumptions. Many of these results are extended to the case of partial pointwise asymptotic stability. The results are then used to extend Zeno solutions to hybrid systems beyond their Zeno times, in a way preserving reasonable dependence of solutions on initial conditions and enabling the analysis of convergence of extended solutions to a compact attractor.

Key words. hybrid system, asymptotic stability, Zeno behavior, Lyapunov conditions, set-valued analysis

AMS subject classifications. 93B52, 34A38, 34K34, 34K20, 37B25, 93D05

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1. Introduction. The goal of the paper is to formally extend Zeno solutions to a hybrid system past their Zeno times in a way that preserves reasonable dependence of extended solutions on initial conditions. A difficulty in guaranteeing such a property is that, in general, the limit of a Zeno solution at its Zeno time does not depend well on initial conditions. As the paper shows, this issue can be overcome when the set of equilibria of a hybrid system has a certain stability property, called pointwise asymptotic stability (PAS), and under a further assumption on (small) Zeno times. Several auxiliary results, including Lyapunov and invariance-based sufficient conditions for (nonpartial and partial) PAS in a hybrid system, are included.¹

1.1. Background. Hybrid dynamical systems are dynamical systems that exhibit features characteristic of continuous-time dynamical systems and features characteristic of discrete-time systems. Modeling of physical systems interacting with computing systems, known as cyber-physical systems, or mechanical systems with impacts, of classical systems in closed-loop with hybrid (logic-based, switching, etc.) control algorithms, solving control problems for continuous-time systems for which no continuous feedback stabilizers exist, among others, motivate the interest in hybrid

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¹Some of the sufficient conditions for PAS in a hybrid system appeared in the conference paper [14] and preliminary work on extending solutions past their Zeno times is in the conference paper [13].

systems in the control theory literature [43], [19], [22], [36]. This paper considers hybrid systems modeled as hybrid inclusions [15], [16]. The hybrid inclusion framework relies on generalized time domains, going back to [10], [41], allows for multivalued dynamics, as pioneered in this setting by [2], and admits a fairly complete asymptotic stability theory [16].

Pointwise asymptotic stability (also called semistability) is a property of a set of equilibria in a dynamical system, where every equilibrium is Lyapunov stable and from a neighborhood of every equilibrium, every solution converges to a possibly different equilibrium. Motivated by applications to consensus algorithms [26], hysteresis [38], chemical processes and thermodynamics [20], and robust control of networks [29], and appropriate for the case when equilibria form a continuum, this stability concept has been analyzed in the setting of differential equations [3], [4] and differential inclusions [27], [28]. Sufficient conditions related to this paper are in [5], where arc-length-based Lyapunov sufficient conditions are proposed. In the setting of difference inclusions, there exist necessary and sufficient conditions involving a set-valued Lyapunov function [11], a concept motivated by [37], which enable robustness results [12]. Some work on semistability has appeared for switching [25], [24] and for a class of impulsive systems modeling consensus over networks [23], but it has not addressed necessary or sufficient conditions for PAS in the hybrid setting.

The Zeno phenomenon in a hybrid system is the occurrence of infinitely many jumps (or events, or switches, or discrete transitions) in a finite amount of (ordinary) time, often called the Zeno time. The phenomenon can arise from modeling abstractions. Early interest in the Zeno phenomenon stemmed, in part, from the challenge it posed to simulations [31], [30]; see a recent and extended discussion in [32]. There exist works on sufficient conditions for the Zeno behavior without relation to equilibria or stability [46]; for Zeno behavior and stability of isolated equilibria [1], [33], [39], [18]; for Zeno behavior at nonisolated equilibria, necessarily without asymptotic stability considerations, [34]; and necessary and sufficient Lyapunov conditions [17] for stability of and Zeno behavior at Zeno equilibria. There also exists a body of work on excluding Zeno and Zeno-like behavior in classes of hybrid and related systems, for example, [40] or [44] and the references therein. A Zeno equilibrium [35] or a compact attractor [17] may have a small (or bounded) ordinary time property, which essentially requires that solutions that start near the equilibrium or attractor have uniformly small (or bounded) Zeno times. PAS and a small ordinary time (SOT) property of the set of equilibria are used in this paper to ensure reasonable dependence of Zeno solutions and their Zeno times on initial conditions.

Continuation of solutions beyond Zeno times is not possible in the hybrid inclusions framework, as the hybrid time domains representing Zeno behavior are already unbounded. Some work on continuation of solutions past Zeno time, in the setting of hybrid inclusions or the less-general hybrid automata, has been done by [47], [39], [9], [32]. In particular, [47], [39] propose a formal “completion,” where Zeno behaviors in a hybrid system are followed by or alternate with continuous evolutions described by a separate set of dynamics. In [32], for simulation purposes, the authors approximate outcomes of potential Zeno behaviors by a set and use the set to initiate post Zeno behavior. In [9] the authors continue a solution past its Zeno time from any point in the ω -limit of the pre-Zeno solution. In these works, the continuation beyond Zeno is achieved by appending another hybrid time domain, to represent the post-Zeno behavior, at “the end” of the Zeno time domain of the pre-Zeno solution. This paper adopts a similar approach. Other works, for example, [8], [21], work with a similar model of a hybrid system/hybrid automaton, but generalize the hybrid time domain to a more

abstract and more general object for the purpose of modeling beyond Zeno. How solutions on such generalized time domains depend on initial conditions is not clear.

The works mentioned in the previous paragraph are not concerned with the dependence of solutions, in particular their post-Zeno parts, on initial conditions. When a Zeno equilibrium is present, as in [39], and solutions converge to a physically meaningful limit so that the post-Zeno behaviors have a natural—and the same—initial condition, that dependence is not of interest. In absence of equilibria, as in [32] or [9], for a single pre-Zeno solution a whole set of potential initial conditions for post-Zeno behaviors is suggested. The dependence of these sets, in particular of ω -limits of solutions as in [9], on initial conditions is not regular. In summary, to the best of the authors’ knowledge, there is no previous work concerned with dependence of past-Zeno solutions on pre-Zeno initial conditions.

Some related modeling frameworks, for example, dynamical systems on time scales [6], transition systems [8], or measure-driven differential equations—see the extensive discussions and references in [7], [45]—allow for solutions with certain Zeno-like and more exotic behaviors as well as for continuation of solutions beyond them. Modeling capabilities of these frameworks are different from those of hybrid inclusions. In particular, hybrid inclusions can model switched systems under certain classes of switching signals, hybrid automata with logical modes, etc., and the available robust stability theory for hybrid inclusions [16] makes them attractive for modeling and design of hybrid control algorithms [15]. A time scale [6] is an arbitrary closed subset of \mathbb{R} , and so can represent multiple Zeno and other behaviors, but is usually fixed a priori. This is not ideal for state-space models, where different solutions may have jumps at different times, and so may require different time scales. Measure-driven differential equations are considered appropriate for mechanical systems with unilateral constraints [7], [45] but are not well-suited to model switching and logical modes or purely discrete-time dynamics. For further discussion, see [7], [16], [15], or [45]. This paper deals with hybrid inclusions only.

1.2. Contribution/outline. The introduction concludes with academic examples motivated by a consensus problem that will be used throughout the paper to illustrate concepts and results. Section 2 presents background on set-valued analysis and recalls the hybrid inclusions framework and the concept of nominal well-posedness of hybrid inclusions that ensures reasonable (outer semicontinuous) dependence of solutions over finite time horizons on initial conditions. Section 3 defines PAS in a hybrid system; gives sufficient conditions for it, some of which rely on set-valued Lyapunov functions; and gives consequences of PAS in a nominally well-posed system. These consequences include reasonable dependence of solutions, not just over finite time horizons, and of their limits on initial conditions. Section 4 extends several results of section 3 to the more general setting of partial PAS. Proofs of some results in section 3 are included for illustration purposes, but the key technical results are proven in the more general setting of section 4. Section 5 proposes several scenarios on how Zeno solutions that converge to a pointwise asymptotically stable set can be continued past their Zeno times, with initial conditions of the continuations depending on the limits of Zeno solutions, shows that reasonable dependence of solutions on initial conditions can apply to the continuations, and lists some consequences of this.

1.3. An example. Consider I agents with states $z_i \in \mathbb{R}^l$ for $i = 1, 2, \dots, I$. Agents communicate, agree on a target a in the convex hull of z_i ’s, and move toward a according to

$$\dot{z}_i = a - z_i$$

for $T > 0$ amount of time. After T amount of time, they communicate again and agree on a new a , and the process is repeated. One can expect that z_i 's converge to a common limit. This property will be shown by using the convex hull of z_1, z_2, \dots, z_I and a , with a involved here to allow for an arbitrary initial value of a , and building from this convex hull a set-valued Lyapunov function that shrinks along solutions. In fact, the set where $z_i = a$ for every $i = 1, 2, \dots, I$ turns out to be partially pointwise asymptotically stable (pPAS)—partially because to keep track of time between communication instants the whole state of the hybrid inclusion modeling this situation involves a timer variable, which does not converge.

A variation on this simple case is this: agents communicate, agree on a target a that is the arithmetic average of z_i 's, and move toward a according to

$$\dot{z}_i = c_i \frac{a - z_i}{\sqrt{\|a - z_i\|}},$$

where $c_i > 0$ are constants and $\|\cdot\|$ is the Euclidean norm, and when one of the agents reaches a , which occurs in finite time due to the assigned velocity of z_i , agents communicate again, agree on a new target a , and the process is repeated infinitely many times. A similar set-valued Lyapunov function can be used here to argue partial Lyapunov stability property of the set of states such that $z_i = a$, $i = 1, 2, \dots, I$, while an appropriate standard Lyapunov function shows that solutions are Zeno and, in fact, the equilibria have the SOT property.²

The stability property of the limits of solutions, as suggested for these two cases, ensures that the limits depend in a regular way on the initial conditions. For the second case, so do the Zeno times of solutions. This regular dependence helps answer the natural question, If the agents, after converging to the same limit in finite time but after infinitely many communication events, take a common action based on that limit, how do the resulting behaviors depend on the initial, pre-Zeno, conditions? Detailed discussion is carried out in Examples 2.1, 2.6, 3.4, 4.7, 4.10, and 5.5.

2. Background.

2.1. Set-valued analysis. The set-valued analysis background presented below follows [42]; see also [16] for an illustration in the setting of hybrid systems.

Let $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, in the sense that for every $x \in \mathbb{R}^m$, $M(x)$ is a subset of \mathbb{R}^n . Then the *domain* of M , denoted $\text{dom}M$, is $\{x \in \mathbb{R}^m \mid M(x) \neq \emptyset\}$ and the *range* of M , denoted $\text{rge}M$, is $\{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^m \text{ with } y \in M(x)\}$. Let $x \in \mathbb{R}^m$. Then M is *outer semicontinuous at x* if for every sequence $x_i \rightarrow x$, every convergent sequence $y_i \in M(x_i)$, one has $\lim_{i \rightarrow \infty} y_i \in M(x)$. It is *continuous at x* if, additionally, for every $y \in M(x)$, every sequence $x_i \rightarrow x$, there exist $y_i \in M(x_i)$ such that the sequence of y_i converges and $\lim_{i \rightarrow \infty} y_i = y$. The mapping M is *locally bounded at x* if there exists a neighborhood U of x such that $M(U) := \bigcup_{z \in U} M(z)$ is bounded. If M has closed values and is *locally bounded at x* , then outer semicontinuous at x is equivalently described as for every $\varepsilon > 0$ there exists $\delta > 0$ such that $M(x + \delta\mathbb{B}) \subset M(x) + \varepsilon\mathbb{B}$, which means that for every $z \in x + \delta\mathbb{B}$, $M(z) \subset M(x) + \varepsilon\mathbb{B}$. Here \mathbb{B} is the closed unit ball in \mathbb{R}^n , and so $x + \delta\mathbb{B}$ is the closed ball of radius δ around x , and $M(x) + \varepsilon\mathbb{B}$ is the ε -sized neighborhood of the set $M(x)$. The additional condition for continuity of such M at x is that for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $z \in x + \delta\mathbb{B}$, $M(x) \subset M(z) + \varepsilon\mathbb{B}$.

²See [23] for conditions guaranteeing finite-time semistability for a consensus problem modeled as an impulsive system.

2.2. Hybrid inclusions. The presentation of hybrid inclusions follows [16]; see also [15] for examples, comparison to other modeling frameworks, etc. A hybrid inclusion is represented by

$$(1) \quad \begin{aligned} x &\in C, & \dot{x} &\in F(x), \\ x &\in D, & x^+ &\in G(x). \end{aligned}$$

Above, $C, D \subset \mathbb{R}^n$ are sets, called, respectively, the *flow set* and the *jump set*, and $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings, called, respectively, the *flow map* and the *jump map*. A special case of (1) is provided by systems where the flow and jump maps are functions, which is represented by

$$(2) \quad \begin{cases} x \in C, & \dot{x} = f(x), \\ x \in D, & x^+ = g(x). \end{cases}$$

For simplicity (but with slight abuse) of notation, given column vectors x and y , we write (x, y) instead of $[x^\top y^\top]^\top$.

Example 2.1. The first example informally discussed in the introduction can be written as the hybrid inclusion

$$(3) \quad \begin{aligned} \dot{z}_i &= a - z_i, \dot{a} = 0, \dot{\tau} = -1 && \text{if} && \tau \geq 0, \\ z_i^+ &= z_i, a^+ \in \text{con}\{z_1, z_2, \dots, z_I\}, \tau^+ = T && \text{if} && \tau = 0, \end{aligned}$$

with state $x = (x_1, x_2) \in \mathbb{R}^{(I+1)l+1}$, where $z = (z_1, z_2, \dots, z_I)$, $x_1 = (z, a)$, and $x_2 = \tau$ is the timer variable. More explicitly, the data (C, F, D, G) is

$$\begin{aligned} C &:= \mathbb{R}^{(I+1)l} \times [0, T], & F(x) &:= (a - z_1, a - z_2, \dots, a - z_I, 0, -1) \quad \forall x \in C, \\ D &:= \mathbb{R}^{(I+1)l} \times \{0\}, & G(x) &:= (z_1, z_2, \dots, z_I, \text{con}\{z_1, z_2, \dots, z_I\}, T) \quad \forall x \in D. \end{aligned}$$

Similarly, the second example discussed in the introduction is modeled as a hybrid inclusion in Example 2.6.

Solutions to (1) are defined on hybrid time domains. A set $E \subset \mathbb{R}^2$ is a *hybrid time domain* if for every $(T, J) \in E$, the truncation of E from $(0, 0)$ to (T, J) , namely, $E \cap ([0, T] \times [0, J])$, has the representation

$$(4) \quad \bigcup_{j=0}^J [t_j, t_{j+1}] \times \{j\}, \quad \text{where } 0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_J \leq t_{J+1} = T.$$

Equivalently, E is a hybrid time domain if it is a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_0 \leq t_1 \leq t_2 \leq \dots$, with the last interval, if it exists, possibly of the form $[t_j, t_{j+1})$ and possibly with $t_{j+1} = \infty$.

A function $\phi : \text{dom } \phi \rightarrow \mathbb{R}^n$ is a *solution to the hybrid system* (1) if the set $\text{dom } \phi \subset \mathbb{R}^2$ is a hybrid time domain, $\phi(0, 0) \in \overline{C} \cup D$, and

- if $I^j := \{t \mid (t, j) \in \text{dom } \phi\}$ has nonempty interior, then $t \mapsto \phi(t, j)$ is locally absolutely continuous on I^j and

$$\phi(t, j) \in C \quad \forall t \in \text{int} I^j \quad \text{and} \quad \frac{d}{dt} \phi(t, j) \in F(\phi(t, j)) \quad \text{for almost all } t \in I^j;$$

- if $(t, j) \in \text{dom } \phi$ and $(t, j+1) \in \text{dom } \phi$, then

$$\phi(t, j) \in D \quad \text{and} \quad \phi(t, j+1) \in G(\phi(t, j)).$$

A solution $\phi : \text{dom } \phi \rightarrow \mathbb{R}^n$ is *maximal* if it cannot be extended (that is, there exists no solution $\psi : \text{dom } \psi \rightarrow \mathbb{R}^n$ such that $\text{dom } \phi$ is a strict subset of $\text{dom } \psi$ and $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \text{dom } \phi$) and *complete* if E is unbounded. The hybrid system (1) is *forward complete* if every maximal solution to (1) is complete.

For conditions guaranteeing that maximal solutions are complete, see [16, Propositions 2.10 and 6.10]. In what follows, \mathcal{S} denotes the set of all maximal solutions to (1), $\mathcal{S}(x)$ denotes the set of maximal solutions to (1) that start from x , and for a set $K \subset \mathbb{R}^n$, $\mathcal{S}(K) := \bigcup_{x \in K} \mathcal{S}(x)$, that is, the set of maximal solutions to (1) that start from a point in K .

A solution $\phi : \text{dom } \phi \rightarrow \mathbb{R}^n$ is *Zeno* if it is complete but $\text{dom } \phi$ is “bounded in the t -direction,” that is, $\sup_t \text{dom } \phi < \infty$, where $\sup_t \text{dom } \phi = \sup\{t \geq 0 \mid \exists j \text{ with } (t, j) \in \text{dom } \phi\}$. In other words, ϕ is Zeno if $\text{length } \text{dom } \phi = \infty$, where $\text{length } \text{dom } \phi = \sup\{t + j \mid (t, j) \in \text{dom } \phi\}$, while $\sup_t \text{dom } \phi < \infty$. For a Zeno solution, $\sup_t \text{dom } \phi$ is referred to as the Zeno time of ϕ . Note that this definition of Zeno solutions does not distinguish between “truly Zeno” solutions, domains of which have infinitely many nontrivial intervals of flow, “instantaneous Zeno,” or discrete, solutions ϕ for which $\sup_t \text{dom } \phi = 0$, etc.

2.3. Nominally well-posed hybrid systems. In a nominally well-posed hybrid system, solutions depend reasonably (outer semicontinuously) on initial conditions when no perturbations are present. On a more technical level, this property results from being able to extract convergent subsequences from certain sequences of solutions and to ensure that the appropriately understood limit is also a solution.

The concept of convergence, which is appropriate for the analysis of sequences of solutions to hybrid systems and which takes the place of uniform convergence usually used for sequences of solutions to differential equations or inclusions, is graphical convergence. For general exposition see [42] or, for details on graphical convergence of solutions to hybrid systems, see [16]. The definitions are not recalled here, but a characterization of graphical convergence for the case of a sequence of solutions to (1) converging to a solution to (1), sufficient for the purposes of this paper, is given below in Theorem 2.8.

The following is, essentially, [42, Theorem 4.18], restated for hybrid arcs in [16, Theorem 6.1].

THEOREM 2.2. *For every sequence $\phi_i \in \mathcal{S}$ with convergent initial conditions $\phi_i(0, 0)$ there exists a graphically convergent subsequence, the graphical limit ϕ of which satisfies $\lim_{i \rightarrow \infty} \phi_i(0, 0) \in \phi(0, 0)$.*

The result above does not ensure that ϕ is a solution. In fact, as the notation suggests, the graphical limit of a sequence of solutions may be set valued. For ϕ to be a solution, more is required of (1). The following definition of a property called *nominal well-posedness* of (1) is simplified, for the purposes of this paper, from the original definition [16, Definition 6.2]. A sequence ϕ_i of solutions to (1) is *locally eventually bounded* if for every $\tau > 0$ there exist i_0 and a compact set $K \subset \mathbb{R}^n$ so that for every $i > i_0$, every $(t, j) \in \text{dom } \phi_i$ with $t + j < \tau$, one has $\phi_i(t, j) \in K$.

DEFINITION 2.3. *The hybrid system (1) is nominally well-posed if for every graphically convergent sequence ϕ_i of solutions to (1) with convergent initial conditions $\phi_i(0, 0)$, either*

- (a) *the sequence ϕ_i is locally eventually bounded and its graphical limit ϕ is a solution to (1) with $\phi(0, 0) = \lim_{i \rightarrow \infty} \phi_i(0, 0)$ and with $\lim_{i \rightarrow \infty} \text{length } \text{dom } \phi_i = \text{length } \text{dom } \phi$, or*

(b) *there exists an unbounded and not complete solution from $\lim_{i \rightarrow \infty} \phi_i(0, 0)$.*

If a nominally well-posed system does not *exhibit finite-time blow-up*, i.e., there is no unbounded and not complete solution to it, then every graphically convergent subsequence ϕ_i as above is locally eventually bounded and (b) above is excluded. In the original definition [16, Definition 6.2], (b) contains a requirement that a sequence ϕ_i that is not locally eventually bounded leads, through graphical convergence of truncations of ϕ_i , to a blowing-up solution.

A reasonably easy to verify sufficient condition for nominal well-posedness, in fact for a stronger property called *well-posedness*, which considers sequences of solutions generated with vanishing perturbations, is described below. It requires that the data (C, F, D, G) have some regularity properties. The subsequent result recalls [16, Theorem 6.8].

DEFINITION 2.4. *The hybrid system (1) satisfies the hybrid basic conditions if its data, which is given by (C, F, D, G) , satisfies the following conditions: the sets $C, D \subset \mathbb{R}^n$ are closed. The set-valued mappings $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are locally bounded and outer semicontinuous at every $x \in C$, respectively, $x \in D$; for every $x \in C$, $F(x)$ is nonempty and convex; for every $x \in D$, $G(x)$ is nonempty.*

THEOREM 2.5. *If (1) satisfies the hybrid basic conditions, then it is nominally well-posed.*

Example 2.6. The second example informally discussed in the introduction can be written as the hybrid inclusion

$$\dot{z}_i = \begin{cases} c_i \frac{a - z_i}{\sqrt{\|a - z_i\|}} & \text{if } a \neq z_i, \\ 0 & \text{if } a = z_i, \end{cases} \quad \dot{a} = 0, \quad \dot{\tau} = -1, \quad \text{if } \tau \geq 0,$$

$$z_i^+ = z_i, \quad a^+ = \frac{1}{I} (z_1 + z_2 + \dots + z_I), \quad \tau^+ = T(z_1, z_2, \dots, z_I, a^+) \quad \text{if } \tau = 0,$$

where $T(z_1, z_2, \dots, z_n, a^+) = 2 \min_{i=1, 2, \dots, I} \frac{\sqrt{\|a^+ - z_i\|}}{c_i}$. Here, the state $x = (x_1, x_2) \in \mathbb{R}^{(I+1)l+1}$, where $x_1 = (z_1, z_2, \dots, z_I, a)$ and $x_2 = \tau$. Explicit formulas for the data (C, F, D, G) can be written as in Example 2.1; in particular, $C := \mathbb{R}^{(I+1)l} \times [0, \infty)$, so that τ can be initialized with an arbitrary positive value. It can be verified that this system satisfies the hybrid basic conditions.

Similarly, the system in Example 2.1 satisfies the hybrid basic conditions.

Graphical convergence in general admits characterizations through uniform bounds, when arguments are restricted to bounded sets. For functions defined on hybrid time domains, these uniform bounds turn to (a) in Definition 2.7. Bounds in (b) and (c) therein apply to solutions to (1) under further assumptions, like PAS.

DEFINITION 2.7. *For a given $\varepsilon > 0$, $\tau > 0$, two hybrid arcs $\phi, \phi' \in \mathcal{S}$ are*

- (a) (τ, ε) -close if
- (i) *for all $(t, j) \in \text{dom } \phi$ with $t + j < \tau \exists (t', j) \in \text{dom } \phi'$ with $|t - t'| < \varepsilon$ and $\|\phi(t, j) - \phi'(t', j)\| < \varepsilon$;*
 - (ii) *for all $(t', j') \in \text{dom } \phi'$ with $t' + j' < \tau \exists (t, j') \in \text{dom } \phi$ with $|t - t'| < \varepsilon$ and $\|\phi'(t', j') - \phi(t, j')\| < \varepsilon$.*
- (b) ε -close to τ -truncations of one another if
- (i) *for all $(t, j) \in \text{dom } \phi \exists (t', j') \in \text{dom } \phi'$, $t' + j' < \tau$ with $\|\phi(t, j) - \phi'(t', j')\| < \varepsilon$;*

- (ii) for all $(t', j') \in \text{dom } \phi' \exists (t, j) \in \text{dom } \phi, t + j < \tau$ with $\|\phi'(t', j') - \phi(t, j)\| < \varepsilon$.
- (c) ε -close if
 - (i) for all $(t, j) \in \text{dom } \phi \exists (t', j) \in \text{dom } \phi'$ with $|t - t'| < \varepsilon$ and $\|\phi(t, j) - \phi'(t', j)\| < \varepsilon$
 - (ii) for all $(t', j') \in \text{dom } \phi' \exists (t, j') \in \text{dom } \phi$ with $|t - t'| < \varepsilon$ and $\|\phi'(t', j') - \phi(t, j')\| < \varepsilon$.

The following summarizes [16, Theorem 5.25] for current purposes.

THEOREM 2.8. *A locally eventually bounded sequence ϕ_i of hybrid arcs graphically converges to a hybrid arc ϕ with closed graph if and only if for every $\varepsilon > 0, \tau > 0$, the arcs ϕ_i and ϕ are (τ, ε) -close for all large enough i .*

3. Pointwise asymptotic stability. PAS is a property of a set. Each element of the set is required to be Lyapunov stable, and from near the set, complete solutions are required to converge to some element of the set. The definition below also allows for maximal solutions to not be complete, and such solutions are required to be bounded.

DEFINITION 3.1. *A set $A \subset \mathbb{R}^n$ is pointwise asymptotically stable if*

- (a) every $a \in A$ is Lyapunov stable, that is, for every $a \in A$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $\text{rge } \phi \subset a + \varepsilon \mathbb{B}$ for every $\phi \in \mathcal{S}(a + \delta \mathbb{B})$, and
- (b) there exists a neighborhood U of A such that, for every $\phi \in \mathcal{S}(U)$, ϕ is bounded and if it is complete, then $\lim_{t+j \rightarrow \infty} \phi(t, j)$ exists and belongs to A .

The basin of pointwise attraction of a pointwise asymptotically stable set A , denoted $\mathcal{B}(A)$, is the set of $x \in \mathbb{R}^n$ such that, for every $\phi \in \mathcal{S}(x)$, ϕ is bounded and if it is complete, then $\lim_{t+j \rightarrow \infty} \phi(t, j)$ exists, and belongs to A .

If A consists of a single point, PAS reduces to the usual asymptotic stability; see, e.g., [16, Definition 7.1]. When A is a set, the concepts differ, since the usual understanding of asymptotic stability of a set does not require that every element of the set be an equilibrium and that solutions have limits. Furthermore, asymptotic stability of a set consisting of equilibria still does not amount to PAS, as in such a case solutions need not have a limit; see [4, Example 1.1].

3.1. Lyapunov sufficient conditions. Classical Lyapunov sufficient conditions for asymptotic stability require that a real-valued function decreases, or at least does not increase, along solutions. Such conditions, on their own, are not enough to ensure PAS. Set-valued Lyapunov functions can be used for this purpose, leading to conditions that at least visually resemble classical Lyapunov inequalities but require that the values of the set-valued Lyapunov functions decrease, as sets, along solutions.

3.1.1. Set-valued Lyapunov functions. Weak set-valued Lyapunov functions for a set, defined below, guarantee Lyapunov stability of elements of the set, as shown in Theorem 3.3. Under a further assumption of nominal well-posedness, they can be used in an invariance-based argument to deduce PAS; see Theorem 3.13. Without nominal well-posedness, a strict decrease as requested in Definition 3.5 leads to PAS in Theorem 3.7.

DEFINITION 3.2. *A set-valued mapping $W : U \rightrightarrows \mathbb{R}^n$ is a weak set-valued Lyapunov function for (1) and a set $A \subset \mathbb{R}^n$ if $U \subset \mathbb{R}^n$ is an open and forward invariant³ neighborhood of A for (1) and*

³A set U is forward invariant (for a hybrid inclusion) if for each $\phi \in \mathcal{S}(U)$, $\text{rge } \phi \subset U$.

- (a) $W(x) = \{x\}$ for every $x \in A$ and $x \in W(x)$ for every $x \in (\overline{C} \cup D \cup G(D)) \cap U$;
- (b) W is locally bounded and, at every $x \in A$, it is outer semicontinuous;
- (c) for every solution $\phi : [0, T] \rightarrow U$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,

$$(5) \quad W(\phi(t)) \subset W(\phi(0)) \quad \forall t \in [0, T].$$

(d)

$$(6) \quad W(G(x)) \subset W(x) \quad \forall x \in D \cap U.$$

If W is a weak set-valued Lyapunov function, as defined above, then for every $\phi \in \mathcal{S}(U)$,

$$(7) \quad \phi(t, j) \in W(\phi(t, j)) \subset W(\phi(0, 0)) \quad \forall (t, j) \in \text{dom } \phi,$$

where the containment comes from (a) and the inclusion comes from (c) and (d) of Definition 3.2.

THEOREM 3.3. *If there exists a weak set-valued Lyapunov function W for (1) and a set $A \subset \mathbb{R}^n$, then every point $a \in A$ is Lyapunov stable.*

Proof. If $A = \emptyset$, there is nothing to prove. Otherwise, pick $a \in A$ and $\varepsilon > 0$. Note that by (a) of Definition 3.2, $W(a) = \{a\}$, and by (b) of Definition 3.2, there exists $\delta > 0$ such that for every $x \in a + \delta\mathbb{B}$, $W(x) \subset W(a) + \varepsilon\mathbb{B} = a + \varepsilon\mathbb{B}$. If δ is small enough so that $a + \delta\mathbb{B} \subset U$, where U is the open neighborhood of A that comes from $W : U \rightrightarrows \mathbb{R}^n$ being a weak set-valued Lyapunov function for (1) and the set A , then for any $\phi \in \mathcal{S}(a + \delta\mathbb{B})$, thanks to (7), one has

$$\phi(t, j) \in W(\phi(0, 0)) \subset W(a + \delta\mathbb{B}) \subset a + \varepsilon\mathbb{B}$$

for every $(t, j) \in \text{dom } \phi$. This verifies Lyapunov stability of a . \square

Example 3.4. In the setting of Example 2.1, consider the set-valued mapping given by

$$w(z, a) = \text{con}\{z_1, z_2, \dots, z_I, a\}.$$

Just as the state x for the system in Example 2.1 is (z, a, τ) , a solution ϕ to (3) at hybrid time (t, j) will be written as $\phi(t, j) = (z(t, j), a(t, j), \tau(t, j))$. If $(t, j) \in \text{dom } \phi$ represents the beginning of an interval of flow for the solution ϕ , for every $t' > t$ such that $(t', j) \in \text{dom } \phi$, one has $z_i(t', j) \in w(z(t, j), a(t, j))$ because z_i moves along a segment contained in $w(z(t, j), a(t, j))$; $a(t', j) \in w(z(t, j), a(t, j))$ because $a(t', j) = a(t, j)$, and hence

$$w(z(t', j), a(t', j)) \subset w(z(t, j), a(t, j)).$$

Similarly, after a jump, $a^+ \in \text{con}\{z_1, z_2, \dots, z_I\}$, $z^+ = z$, and so for every solution ϕ , every $(t, j) \in \text{dom } \phi$ such that $(t, j + 1) \in \text{dom } \phi$,

$$w(z(t, j + 1), a(t, j + 1)) \subset w(z(t, j), a(t, j)).$$

These two weak decrease properties of w remain true even if, initially, a is not in the convex hull of the z_i 's. It is thus a natural idea to try to build a weak set-valued Lyapunov function from w as above. However, for the system in Example 2.1, the variable τ is not expected to converge, and only partial PAS can be expected. For this purpose, an appropriate partial set-valued Lyapunov function is given in Example 4.7.

Below, a function $\gamma : U \rightarrow \mathbb{R}$ is *positive definite with respect to* A , where $A \subset U$, if $\gamma(x) \geq 0$ for every $x \in U$ and $\gamma(x) = 0$ if and only if $x \in A$. Given a hybrid inclusion with data (C, F, D, G) , a function $\gamma : U \rightarrow \mathbb{R}$ is *positive definite with respect to* A on $\overline{C \cup D} \cap U$ if $\gamma(x) \geq 0$ for every $x \in \overline{C \cup D} \cap U$, and $\gamma(x) = 0$ if and only if $x \in \overline{C \cup D} \cap A$. Finally, given a hybrid inclusion with data (C, F, D, G) , a function $\gamma : U \rightarrow \mathbb{R}$ is *positive definite with respect to* A on $\overline{C \cup D} \cap \overline{U}$ if it is positive definite with respect to A on $\overline{C \cup D} \cap U$ and, additionally, for any convergent sequence of points $x_i \in U$ satisfying $\lim_{i \rightarrow \infty} x_i =: x \notin A$, one has $\liminf_{i \rightarrow \infty} \gamma(x_i) > 0$.

DEFINITION 3.5. A set-valued mapping $W : U \rightrightarrows \mathbb{R}^n$ is a set-valued Lyapunov function for (1) and a set $A \subset \mathbb{R}^n$ if it is a weak set-valued Lyapunov function for (1) and A and there exist continuous functions $c, d : U \rightarrow [0, \infty)$, positive definite with respect to A on $(\overline{C \cup D}) \cap U$, such that the following two conditions hold:

- (c) for every solution $\phi : [0, T] \rightarrow U$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,

$$(8) \quad W(\phi(t)) + \left(\int_0^t c(\phi(s)) ds \right) \mathbb{B} \subset W(\phi(0)) \quad \forall t \in [0, T];$$

- (d)

$$(9) \quad W(G(x)) + d(x)\mathbb{B} \subset W(x) \quad \forall x \in D \cap U;$$

and either $\overline{W(x)} \subset U$ for every $x \in (C \cup D) \cap U$ or the functions c and d are positive definite with respect to A on $\overline{C \cup D} \cap \overline{U}$.

In general, continuity of c and d and their positive definiteness with respect to A on $(\overline{C \cup D}) \cap U$ imply that A is relatively closed in U .

Remark 3.6. The definition of a set-valued Lyapunov function simplifies significantly if $U = \mathbb{R}^n$, that is, when W is a global set-valued Lyapunov function. It simplifies further if the sets C and D are closed. (This is the case when (1) satisfies the hybrid basic conditions.) Under such additional assumptions, $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued Lyapunov function if

- (a) $W(x) = \{x\}$ for every $x \in A$ and $x \in W(x)$ for every $x \in C \cup D \cup G(D)$;

- (b) W is locally bounded and, at every $x \in A$, it is outer semicontinuous;

and there exist continuous functions $c, d : \mathbb{R}^n \rightarrow [0, \infty)$, positive definite with respect to A on $C \cup D$, such that the following two conditions hold:

- (c) for every solution $\phi : [0, T] \rightarrow C$ to $\dot{x} \in F(x)$, (8) holds;

- (d) $W(G(x)) + d(x)\mathbb{B} \subset W(x)$ for all $x \in D$.

The following result establishes that the existence of a set-valued Lyapunov function W implies PAS of the associated set. A key property for this result is the following. Suppose W is a set-valued Lyapunov function, as in Definition 3.5. Pick any $\phi \in \mathcal{S}(U)$ and any $(T, J) \in \text{dom } \phi$, and let (4) represent $\text{dom } \phi$ from $(0, 0)$ to (T, J) . Then, (c) and (d) of Definition 3.2 imply that

$$(10) \quad W(\phi(T, J)) + \left(\sum_{j=0}^J \int_{t_j}^{t_{j+1}} c(\phi(s, j)) ds + \sum_{j=1}^J d(\phi(t_j, j-1)) \right) \mathbb{B} \subset W(\phi(0, 0)).$$

THEOREM 3.7. If there exists a set-valued Lyapunov function $W : U \rightrightarrows \mathbb{R}^n$ for (1) and a set $A \subset \mathbb{R}^n$, then A is pointwise asymptotically stable and $U \subset \mathcal{B}(A)$.

Proof. Lyapunov stability of $a \in A$ was shown in Theorem 3.3. Take any $\phi \in \mathcal{S}(U)$. By (b) of Definition 3.2, W is locally bounded. In particular, $W(\phi(0, 0))$ is bounded, by (7) one has $\phi(t, j) \in W(\phi(0, 0))$ for every $(t, j) \in \text{dom } \phi$, and hence ϕ is bounded. If ϕ is not complete, there is nothing left to do. Suppose that ϕ is complete. The decrease condition (10) implies that

$$\sum_{j=0}^J \int_{t_j}^{t_{j+1}} c(\phi(s, j)) ds + \sum_{j=1}^J d(\phi(t_j, j-1))$$

is bounded over all $(T, J) \in \text{dom } \phi$. Then, since ϕ is complete, there exists a sequence $(t_k, j_k) \in \text{dom } \phi$, with $t_k + j_k \rightarrow \infty$ as $k \rightarrow \infty$, such that either $c(\phi(t_k, j_k))$ or $d(\phi(t_k, j_k))$ converges to 0 as $k \rightarrow \infty$. Because ϕ is bounded, without loss of generality one may assume that the sequence $\phi(t_k, j_k)$ converges. Denote the limit by x and note that $x \in \overline{W(\phi(0, 0))}$, thanks to (7), and $x \in \overline{C \cup D}$, by the definition of a solution and the fact that ϕ is complete. If $\overline{W(\phi(0, 0))} \subset U$, then $x \in U$, either $c(x) = 0$ or $d(x) = 0$, and positive definiteness of c and d with respect to $(\overline{C \cup D}) \cap U$ implies that $x \in A$. If c and d are positive definite with respect to A on $(\overline{C \cup D}) \cap \overline{U}$, then by definition of this property, $x \in A$. Lyapunov stability of every $x \in A$ implies that $\lim_{t+j \rightarrow \infty} \phi(t, j) = x$. \square

3.1.2. Weakened Lyapunov sufficient conditions. When the decrease of a set-valued Lyapunov function is strict only along flows, or only at jumps, as specified in Definition 3.8 below, PAS can still be deduced under further conditions on the amount of flow, or the amount of jumps for each solution.

DEFINITION 3.8. Let $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a weak set-valued Lyapunov function for (1) and a set $A \subset \mathbb{R}^n$. Then

- (a) W is strict along flows if there exists a continuous function $c : U \rightarrow [0, \infty)$, positive definite with respect to A on $\overline{C} \cap U$, such that (c) of Definition 3.5 holds and either $\overline{W(x)} \subset U$ for every $x \in C \cap U$ or the function c is positive definite with respect to A on $\overline{C} \cap \overline{U}$;
- (b) W is strict along jumps if there exists a continuous function $d : U \rightarrow [0, \infty)$, positive definite with respect to A on $\overline{D} \cap U$, such that (d) of Definition 3.5 holds and either $\overline{W(x)} \subset U$ for every $x \in D \cap U$ or d is positive definite with respect to A on $\overline{D} \cap \overline{U}$.

THEOREM 3.9. If there exists a weak set-valued Lyapunov function W for (1) and a closed set $A \subset \mathbb{R}^n$, and either

- (a) W is strict along flows and, if a solution ϕ to (1) is complete, then $\sup_t \text{dom } \phi = \infty$, or
- (b) W is strict along jumps and, if a solution ϕ to (1) is complete, then $\sup_j \text{dom } \phi = \infty$,

then A is pointwise asymptotically stable.

Proof. Lyapunov stability was shown in Theorem 3.3 so only convergence needs to be argued. Suppose that (a) holds. The case of (b) is similar. Proceeding as in the proof of Theorem 3.7, take any complete $\phi \in \mathcal{S}(U)$. As before, it turns out to be bounded. By assumption, ϕ satisfies $\sup_t \text{dom } \phi = \infty$, and hence without loss of generality, one can assume that $\phi(0, 0) \in C$. As before, one obtains that

$$(11) \quad \sum_{j=0}^J \int_{t_j}^{t_{j+1}} c(\phi(s, j)) ds$$

is bounded over all $(T, J) \in \text{dom } \phi$. Letting $T = t_{j+1} \rightarrow \infty$, which is possible since $\sup_t \text{dom } \phi = \infty$, one obtains the existence of a sequence $(t_k, j_k) \in \text{dom } \phi$, with $t_k \rightarrow \infty$ as $k \rightarrow \infty$, such that $c(\phi(t_k, j_k))$ converges to 0 as $k \rightarrow \infty$. Because ϕ is bounded, without loss of generality one may assume that the sequence $\phi(t_k, j_k)$ converges. Denote the limit by x and note that $x \in \overline{W(\phi(0, 0))}$, thanks to (7), and $x \in \overline{C}$. If $\overline{W(\phi(0, 0))} \subset U$, then $x \in U$, $c(x) = 0$, and positive definiteness of c with respect to $\overline{C} \cap U$ implies that $x \in A$. If c is positive definite with respect to A on $\overline{C} \cap \overline{U}$, then by definition of this property, $x \in A$. Lyapunov stability of every $x \in A$ implies that $\lim_{t+j \rightarrow \infty} \phi(t, j) = x$. \square

3.1.3. Sufficient conditions based on solution length. A different kind of sufficient conditions, using real-valued Lyapunov functions that limit the length of solutions, is featured in this section. As the proof of Corollary 3.12 shows, such functions are closely related to the strict set-valued Lyapunov functions used before.

DEFINITION 3.10. *A continuously differentiable function $V : U \rightarrow [0, \infty)$ is a finite-length Lyapunov function for (1) and a closed set $A \subset \mathbb{R}^n$ if U is an open neighborhood of A forward invariant with respect to (1), V is positive definite with respect to A , and there exist continuous functions $c, d : U \rightarrow [0, \infty)$, positive definite with respect to A on $\overline{C} \cup \overline{D} \cap \overline{U}$, such that the following two conditions hold:*

(c)

$$(12) \quad \nabla V(x) \cdot f \leq -c(x) - \|f\| \quad \forall x \in C, f \in F(x),$$

(d)

$$(13) \quad V(g) - V(x) \leq -d(x) - \|g - x\| \quad \forall x \in D, g \in G(x).$$

Remark 3.11. The name “finite-length Lyapunov function” comes from the fact that V as in Definition 3.10 implies that for every $\phi \in \mathcal{S}$ with $\text{rge } \phi \subset \text{dom } V$, for every $(T, J) \in \text{dom } \phi$, one has

$$V(\phi(T, J)) + \sum_{j=0}^J \int_{t_j}^{t_{j+1}} \|\dot{\phi}(t, j)\| ds + \sum_{j=1}^J \|\phi(t_j, j) - \phi(t_j, j-1)\| \leq V(\phi(0, 0))$$

and thus the quantity that can be considered the length of ϕ from $(0, 0)$ to (T, J) , namely,

$$\sum_{j=0}^J \int_{t_j}^{t_{j+1}} \|\dot{\phi}(t, j)\| ds + \sum_{j=1}^J \|\phi(t_j, j) - \phi(t_j, j-1)\|,$$

is uniformly bounded above by $V(\phi(0, 0))$ over all $(T, J) \in \text{dom } \phi$. This is the result of including the terms $\|f\|$ and $\|g - x\|$ on the right-hand side of Lyapunov inequalities (12), (13), which is inspired by [5]. There, the inequality $\nabla V(x) \cdot f(x) \leq -\|f(x)\|$ was used in sufficient conditions for PAS in a continuous-time system $\dot{x} = f(x)$.

COROLLARY 3.12. *If there exists a finite-length Lyapunov function $V : U \rightarrow [0, \infty)$ for (1) and a closed set $A \subset \mathbb{R}^n$, then A is pointwise asymptotically stable and $U \subset \mathcal{B}(A)$.*

Proof. Consider the set-valued mapping $W : U \rightrightarrows \mathbb{R}^n$ given by

$$(14) \quad W(x) = x + V(x)\mathbb{B} \quad \forall x \in U.$$

It is straightforward that (a) and (b) of Definition 3.5 are satisfied. Indeed, continuity of V implies local boundedness of W and continuity of W at every $x \in \text{dom } W$, not just outer semicontinuity of W at $x \in A$. By assumption, $\text{dom } W = U$ is forward invariant. From (c) in Definition 3.10, by integrating, one obtains that for every solution $\phi : [0, T] \rightarrow \text{dom } W$ to $\dot{x} \in F(x)$ with $\phi(t) \in C$ for all $t \in (0, T)$, one has that for each $t \in \text{dom } \phi$,

$$V(\phi(t)) - V(\phi(0)) \leq - \int_0^t c(\phi(s)) ds - \int_0^t \|f(\phi(s))\| ds.$$

Because $\|\phi(t) - \phi(0)\| \leq \int_0^t \|f(\phi(s))\| ds = \int_0^t \|\dot{\phi}(s)\| ds$, one obtains

$$V(\phi(t)) + \int_0^t c(\phi(s)) ds + \|\phi(t) - \phi(0)\| \leq V(\phi(0)).$$

This implies that

$$\phi(t) + V(\phi(t))\mathbb{B} + \int_0^t c(\phi(s))\mathbb{B} \subset \phi(0) + V(\phi(0))\mathbb{B},$$

and hence (c) of Definition 3.5 holds. Similarly, but without integration, (d) in Definition 3.10 implies (d) in Definition 3.5. Thus W in (14) is a set-valued Lyapunov function on U for A . Theorem 3.7 completes the proof. \square

3.1.4. Invariance-based sufficient conditions. A weak set-valued Lyapunov function for a set A ensures that every $a \in A$ is Lyapunov stable, as shown in Theorem 3.3. It does not ensure convergence of complete solutions to points in A . For nominally well-posed systems, under further invariance-based assumptions, convergence is guaranteed.

THEOREM 3.13. *Suppose that (1) is nominally well-posed. If there exist a continuous weak set-valued Lyapunov function $W : U \rightrightarrows \mathbb{R}^n$ for (1) and a set $A \subset \mathbb{R}^n$, and every complete solution to (1) on which W is constant is contained in A , then A is pointwise asymptotically stable and $U \subset \mathcal{B}(A)$.*

The proof is not given, as a more general result for partial PAS, Theorem 4.6, is proven later.

3.2. Small ordinary time stability. The SOT property of a Lyapunov stable point calls for a uniform small bound on $\sup_t \text{dom } \phi$ over solutions starting near enough to that point. If all such solutions are complete, then they are Zeno and there exists a uniform small bound on their Zeno times.

DEFINITION 3.14.

- (a) A point $a \in \mathbb{R}^n$ is SOT stable if it is Lyapunov stable and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_t \text{dom } \phi < \varepsilon$ for every $\phi \in \mathcal{S}(a + \delta\mathbb{B})$.
- (b) A set $A \subset \mathbb{R}^n$ is pointwise small ordinary time asymptotically stable (PSO-TAS) if it is pointwise asymptotically stable and every $a \in A$ is SOT stable.

This property is generalized to partial SOT in Definition 4.8, for which a Lyapunov sufficient condition is given in Proposition 4.9.

3.3. Consequences of PAS in well-posed systems. Limits of solutions to a hybrid system and, in fact, to a simple differential equation, need not exist and when they do, the limits need not depend regularly on initial conditions. For example, for

$\dot{x} = -x(x-1)^2$, limits equal 0 for solutions from $(-\infty, 1)$ and equal 1 otherwise. Limits of solutions depend discontinuously on initial conditions at $x = 1$. Note that the smallest globally asymptotically stable set here is $[0, 1]$, and the discontinuity occurs at a point in A . For the system $\dot{x} = -x(x-1)^2$, $\dot{y} = -y$ in the xy -plane, the compact set $[0, 1] \times \{0\}$ is the smallest globally asymptotically stable (but not pointwise asymptotically stable) set, the limits have a similar pattern as in the previous example and depend discontinuously on initial conditions at every point (x, y) with $x = 1$.

In fact, the limits may depend discontinuously on initial conditions even when the hybrid inclusion has affine flow and jump maps. Consider $C = [1, \infty)$, $f(x) = 1 - x$, $D = (-\infty, 1]$, and $g(x) = 0$. Then every maximal solution from $C \setminus \{1\}$ has limit equal to 1, while every maximal solution from $D \setminus \{1\}$ has limit equal to 0. From $x = 1$ there are maximal solutions with limit equal to 1 or 0. Also, the set $\mathcal{A} = \{0, 1\}$ is globally asymptotically stable but not globally pointwise asymptotically stable since $a = 1 \in \mathcal{A}$ is not Lyapunov stable.

In the presence of a pointwise asymptotically stable set, the limits of solutions to a hybrid system do depend regularly on initial conditions, and so do Zeno times of solutions under a further SOT assumption. These facts are included in the next result.

THEOREM 3.15. *Suppose that (1) is nominally well-posed and forward complete. For every sequence of $\phi_i \in \mathcal{S}$ with $\phi_i(0, 0)$ convergent, there exists a graphically convergent subsequence, which is not relabeled, such that*

- (a) *the graphical limit ϕ of the graphically convergent subsequence ϕ_i is a complete solution to (1).*

If, additionally, the closed set $A \subset \mathbb{R}^n$ is pointwise asymptotically stable and $\lim_{i \rightarrow \infty} \phi_i(0, 0) \in \mathcal{B}(A)$, then

- (b) *for all large enough i , $\lim_{t+j \rightarrow \infty} \phi_i(t, j) =: a_i$ exists and belongs to A ;*
- (c) *$\lim_{i \rightarrow \infty} a_i = \lim_{t+j \rightarrow \infty} \phi(t, j)$;*
- (d) *convergence of ϕ_i to ϕ is uniform in the following sense: for every $\varepsilon > 0$ there exists $\tau > 0$ such that, for every large enough i , ϕ_i and ϕ are ε -close to τ -truncations of one another.*

If, additionally, A is PSOTAS, then

- (e) *ϕ and ϕ_i for all large enough i are Zeno, and $\lim_{i \rightarrow \infty} \sup_t \text{dom } \phi_i = \sup_t \text{dom } \phi$;*
- (f) *convergence of ϕ_i to ϕ is uniform in the following sense: for every $\varepsilon > 0$ and every large enough i , ϕ_i and ϕ are ε -close to one another.*

The proofs of Theorem 3.15 and the resulting Corollary 3.16 are not given, as more general results for partial PAS, Theorem 4.11, and Corollary 4.12 are proven later.

COROLLARY 3.16. *Suppose that (1) is nominally well-posed and forward complete and the closed set $A \subset \mathbb{R}^n$ is PAS. Then,*

- (a) *the basin of pointwise attraction of A , $\mathcal{B}(A)$, is an open neighborhood of A ;*
- (b) *the set-valued mapping $\mathcal{L} : \mathcal{B}(A) \rightrightarrows \mathbb{R}^n$ defined by*

$$(15) \quad \mathcal{L}(x) = \left\{ \lim_{t+j \rightarrow \infty} \phi(t, j) \mid \phi \in \mathcal{S}(x) \right\}$$

is outer semicontinuous and locally bounded;

- (c) *the set-valued mapping $\overline{\mathcal{R}}_\infty : \mathcal{B}(A) \rightrightarrows \mathbb{R}^n$ defined by*

$$(16) \quad \overline{\mathcal{R}}_\infty(x) = \overline{\mathcal{R}_\infty(x)}$$

is outer semicontinuous and locally bounded and

$$\overline{\mathcal{R}}_\infty(x) = \mathcal{R}_\infty(x) \cup \mathcal{L}(x)$$

for every $x \in \mathcal{B}(A)$, where the set-valued mapping $\mathcal{R}_\infty : \mathcal{B}(A) \rightrightarrows \mathbb{R}^n$ is defined by

$$\mathcal{R}_\infty(x) = \{\text{rge}\phi \mid \phi \in \mathcal{S}(x)\}.$$

If, additionally, A is PSOTAS, then

(d) the function $\text{Length}_t : \mathcal{B}(A) \rightarrow [0, \infty]$ defined by

$$(17) \quad \text{Length}_t(x) = \sup \left\{ \sup_t \text{dom } \phi \mid \phi \in \mathcal{S}(x) \right\}$$

is upper semicontinuous and locally bounded (in particular, finite-valued), and the sup defining $\text{Length}_t(x)$ is attained for every $x \in \mathcal{B}(A)$.

The mappings \mathcal{L} in (15) and $\overline{\mathcal{R}}_\infty$ in (16) can be viewed as mappings from \mathbb{R}^n , not just from $\mathcal{B}(A)$. In such a case, $\mathcal{L}(x)$ and $\overline{\mathcal{R}}_\infty(x)$ are defined as empty sets for $x \notin \mathcal{B}(A)$.

It must be noted that while PAS in a well-posed system ensures that limits of solutions depend outer semicontinuously on initial conditions (continuously if solutions are unique), converse implications fail.

Example 3.17. Consider the differential inclusion in one dimension

$$\dot{x} \in F(x) := \begin{cases} 1 & \text{if } x \in (-\infty, -1) \cup (0, 1), \\ -1 & \text{if } x \in (-1, 0) \cup (1, \infty), \\ [-1, 1] & \text{if } x \in \{-1, 0, 1\}. \end{cases}$$

Then every solution is convergent and

$$\mathcal{L}(x) = \begin{cases} -1 & \text{if } x \in (-\infty, 0), \\ 1 & \text{if } x \in (0, \infty), \\ \{-1, 0, 1\} & \text{if } x = 0 \end{cases}$$

is outer semicontinuous but 0 is not Lyapunov stable; in fact it is not an equilibrium. This is not due to set-valuedness of the dynamics, as a similar outcome is generated by a differential equation, with nonunique solutions, given around 0 by $\dot{x} = \text{sign } x \sqrt{|x|}$ and built up appropriately elsewhere. For another example, consider a differential equation in two dimensions, given in polar coordinates by $\dot{r} = 0$, $\dot{\theta} = \theta(2\pi - \theta)$ for $\theta \in [0, 2\pi)$. Then $\mathcal{L}(r, \theta) = (r, 0)$ is continuous, with points $(r, 0)$ for $r \geq 0$ forming the set of equilibria, but only the origin is Lyapunov stable.

COROLLARY 3.18. *Suppose that (C, F, D, G) is nominally well-posed and forward complete and the closed set $A \subset \mathbb{R}^n$ is PAS. For every compact set $K \subset \mathcal{B}(A)$ and every $\varepsilon > 0$, there exist $\delta > 0$ and $\tau \geq 0$ with the following property: for every $\phi \in \mathcal{S}(K + \delta\mathbb{B})$ there exists $\phi' \in \mathcal{S}(K)$ such that ϕ and ϕ' are ε -close to τ -truncations of one another. If, additionally, A is PSOTAS, then for every compact set $K \subset \mathcal{B}(A)$ and every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: for every $\phi \in \mathcal{S}(K + \delta\mathbb{B})$ there exists $\phi' \in \mathcal{S}(K)$ such that ϕ' and ϕ are ε -close.*

The corollary can be shown by arguments very similar to those proving [16, Proposition 6.14], with ε -closeness to τ -truncations concept as in (d) of Theorem 3.15 replacing the (τ, ε) -closeness concept used in [16].

4. Partial PAS. This section generalizes several results from the previous one to the setting of partial PAS. Throughout the section, $n = n_1 + n_2$; for $x \in \mathbb{R}^n$, $x = (x_1, x_2)$ with $x_i \in \mathbb{R}^{n_i}$; and $A = A_1 \times \mathbb{R}^{n_2}$, where $A_1 \subset \mathbb{R}^{n_1}$. Similarly, for a solution ϕ , $\phi(t, j) = (\phi_1(t, j), \phi_2(t, j))$.

DEFINITION 4.1. *A set $A \subset \mathbb{R}^n$ is pPAS if*

- (a) *every $a \in A$ is partially Lyapunov stable, that is, for every $a_1 \in A_1$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $\text{rge}\phi_1 \subset a_1 + \varepsilon\mathbb{B}$ for every $\phi \in \mathcal{S}((a_1 + \delta\mathbb{B}) \times \mathbb{R}^{n_2})$, and*
- (b) *there exists a neighborhood U_1 of A_1 such that for every $\phi \in \mathcal{S}(U_1 \times \mathbb{R}^{n_2})$, ϕ_1 is bounded, and if ϕ is complete, then $\lim_{t+j \rightarrow \infty} \phi_1(t, j)$ exists and belongs to A_1 .*

The basin of partial pointwise attraction of a pPAS set A , denoted $p\mathcal{B}(A)$, is the set of points $x \in \mathbb{R}^n$ such that for every $\phi \in \mathcal{S}(x)$, ϕ_1 is bounded and if ϕ is complete, then $\lim_{t+j \rightarrow \infty} \phi_1(t, j)$ exists and belongs to A_1 .

The usefulness of partial stability was already pointed out in the informal discussion of the first example in the introduction. Another, natural situation where partial stability may need to be considered is when (1) with state $x = (q, z)$ models a hybrid automaton with different so-called discrete states or logical modes q (see [16, section 1.4.1]), when the different logical modes q_1, q_2, \dots, q_l share the same “continuous variable” equilibrium \bar{z} , and then the reset (jump) maps have the set $\{q_1, q_2, \dots, q_l\} \times \{\bar{z}\}$ invariant. In such a situation, stability can be desired for z but not for q .⁴ Additionally, timer variables or logical modes introduced in a control algorithm may be irrelevant for stability consideration for the controlled variable; Example 4.7 illustrates this.

4.1. Sufficient conditions. Definition 4.2 extends Definition 3.2 to the partial setting. Similarly, Theorems 4.3, 4.5, and 4.6 extend Theorems 3.3, 3.7, and 3.13 to such setting, respectively.

DEFINITION 4.2. *A set-valued mapping $W : U_1 \times \mathbb{R}^{n_2} \rightrightarrows \mathbb{R}^{n_1}$ is a partial weak set-valued Lyapunov function for (1) and the set A if $U_1 \subset \mathbb{R}^{n_1}$ is an open neighborhood of A_1 , $U_1 \times \mathbb{R}^{n_2}$ is forward invariant, and*

- (a) *$W(x) = \{x_1\}$ for every $x \in A_1 \times \mathbb{R}^{n_2}$ and $x_1 \in W(x)$ for every $x \in (\bar{C} \cup D \cup G(D)) \cap (U_1 \times \mathbb{R}^{n_2})$;*
- (b) *W is locally bounded and $W(x_1, x_2)$ is outer semicontinuous in x_1 at every $x_1 \in A_1$, uniformly in $x_2 \in \mathbb{R}^{n_2}$;*

and (c) and (d) in Definition 3.2 hold.

The outer semicontinuity assumption in (b) of Definition 4.2 means that for every $x_1 \in A_1$ and every $\varepsilon > 0$ there exists $\delta > 0$ so that for every $x_2 \in \mathbb{R}^{n_2}$, $W(x_1 + \delta\mathbb{B}, x_2) \subset W(x_1, x_2) + \varepsilon\mathbb{B}$. As expected, if W is a partial weak set-valued Lyapunov function, as defined above, then for every $\phi \in \mathcal{S}(U_1 \times \mathbb{R}^{n_2})$,

$$(18) \quad \phi_1(t, j) \in W(\phi(t, j)) \subset W(\phi(0, 0)).$$

THEOREM 4.3. *If there exist a partial weak set-valued Lyapunov function $W : U_1 \times \mathbb{R}^{n_2} \rightrightarrows \mathbb{R}^{n_1}$ for (1) and the set A , then every point $a \in A$ is partially Lyapunov stable.*

⁴In [33], [34], Zeno equilibria consisting of $\bigcup(q_i, \bar{z}_i)$, with different equilibrium states \bar{z}_i for different logical modes, were studied, without stability considerations. The question whether partial stability ideas can extend to such situations is not pursued here.

Proof. If $A_1 = \emptyset$, there is nothing to prove. Otherwise, pick $a = (a_1, x_2) \in A$, so that $a_1 \in A_1$, and $\varepsilon > 0$. Note that by (a) of Definition 4.2, $W(a) = \{a_1\}$, and by (b) of Definition 4.2, there exists $\delta > 0$ such that for every $x_1 \in a_1 + \delta\mathbb{B}$, every $x_2 \in \mathbb{R}^{n_2}$, $W(x_1, x_2) \subset W(a_1, x_2) + \varepsilon\mathbb{B} = a_1 + \varepsilon\mathbb{B}$. If δ is small enough so that $a_1 + \delta\mathbb{B} \subset U_1$, then for any $\phi \in \mathcal{S}(a + \delta\mathbb{B}, x_2)$, thanks to (18), one has

$$\phi_1(t, j) \in W(\phi(0, 0)) \subset W(a + \delta\mathbb{B}) \subset a_1 + \varepsilon\mathbb{B}$$

for every $(t, j) \in \text{dom } \phi$. This verifies Lyapunov stability of a . \square

DEFINITION 4.4. A set-valued mapping $W : U_1 \times \mathbb{R}^{n_2} \rightrightarrows \mathbb{R}^n$ is a partial set-valued Lyapunov function for (1) and a set A if it is a partial weak set-valued Lyapunov function for (1) and the set A and there exist continuous functions $c, d : U_1 \rightarrow [0, \infty)$, positive definite with respect to A_1 on $(\overline{C \cup D}) \cap U$, where $U = U_1 \times \mathbb{R}^{n_2}$, such that

(c) for every solution $\phi : [0, T] \rightarrow U$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,

$$(19) \quad W(\phi(t)) + \int_0^t c(\phi_1(s)) ds\mathbb{B} \subset W(\phi(0)) \quad \forall t \in [0, T];$$

(d)

$$(20) \quad W(G(x)) + d(x_1)\mathbb{B} \subset W(x) \quad \forall x \in D \cap U,$$

and either $\overline{W(x)} \subset U$ for every $x \in (C \cup D) \cap U$ or the functions c and d are positive definite with respect to A on $(\overline{C \cup D}) \cap \overline{U}$.

THEOREM 4.5. If there exist a partial set-valued Lyapunov function $W : U_1 \times \mathbb{R}^{n_2} \rightrightarrows \mathbb{R}^n$ for (1) and the set A , then A is pPAS and $U_1 \times \mathbb{R}^{n_2} \subset p\mathcal{B}(A)$.

The proof is similar to that of Theorem 3.7 and is left to the reader.

THEOREM 4.6. Suppose that (1) is nominally well-posed, that there exist a continuous partial weak set-valued Lyapunov function $W : U_1 \rightrightarrows \mathbb{R}^{n_1}$ for (1) and the set A , and that every complete solution to (1) on which W is constant is contained in A . Then, for every $\phi \in \mathcal{S}(U_1 \times \mathbb{R}^{n_2})$, ϕ_1 is bounded, and if ϕ is also complete and ϕ_2 is bounded, then $\lim_{t+j \rightarrow \infty} \phi_1(t, j)$ exists and belongs to A_1 . In particular, if ϕ_2 is bounded for every $\phi \in \mathcal{S}(U_1 \times \mathbb{R}^{n_2})$, then A is pPAS and $U_1 \times \mathbb{R}^{n_2} \subset p\mathcal{B}(A)$:

Proof. Partial Lyapunov stability of every $a \in A$ follows from Theorem 4.3. Take $\phi \in \mathcal{S}(U_1 \times \mathbb{R}^{n_2})$. By (18), $\text{rge } \phi_1 \subset W(\phi(0, 0))$, and since W has bounded values, ϕ_1 is bounded. Hence, if ϕ_2 is also bounded, ϕ is bounded. Suppose that ϕ is complete. Then its Ω -limit $\omega(\phi)$, defined by

$$\omega(\phi) = \left\{ x \in \mathbb{R}^n \mid \exists (t_k, j_k) \in \text{dom } \phi, t_k + j_k \rightarrow \infty \text{ as } k \rightarrow \infty \text{ so that } \lim_{k \rightarrow \infty} \phi(t_k, j_k) = x \right\},$$

is nonempty and compact, and since (1) is nominally well-posed, it is viable/weakly invariant; see [16, Proposition 6.21]. Let

$$K = \bigcap_{(t, j) \in \text{dom } \phi} \overline{W(\phi(t, j))}.$$

Since the sets $W(\phi(t, j))$ are nonincreasing as $(t, j) \in \text{dom } \phi$ increases, for every sequence $(t_k, j_k) \in \text{dom } \phi$ such that $t_k + j_k \rightarrow \infty$ as $k \rightarrow \infty$, $\lim_{k \rightarrow \infty} W(\phi(t_k, j_k))$

exists and equals $\bigcap_{k=1}^{\infty} \overline{W(\phi(t_k, j_k))}$, by [42, Exercise 4.2]. It easily follows that $\lim_{k \rightarrow \infty} W(\phi(t_k, j_k)) = K$. Because W is continuous, it follows that for every $x \in \omega(\phi)$, $W(x) = K$. In other words, W is constant on $\omega(\phi)$, and, since viability/weak invariance can be verified by complete solutions, $\omega(\phi) \subset A$. Take any $x = (x_1, x_2) \in \omega(\phi) \subset A = A_1 \times \mathbb{R}^{n_2}$ and $\varepsilon > 0$, and choose $\delta > 0$ based on partial Lyapunov stability of x . By the definition of $\omega(\phi)$, there exists $(T, J) \in \text{dom } \phi$ such that $\phi_1(T, J) \in x_1 + \delta \mathbb{B}$. Then $\phi_1(t, j) \in x_1 + \varepsilon \mathbb{B}$ for every $(t, j) \in \text{dom } \phi$ with $t + j \geq T + J$. As $\varepsilon > 0$ is arbitrary, this implies that $\lim_{t+j \rightarrow \infty} \phi_1(t, j)$ exists and equals $x_1 \in A_1$. \square

Example 4.7. For the hybrid system in Example 2.1, the set

$$(21) \quad A = A_1 \times \mathbb{R}, \quad \text{where } A_1 = \{x_1 \in \mathbb{R}^{(I+1)l} \mid z_1 = z_2 = \dots = z_I = a\},$$

is pPAS. Recall $w(z, a) = \text{con}\{z_1, z_2, \dots, z_I, a\}$ discussed in Example 3.4 and define

$$(22) \quad W(x) = W(x_1) = \underbrace{w(z, a) \times w(z, a) \times \dots \times w(z, a)}_{I+1 \text{ copies}}.$$

It is easy to check that W is a weak partial set-valued Lyapunov function for A ; in fact the weak decrease of W follows from Example 3.4. Hence, Theorem 4.3 implies partial Lyapunov stability of A . Furthermore, the data of the associated hybrid inclusion satisfies the hybrid basic conditions, as defined in Example 2.1 and noted below Example 2.6, so the system is nominally well-posed and W is continuous. To conclude partial PAS of A from Theorem 4.6 one only needs to argue that if W is constant along a solution, then the solution is contained in A_1 , which is the case.

4.2. Partial small ordinary time property. Definition 4.8 extends Definition 3.14 and a sufficient condition for the defined property is given in Proposition 4.9.

DEFINITION 4.8. *The set $A \subset \mathbb{R}^n$ is partially pointwise small ordinary time asymptotically stable (pPSOTAS) if it is pPAS and*

- (a) *for every $a_1 \in A_1$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_t \text{dom } \phi < \varepsilon$ for every $\phi \in \mathcal{S}((a_1 + \delta \mathbb{B}) \times \mathbb{R}^{n_2})$.*

The following sufficient condition for pPSOTAS is inspired by [17, Proposition 3.2].

PROPOSITION 4.9. *Suppose that a closed set $A \subset \mathbb{R}^n$ is pPAS for (1) and there exists a continuously differentiable $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $x_1 \mapsto V(x_1, x_2)$ is continuous in x_1 uniformly in x_2 and*

- (a) *V is positive definite with respect to A ;*
- (b) *there exist $c > 0$ and $\rho \in [0, 1)$ such that*

$$\nabla V(x) \cdot f \leq -c(V(x))^\rho \quad \forall x \in C, f \in F(x);$$

- (c) *$V(g) \leq V(x)$ for all $x \in D, g \in G(x)$;*
and there exist no nontrivial flowing solutions ϕ with $\text{rge} \phi_1 \subset A_1$. Then A is pPSOTAS.

Proof. For every $\phi \in \mathcal{S}$, every $(t, j) \in \text{dom } \phi$, thanks to (b) and (c) one has

$$V(\phi(t, j)) \leq (V(\phi(0, 0))^{1-\rho} - c(1-\rho)t)^{\frac{1}{1-\rho}},$$

as long as $t \leq \frac{1}{c(1-\rho)}V(\phi(0,0))^{1-\rho}$ and $V(\phi(t,j)) = 0$ if $t \geq \frac{1}{c(1-\rho)}V(\phi(0,0))^{1-\rho}$. Because $V(x) = 0$ implies $x \in A$ and because there are no nontrivial flowing solutions in A ,

$$\sup_t \text{dom } \phi \leq \frac{1}{c(1-\rho)}V(\phi(0,0))^{1-\rho}$$

for every solution. Pick $a \in A$ and $\varepsilon > 0$. Using continuity of V that is uniform in x_2 , find $\delta > 0$ so that $\frac{1}{c(1-\rho)}V(x)^{1-\rho} < \varepsilon$ when $x_1 \in a_1 + \delta\mathbb{B}$. Then $\sup_t \text{dom } \phi < \varepsilon$ for every $\phi \in \mathcal{S}((a_1 + \delta\mathbb{B}) \times \mathbb{R}^{n_2})$, and the partial SOT property of every $a \in A$ is proven. \square

Example 4.10. For the hybrid system in Example 2.6, the set (21) is partially Lyapunov stable. In fact, (22) is a weak partial set-valued Lyapunov function for A in this system as well, and arguments similar to those in Example 4.7 apply.

For this system, A also turns out to have the partial SOT property. Consider

$$V(x) = \frac{1}{2} \sum_{i=1}^I \|z_i - a\|^2.$$

Clearly, it is positive definite with respect to A in (21) and continuously differentiable, continuous in x_1 uniformly in x_2 (as it does not depend on $x_2 = \tau$). Furthermore, $a^+ = \frac{1}{I}(z_1 + z_2 + \dots + z_I)$ is the unique minimizer of $y \mapsto \frac{1}{2} \sum_{i=1}^I \|z_i - y\|^2$, and because $z_i^+ = z_i$, V does not increase during jumps, i.e., (c) of Proposition 4.9 holds. During flows, for x such that $z_i \neq a$ for each i , with

$$f = \left(c_1 \frac{a - z_1}{\sqrt{\|a - z_1\|}}, c_2 \frac{a - z_2}{\sqrt{\|a - z_2\|}}, \dots, c_I \frac{a - z_I}{\sqrt{\|a - z_I\|}}, 0, -1 \right),$$

one obtains

$$\begin{aligned} \nabla V(x) \cdot f &= \sum_{i=1}^I (z_i - a) c_i \frac{a - z_i}{\sqrt{\|a - z_i\|}} \leq - \min_{i=1,2,\dots,I} c_i \sum_{i=1}^I \|a - z_i\|^{3/2} \\ &\leq - \min_{i=1,2,\dots,I} c_i \left(\sum_{i=1}^I \|a - z_i\|^2 \right)^{3/4} = -2^{3/4} \min_{i=1,2,\dots,I} c_i (V(x))^{3/4}, \end{aligned}$$

where the second inequality holds because the $l_{2/3}$ “norm” is greater than or equal to the l_2 norm. Proposition 4.9 finishes the argument.

Finally, for the case of two agents, $I = 2$, the set (21) is pPAS. The trouble with $I > 2$ is that $T(z_1, \dots, z_n, a^+)$ may equal 0 for states not in A , leading to instantaneous Zeno solutions along which W remains constant. For the case of two agents, Theorem 4.6 can be applied.

4.3. Consequences of PAS in well-posed systems. The key technical results, extending those in Theorem 3.15, are now stated and proven.

THEOREM 4.11. *Suppose that (1) is nominally well-posed and forward complete. For every sequence $\phi_i = (\phi_{1,i}, \phi_{2,i}) \in \mathcal{S}$ with $\phi_i(0,0)$ convergent, there exists a graphically convergent subsequence, which is not relabeled, such that*

- (a) *the graphical limit ϕ of the graphically convergent subsequence ϕ_i is a complete solution to (1).*

If, additionally, A is pPAS and $\lim_{i \rightarrow \infty} \phi_i(0,0) \in p\mathcal{B}(A)$, then

- (b) *for all large enough i , $\lim_{t+j \rightarrow \infty} \phi_{1,i}(t,j)$ exists and belongs to A_1 ;*
(c) *$\lim_{i \rightarrow \infty} \lim_{t+j \rightarrow \infty} \phi_{1,i}(t,j) = \lim_{t+j \rightarrow \infty} \phi_1(t,j)$;*

- (d) convergence of $\phi_{1,i}$ to ϕ_1 is uniform in the following sense: for every $\varepsilon > 0$ there exists $\tau > 0$ such that, for every large enough i , $\phi_{1,i}$ and ϕ_1 are ε -close to τ -truncations of one another.

If, additionally, A is pPSOTAS, then

- (e) ϕ and ϕ_i for all large enough i are Zeno, and $\lim_{i \rightarrow \infty} \sup_t \text{dom } \phi_i = \sup_t \text{dom } \phi$;
(f) convergence of $\phi_{1,i}$ to ϕ_1 is uniform in the following sense: for every $\varepsilon > 0$ and every large enough i , $\phi_{1,i}$ and ϕ_1 are ε -close to one another.

Proof. Take a sequence of solutions $\phi_i \in \mathcal{S}$ with $\phi_i(0,0)$ convergent. By [42, Theorem 5.36], restated for hybrid systems in [16, Theorem 6.1], there exists a graphically convergent subsequence. From now on, let ϕ_i be that subsequence. As (1) is forward complete, it does not have solutions that are unbounded, hence maximal, but not complete. Then (a) is essentially a restatement of nominal well-posedness.

In what follows, let A be pPAS. If $\phi(0,0) = \lim_{i \rightarrow \infty} \phi_i(0,0) \in p\mathcal{B}(A)$, then the limit $a_1 := \lim_{t+j \rightarrow \infty} \phi_1(t,j)$ exists and belongs to A_1 . Thus, for some $(t_0, j_0) \in \text{dom } \phi$, $\phi_1(t_0, j_0) \in U_1$, where U_1 is an open neighborhood of A_1 as in the definition of partial PAS, Definition 4.1(b). Graphical convergence of ϕ_i to ϕ implies, in particular, that there exist $(t_i, j_i) \in \text{dom } \phi_i$ such that $(t_i, j_i) \rightarrow (t_0, j_0)$ and $\phi_{1,i}(t_i, j_i) \rightarrow \phi_1(t_0, j_0)$ as $i \rightarrow \infty$. Then, for all large enough i , $\phi_{1,i}(t_i, j_i) \in U_1$. Because tails of solutions ϕ_i , i.e., the hybrid arcs $(t, j) \rightarrow \phi_i(t+t_i, j+j_i)$, are solutions, the properties of U_1 ensure that, for all large enough i , $\lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j)$ exist and belong to A_1 . This verifies (b).

Pick $\varepsilon > 0$ and let $a_1 \in A_1$ be as in the paragraph above. Pick $\delta > 0$ using partial Lyapunov stability, small enough to satisfy $\delta < \varepsilon/2$ and $a_1 + \delta\mathbb{B} \subset U_1$, so that $\text{rge}\psi_1 \subset a_1 + \varepsilon/2\mathbb{B}$ for every $\psi \in \mathcal{S}(a_1 + \delta\mathbb{B}, \mathbb{R}^{n_2})$. The time (t_0, j_0) in the paragraph above can be picked so that $\phi_1(t_0, j_0) \in a_1 + \delta/2\mathbb{B}$. Then, for all large enough i , $\phi_{1,i}(t_i, j_i) \in a_1 + \delta\mathbb{B}$ and hence $\phi_{1,i}(t, j) \in a_1 + \varepsilon\mathbb{B}$ for all $(t, j) \in \text{dom } \phi_i$, $t+j > t_i + j_i$. In particular, for all large enough i , $\lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j) \in a_1 + \varepsilon\mathbb{B}$. As $\varepsilon > 0$ is arbitrary, this means that $\lim_{i \rightarrow \infty} \lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j) = a_1$. This verifies (c).

Furthermore, with (t_0, j_0) as in the paragraph above, let $\tau = t_0 + j_0 + 1$. By Theorem 2.8, $\phi_{1,i}$ and ϕ_1 are (τ, ε) -close for all large enough i . In particular, for all large enough i there exist $(t_i, j_i) \in \text{dom } \phi_i$, $(t_i, j_i) \rightarrow (t_0, j_0)$, and hence $t_i + j_i < \tau$, so that $\phi_{1,i}(t_i, j_i) \in a_1 + \delta\mathbb{B}$. By choice of δ , for all large enough i , for all $(t, j) \in \text{dom } \phi_i$ with $t+j \geq \tau$, $\phi_{1,i}(t, j) \in a_1 + \varepsilon/2\mathbb{B}$. Hence, for each such (t, j) , $\phi_{1,i}(t, j) \in \phi_1(t_0, j_0) + \varepsilon\mathbb{B}$. Similarly, for all $(t, j) \in \text{dom } \phi$ with $t+j \geq \tau$, $\phi_1(t, j) \in \phi_{1,i}(t_i, j_i) + \varepsilon\mathbb{B}$. Combined with the (τ, ε) -closeness established before, one obtains that for large enough i , $\phi_{1,i}$ and ϕ_1 are ε -close to τ -truncations of one another. This verifies (d).

In what follows assume, additionally, that a_1 has the SOT property (a) in Definition 4.8. Clearly, ϕ and, for large enough i , ϕ_i are Zeno. Take $\delta > 0$ as two paragraphs above and, by using the SOT property, so that also $\sup_t \text{dom } \psi < \varepsilon/2$ for every $\psi \in \mathcal{S}(a_1 + \delta\mathbb{B}, \mathbb{R}^{n_2})$. With $(t_0, j_0) \in \text{dom } \phi$ as before, $\phi(t_0, j_0) \in a_1 + \delta/2\mathbb{B}$, and hence $t_0 \leq \sup_t \text{dom } \phi \leq t_0 + \varepsilon/2$. Similarly, for large enough i , $|t_0 - t_i| < \varepsilon/2$, $j_i = j_0$, $\phi_{1,i}(t_i, j_i) \in a_1 + \delta\mathbb{B}$, and hence $t_i \leq \sup_t \text{dom } \phi_i \leq t_i + \varepsilon/2$. Then

$$\sup_t \text{dom } \phi_i - \varepsilon \leq t_i - \varepsilon/2 < t_0 \leq \sup_t \text{dom } \phi \leq t_0 + \varepsilon/2 < t_i + \varepsilon/2 + \varepsilon/2 \leq \sup_t \text{dom } \phi_i + \varepsilon$$

which implies that $\sup_t \text{dom } \phi_i$ converge to $\sup_t \text{dom } \phi$. This proves (e).

Finally, with all notions already defined, note that $\phi_1(t, j) \in a_1 + \varepsilon\mathbb{B}$ for $(t, j) \in \text{dom } \phi$ with $t+j \geq \tau$ and for all large enough i , $\phi_{1,i}(t, j) \in a_1 + \varepsilon\mathbb{B}$ for all $(t, j) \in \text{dom } \phi_i$, $t+j \geq \tau > t_i + j_i$. Hence $\phi_1(t, j)$ and $\phi_{1,i}(t', j')$ are within 2ε of one another for large enough i and $t+j \geq \tau$, $t'+j' \geq \tau$. Furthermore, $(t, j) \in \text{dom } \phi$ with $t+j \geq \tau$ implies

$0 \leq \sup_t \text{dom } \phi - t < \varepsilon/2$; for all large enough i , $(t', j') \in \text{dom } \phi_i$ with $t' + j' \geq \tau$ implies $0 \leq \sup_t \text{dom } \phi_i - t_i < \varepsilon/2$; and since $\sup_t \text{dom } \phi_i$ converge to $\sup_t \text{dom } \phi$, one has $|t - t'| < 2\varepsilon$ for $(t, j) \in \text{dom } \phi$ with $t + j \geq \tau$ and $(t', j') \in \text{dom } \phi_i$ with $t' + j' \geq \tau$. This implies that ϕ_1 and $\phi_{1,i}$ are 2ε -close to one another for large i , since ϕ and ϕ_i are Zeno, and since ϕ_1 and $\phi_{1,i}$ are (τ, ε) -close (so also $(\tau, 2\varepsilon)$ -close) to one another. As $\varepsilon > 0$ is arbitrary, this proves (f). \square

COROLLARY 4.12. *Suppose that (1) is nominally well-posed and forward complete and A is pPAS. Then,*

- (a) *the basin of partial pointwise attraction of A , $p\mathcal{B}(A)$, is an open neighborhood of A ;*
- (b) *the set-valued mapping $\mathcal{L}_1 : p\mathcal{B}(A) \rightrightarrows \mathbb{R}^{n_1}$ defined by*

$$(23) \quad \mathcal{L}_1(x) = \left\{ \lim_{t+j \rightarrow \infty} \phi_1(t, j) \mid \phi \in \mathcal{S}(x) \right\}$$

is outer semicontinuous and locally bounded;

- (c) *the set-valued mapping $\overline{\mathcal{R}}_{\infty,1} : p\mathcal{B}(A) \rightrightarrows \mathbb{R}^{n_1}$ defined by*

$$(24) \quad \overline{\mathcal{R}}_{\infty,1}(x) = \overline{\mathcal{R}_{\infty,1}(x)}$$

is outer semicontinuous and locally bounded and

$$(25) \quad \overline{\mathcal{R}}_{\infty,1}(x) = \mathcal{R}_{\infty,1}(x) \cup \mathcal{L}_1(x)$$

for every $x \in p\mathcal{B}(A)$, where the set-valued mapping $\mathcal{R}_{\infty,1} : p\mathcal{B}(A) \rightrightarrows \mathbb{R}^{n_1}$ is defined by

$$\mathcal{R}_{\infty,1}(x) = \{\text{rge } \phi_1 \mid \phi \in \mathcal{S}(x)\}.$$

If, additionally, A is pPSOTAS, then

- (d) *the function $\text{Length}_t : p\mathcal{B}(A) \rightarrow [0, \infty]$ defined by (17) is upper semicontinuous, locally bounded (in particular, finite-valued), and the sup defining $\text{Length}_t(x)$ is attained for every $x \in p\mathcal{B}(A)$ by some $\phi \in \mathcal{S}(x)$.*

The mappings \mathcal{L}_1 in (23) and $\overline{\mathcal{R}}_{\infty}$ in (24) can be viewed as mappings from \mathbb{R}^n , not just from $\mathcal{B}(A)$. In such a case, $\mathcal{L}_1(x)$ and $\overline{\mathcal{R}}_{\infty}(x)$ are considered to be empty for $x \notin \mathcal{B}(A)$.

Proof. Since $U_1 \times \mathbb{R}^{n_2}$ is a neighborhood of A , and $U_1 \times \mathbb{R}^{n_2} \subset p\mathcal{B}(A)$, $p\mathcal{B}(A)$ is a neighborhood of A . Suppose that $p\mathcal{B}(A)$ is not open: there exist $x_i \notin p\mathcal{B}(A)$, $x_i \rightarrow x \in p\mathcal{B}(A)$ as $i \rightarrow \infty$. Because $x_i \notin \mathcal{B}(A)$, there exist $\phi_i \in \mathcal{S}(x_i)$ so that either each $\phi_{1,i}$ is not bounded, or it does not converge, or it converges and the limit is not in A_1 . Applying Theorem 4.11 to the sequence ϕ_i , in particular conclusion (b), leads to a contradiction. Hence, (a) is proven.

Similarly, if \mathcal{L}_1 is not locally bounded, then there exist $\phi_i \in \mathcal{S}(x_i)$ with

$$\left\| \lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j) \right\| \rightarrow \infty,$$

as $i \rightarrow \infty$ while, without loss of generality, $x_i \rightarrow x \in \mathcal{B}(A)$. Theorem 4.11, in particular conclusion (c), leads to a contradiction. Hence \mathcal{L}_1 is locally bounded. To see that \mathcal{L}_1 is outer semicontinuous, take $x_i, x \in \mathcal{B}(A)$ with $x_i \rightarrow x$ as $i \rightarrow \infty$. Pick $y_i \in \mathcal{L}_1(x_i)$, so that $y_i = \lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j)$ for some $\phi_i \in \mathcal{S}(x_i)$, and suppose that $y_i \rightarrow y$. Then (c) of Theorem 4.11 implies that $y = \lim_{t+j \rightarrow \infty} \phi_1(t, j)$ for some $\phi \in \mathcal{S}(x)$ and hence $y \in \mathcal{L}_1(x)$. This proves outer semicontinuity of \mathcal{L}_1 and hence finishes the proof of (b).

Local boundedness of $\mathcal{R}_{\infty,1} : p\mathcal{B}(A) \rightrightarrows \mathbb{R}^{n_1}$ is proven similarly to how that property was proven for \mathcal{L}_1 , with the (d) of Theorem 4.11 replacing (c). Hence

$\overline{\mathcal{R}}_{\infty,1} : p\mathcal{B}(A) \rightrightarrows \mathbb{R}^{n_1}$ is locally bounded as well. For outer semicontinuity, it is enough to consider $x_i \rightarrow x \in p\mathcal{B}(A)$, $y_i \in \mathcal{R}_{\infty,1}(x_i)$ convergent to some $y \in \mathbb{R}^{n_2}$, and argue that $y \in \overline{\mathcal{R}}_{\infty,1}(x)$. Indeed, points in $\overline{\mathcal{R}}_{\infty,1}(x_i)$ can be approximated by points in $\mathcal{R}_{\infty,1}(x_i)$. This will in fact prove (25) along the way. For x_i , x , y_i , and y as considered, there exist $\phi_i \in \mathcal{S}(x_i)$ and $(t_i, j_i) \in \text{dom } \phi_i$ so that $y_i = \phi_{1,i}(t_i, j_i)$. By (a) of Theorem 4.11, without loss of generality one can assume that ϕ_i graphically converge to $\phi \in \mathcal{S}(x)$. If $t_i + j_i$ do not diverge to ∞ as $i \rightarrow \infty$, without loss of generality one can further assume that (t_i, j_i) converge, say, to (t, j) . Graphical convergence implies that $\phi_i(t_i, j_i) \rightarrow \phi(t, j)$, in particular $\phi_{1,i}(t_i, j_i) \rightarrow \phi_1(t, j)$. Then, $y_i \rightarrow y$ implies that $y = \phi_1(t, j)$, and hence $y \in \mathcal{R}_{\infty,1}(x) \subset \overline{\mathcal{R}}_{\infty,1}(x)$. If $t_i + j_i \rightarrow \infty$ as $i \rightarrow \infty$, then arguments as in the proof of (d) of Theorem 4.11 show that $\phi_{1,i}(t_i, j_i) \rightarrow \lim_{t+j \rightarrow \infty} \phi_1(t, j)$. Indeed, for every $\varepsilon > 0$, all large enough i , and all large enough $t + j$, $\phi_{1,i}(t, j) \in a_1 + \varepsilon/2\mathbb{B}$, where $a_1 = \lim_{t+j \rightarrow \infty} \phi_1(t, j)$, which verifies the claim. In this case, $y = \lim_{t+j \rightarrow \infty} \phi_1(t, j)$ and so $y \in \mathcal{L}_1(x) \subset \overline{\mathcal{R}}_{\infty,1}(x)$. The proof of outer semicontinuity is complete, and (25) follows by considering $x_i = x$ in the above argument.

Under the additional SOT property, local boundedness of Length_t is proven similarly to what was done above for \mathcal{L}_1 , with the use of Theorem 3.15(e). Let $x_i \rightarrow x \in p\mathcal{B}(A)$ and $\phi_i \in \mathcal{S}(x_i)$ be such that $\sup_t \text{dom } \phi_i > \text{Length}_t(x_i) - 1/i$. Theorem 4.11(e) applied to ϕ_i yields $\phi \in \mathcal{S}(x)$ such that $\sup_t \text{dom } \phi \geq \limsup_{i \rightarrow \infty} \text{Length}_t(x_i)$. Considering $x_i = x$ implies that the supremum defining $\text{Length}_t(x)$ is attained (by ϕ). Considering the general case verifies upper semicontinuity, because $\text{Length}_t(x) \geq \sup_t \text{dom } \phi$. \square

5. Past Zeno. This section illustrates how the results established earlier, about the structure of solution sets under PAS, can be used to extend Zeno solutions to a hybrid inclusion past their Zeno times and maintain reasonable dependence of the extended solutions on initial conditions. Two scenarios are considered. The first scenario involves two hybrid systems. One has a pPSOTAS set, and the Zeno solutions from its basin of attraction are formally extended using the other hybrid system, with initial conditions of the extensions dependent on the limits of Zeno solutions. The second scenario involves one hybrid system with a PSOTAS set, where the Zeno solutions from its basin of attraction are “restarted” after their Zeno times in that basin, and the process is repeated infinitely many times.

For both scenarios, results related to well-posedness of the extended solutions are given as well as some sample results from stability theory for the scenarios. This small collection of the results is not meant to be comprehensive but should illustrate what is possible in the presence of PAS of the set of equilibria.

5.1. Scenario 1: Past Zeno, with pPAS, to a compact attractor. In this section, the hybrid system with data (C_1, F_1, G_1, D_1) is the system with complete solutions $\phi = (\phi_1, \phi_2)$ that are Zeno and the state components in ϕ_1 converge to the set A . The hybrid system with data (C_2, F_2, G_2, D_2) describes post-Zeno behavior. Its solutions, denoted ψ , are initialized from points in the limit of ϕ_1 passed through the set-valued map Ψ , namely, from $\Psi(\lim_{t+j \rightarrow \infty} \phi_1(t, j))$. In this way, the mapping Ψ introduces some degree of control on the points from where solutions are continued after Zeno. Throughout this section, the following assumption is in place.

ASSUMPTION 5.1.

- (C_1, F_1, G_1, D_1) is a nominally well-posed and forward complete hybrid system in \mathbb{R}^m ;

- A is a closed p PSOTAS set for (C_1, F_1, G_1, D_1) ;
- (C_2, F_2, G_2, D_2) is a nominally well-posed and preforward complete hybrid system in \mathbb{R}^n ;
- $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous and locally bounded set-valued mapping.

As before, it is considered that $A = A_1 \times \mathbb{R}^{m_2}$, where $A_1 \subset \mathbb{R}^{m_1}$ and $m_1 + m_2 = m$. Let $p\mathcal{B}_1(A) \subset \mathbb{R}^m$ be the basin of partial pointwise attraction of A for (C_1, F_1, G_1, D_1) .

Definition 5.2 combines the solutions ϕ to (C_1, F_1, G_1, D_1) and ψ to (C_2, F_2, G_2, D_2) to define the concept of solution to the pair $((C_1, F_1, G_1, D_1), (C_2, F_2, G_2, D_2))$ for going beyond Zeno in system (C_1, F_1, G_1, D_1) .

DEFINITION 5.2. *A solution to the pair of systems $((C_1, F_1, G_1, D_1), (C_2, F_2, G_2, D_2))$ is a pair (ϕ, ψ) such that*

- ϕ is a maximal/complete solution to (C_1, F_1, G_1, D_1) ;
- $\phi(0, 0) \in p\mathcal{B}_1(A)$
- $\psi(T, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi_1(t, j))$, where $T = \sup_t \text{dom } \phi$;
- ψ is a solution to (C_2, F_2, G_2, D_2) defined on a hybrid time domain from $(T, 0)$.

Above, a *hybrid time domain from $(T, 0)$* is a set S such that $S - (T, 0)$ is a hybrid time domain in the usual sense (in other words, a hybrid time domain from $(0, 0)$). Similarly, a *solution to (C_2, F_2, G_2, D_2) defined on a hybrid time domain from $(T, 0)$* is a function ψ such that $(t, k) \mapsto \psi(t + T, k)$ is a solution to (C_2, F_2, G_2, D_2) on a hybrid time domain $S - (T, 0)$. In what follows, \mathcal{S} is the set of solutions as defined above, \mathcal{S}_1 is the set of solutions to (C_1, F_1, G_1, D_1) , and \mathcal{S}_2 is the set of solutions to (C_2, F_2, G_2, D_2) . $\mathcal{S}(x)$ is the set of solutions (ϕ, ψ) as above such that $\phi(0, 0) = x$.

The first result below, and its corollary, extend in a sense the concept of nominal well-posedness to solutions to the pair $((C_1, F_1, G_1, D_1), (C_2, F_2, G_2, D_2))$. Then, a compact attractor for the second system, (C_2, F_2, G_2, D_2) , is considered.

THEOREM 5.3. *Suppose Assumption 5.1 holds. Let $(\phi_i, \psi_i) \in \mathcal{S}(x_i)$ with $x_i \rightarrow x \in p\mathcal{B}_1(A)$. Then, there exists a graphically convergent subsequence of (ϕ_i, ψ_i) , the limit (ϕ, ψ) of which is a solution in $\mathcal{S}(x)$. Furthermore, for every $\varepsilon > 0$, $\tau > 0$ and all large enough i , ψ_i and ψ are (τ, ε) -close, in the sense of Definition 2.7(a).*

Proof. Theorem 4.11 yields a graphically convergent subsequence of ϕ_i , not relabeled, the graphical limit ϕ of which satisfies $\phi_i \in \mathcal{S}(x)$, and such that for all large enough i , $\lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j)$ exists and belongs to A_1 ; $\lim_{i \rightarrow \infty} \lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j) = \lim_{t+j \rightarrow \infty} \phi_1(t, j)$; and ϕ and ϕ_i for all large enough i are Zeno, and $\lim_{i \rightarrow \infty} T_i = T$, where $T_i := \sup_t \text{dom } \phi_i$, $T := \sup_t \text{dom } \phi$.

Because $\psi_i(T_i, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi_{1,i}(t, j))$ and Ψ is locally bounded, the sequence $\psi_i(T_i, 0)$ is bounded and, without loss of generality, it can be assumed to converge. By Theorem 2.2, a further graphically convergent subsequence of $(t, k) \mapsto \psi_i(t + T_i, k)$ can be picked, the limit $(t, k) \mapsto \psi(t + T, k)$ of which is, by nominal well-posedness, a solution to (C_2, F_2, G_2, D_2) . Then $\text{dom } \psi$ starts at $(T, 0)$ and $\lim_{i \rightarrow \infty} \psi_i(T_i, 0) = \psi(T, 0)$. Outer semicontinuity of Ψ implies that $\psi(T, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi_1(t, j))$. Hence, the pair (ϕ, ψ) is a solution in $\mathcal{S}(x)$. Because $(t, k) \mapsto \psi_i(t + T_i, k)$ graphically converge to $(t, k) \mapsto \psi(t + T, k)$ and because $T_i \rightarrow T$, ψ_i graphically converge to ψ .

Preforward completeness of (C_2, F_2, G_2, D_2) ensures, through nominal well-posedness, that the graphically convergent sequence $(t, k) \mapsto \psi_i(t + T_i, k)$ is locally eventually bounded. In that case, graphical convergence is equivalent [16, Theorem 5.25] to $(\tau, \varepsilon/2)$ -closeness of $(t, k) \mapsto \psi_i(t + T_i, k)$ and $(t, k) \mapsto \psi(t + T, k)$,

for large enough i . Also for large enough i , $|T - T_i| < \varepsilon/2$. This implies the claim about (τ, ε) -closeness of ψ_i and ψ . \square

COROLLARY 5.4. *Suppose Assumption 5.1 holds. For every compact set $K \subset p\mathcal{B}_1(A)$ and every $\tau > 0$, $\varepsilon > 0$, there exist $\delta > 0$ with the following property: for every $(\phi, \psi) \in \mathcal{S}(K + \delta\mathbb{B})$ there exists $(\phi', \psi') \in \mathcal{S}(K)$ such that ψ and ψ' are (τ, ε) -close.*

Proof. Suppose that the conclusion fails. Then, there exist a compact $K \subset p\mathcal{B}_1(A)$, $\tau > 0$, $\varepsilon > 0$ and, for each $i \in \mathbb{N}$, solutions $(\phi_i, \psi_i) \in \mathcal{S}(K + 1/i\mathbb{B})$ such that either (i) or (ii) in Definition 2.7 (a) fails for (ϕ_i, ψ_i) and every $(\phi, \psi) \in \mathcal{S}(K)$. Without loss of generality, suppose $\phi_i(0, 0)$ converge to a point in K and, using Theorem 5.3, that (ϕ_i, ψ_i) graphically converge to some $(\phi, \psi) \in \mathcal{S}(K)$ and that for large enough i , ψ_i and ψ are (τ, ε) -close. This is a contradiction. \square

Example 5.5. The second example informally discussed in the introduction was written as a hybrid inclusion in Example 2.6, where it was noted that the hybrid basic conditions are satisfied. In Example 4.10 it was shown that the set (21) is PSOTAS for that system, when two agents z_1, z_2 are considered. Let (C_1, F_1, D_1, G_1) be that system and A be the set (21). Recall that, for that system, $x = (x_1, x_2)$ with $x_1 = (z_1, z_2, a)$, $x_2 = \tau$, and let $\Psi : \mathbb{R}^{m_1} \rightarrow \mathbb{R}^n$ be given by $\Psi(x_1) = a$. Let (C_2, F_2, D_2, G_2) have $C_2 = \mathbb{R}^n$ and empty D_2 , so it represents an unconstrained continuous-time system, and let F_2 be a Lipschitz continuous function. This can be thought of as agents reaching consensus and then performing a task encoded by F_2 together. Let (ϕ, ψ) be a solution, with ψ maximal and hence complete, and let $T = \sup_t \text{dom } \phi$. Corollary 5.4, applied with $K = \{\phi(0, 0)\}$ and $\tau > T$, yields for any $\varepsilon > 0$ a $\delta > 0$ so that any maximal solution (ϕ', ψ') with $\|\phi'(0, 0) - \phi(0, 0)\| < \delta$ satisfies $\|\psi'(\tau', 0) - \psi(\tau, 0)\| < \varepsilon$ for some τ' with $\|\tau' - \tau\| < \varepsilon$. Lipschitz continuity of F can improve the conclusion to the existence of $\delta > 0$ so that $\|\psi'(\tau, 0) - \psi(\tau, 0)\| < \varepsilon$.

ASSUMPTION 5.6. *K is a compact preasymptotically stable set for (C_2, F_2, G_2, D_2) .*

Let $\mathcal{B}_2(K)$ be the basin of attraction of K for (C_2, F_2, G_2, D_2) . Note that $\mathcal{B}_2(K)$ is open by [16, Proposition 7.4].

DEFINITION 5.7. *Under Assumptions 5.1 and 5.6, the basin of attraction $\mathcal{B}_1(K) \subset \mathbb{R}^m$ is the set of all $x \in \mathbb{R}^m$ such that every $(\phi, \psi) \in \mathcal{S}(x)$ is bounded and if ψ is complete, then $\lim_{t+j \rightarrow \infty} d_K(\psi(t, j)) = 0$.*

Above, d_K is the distance from the set K , so that $d_K(x) = \inf_{k \in K} \|x - k\|$. Directly from the definitions, one can gather that $\mathcal{B}_1(K) \subset p\mathcal{B}_1(A)$ and $\Phi(\mathcal{L}_1(\mathcal{B}_1(K))) \subset \mathcal{B}_2(K)$. Here, \mathcal{L}_1 is the limit mapping (15) for (C_1, F_1, G_1, D_1) .

PROPOSITION 5.8. *Suppose that Assumptions 5.1 and 5.6 hold. Then,*

- (a) *the basin of attraction $\mathcal{B}_1(K)$ is open;*
- (b) *convergence from $\mathcal{B}_1(K)$ to K is uniform, in the sense that for every compact set $K_1 \subset \mathcal{B}_1(K)$ and every neighborhood U_2 of K there exists $T > 0$ such that, for every $(\phi, \psi) \in \mathcal{S}(K_1)$, every $(t, k) \in \text{dom } \psi$ with $t + k > T$, $\psi(t, k) \in U_2$.*

For a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a set $O \subset \mathbb{R}^n$, define the set $M^{-1}(O)$ by

$$M^{-1}(O) = \{x \in \mathbb{R}^m \mid M(x) \subset O\}.$$

Note that, in contrast, $M^{-1}(O)$ is $\{x \in \mathbb{R}^m \mid M(x) \cap O \neq \emptyset\}$. If M is locally bounded and outer semicontinuous and O is open, then $M^{-1}(O)$ is open; this follows from [42, Theorem 5.19].

Proof. For (a), note that

$$\mathcal{B}_1(K) = \mathcal{L}_1^{-1}(\Psi_A^{-1}(\mathcal{B}_2(K))),$$

where \mathcal{L} is the limit mapping (15) and Ψ_A is the mapping Ψ altered to have empty values outside of A . $\mathcal{B}_2(K)$ is open by [16, Proposition 7.4]. Ψ_A is outer semicontinuous on \mathbb{R}^m because Ψ is and A is closed. Ψ_A is also locally bounded because Ψ is. Then $\Psi_A^{-1}(\mathcal{B}_2(K))$ is an open set. (It is the union of $\mathbb{R}^m \setminus A$ and of a relatively open subset of A .) Then \mathcal{L}^{-1} of that set is open as well. This proves (a).

For (b), note that because Length_t is upper semicontinuous on $p\mathcal{B}_1(A)$ and $\mathcal{B}_1(K) \subset p\mathcal{B}_1(A)$, Length_t is bounded above on K_1 . Let the bound be $T_1 > 0$. Also note that because \mathcal{L} is osc and locally bounded, $\mathcal{L}(K_1)$ is a compact subset of A , and because Ψ is outer semicontinuous and locally bounded, $K_2 := \Psi(\mathcal{L}(K_1))$ is compact, and a subset of $\mathcal{B}_2(K)$. Preattractivity of K from $\mathcal{B}_2(K)$ is uniform (see [16, Lemma 7.8]), and hence there exists $T_2 > 0$ such that, for every $\psi \in \mathcal{S}_2(K_2)$, for every $(t, k) \in \text{dom } \psi$ with $t + k > T_2$, $\psi(t, k) \in U_2$. Then $T_1 + T_2$ is the T needed to verify (b). \square

5.2. Scenario 2: Past Zeno repeatedly. In this scenario, the hybrid system with data (C, F, D, G) has a closed attractor A which is PSOTAS. Solutions are Zeno, converge to A in Zeno time, and are reinitialized, and the process is repeated infinitely many times. The reinitialization, as in Scenario 1, is from the limit of a solution passed through the set-valued map Ψ . Throughout this section, the following assumption is in place.

ASSUMPTION 5.9.

- (C, F, D, G) is a nominally well-posed and forward complete hybrid system in \mathbb{R}^n ;
- A is a closed PSOTAS set for (C, F, D, G) ;
- $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous and locally bounded set-valued mapping, with $\text{rge}\Psi \subset \mathcal{B}(A)$.

Above, as before, $\mathcal{B}(A)$ is the basin of pointwise attraction of A . Below, a solution is defined as, roughly, an infinite sequence of solutions to (C, F, D, G) , concatenated in an appropriate way. The index k represents which Zeno behavior is occurring. In particular, initially, $k = 1$. Assumption 5.9 imposes that the range of Ψ is in the basin of pointwise attraction $\mathcal{B}(A)$, which guarantees that each post-Zeno solution converges to A , and the process repeats infinitely many times. Note that by the definition below, every solution is maximal (it cannot be extended).

DEFINITION 5.10. *A repeatedly Zeno solution to (C, F, D, G) from $x \in \mathcal{B}(A)$, denoted $\phi \in \mathcal{S}(x)$, is a function $\phi : \mathbb{N} \times \mathbb{R}^2$ such that $\phi(1, 0, 0) = x$ and, for every $k \in \mathbb{N}$,*

- (a) $(t, j) \mapsto \phi(k, t, j)$ is a maximal/complete solution to (C, F, D, G) ;
- (b) $\phi(k + 1, 0, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi(k, t, j))$.

The theorem below gives, in a sense, nominal well-posedness of solutions in the current scenario. Then, the result is used to describe omega-limits of solutions and to propose an invariance principle for the current scenario.

THEOREM 5.11. *Suppose Assumption 5.9 holds. Let $\phi_i \in \mathcal{S}(x_i)$ with $x_i \rightarrow x \in \mathcal{B}(A)$. Then, there exists a graphically convergent subsequence of ϕ_i , the limit ϕ of which is a solution in $\mathcal{S}(x)$.*

Proof. By Theorem 3.15, there exists a subsequence, not relabeled below, so that $\phi_i(1, \cdot, \cdot)$ converge graphically, the limit $\phi(1, \cdot, \cdot)$ is a complete solution to (1), and $\phi(1, 0, 0) = x$. By Theorem 5.3, a further subsequence, also not relabeled, can be picked so that also $\phi_i(2, \cdot, \cdot)$ converge graphically, the limit $\phi(2, \cdot, \cdot)$ is a solution to (1), and $\phi(2, 0, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi(1, t, j))$. Repeating this procedure, using Theorems 3.15 and 5.3, a further subsequence can be picked so that $\phi_i(2, \cdot, \cdot)$ converge graphically, the limit $\phi(2, \cdot, \cdot)$ is a complete solution to (1), $\phi_i(3, \cdot, \cdot)$ converge graphically, the limit $\phi(3, \cdot, \cdot)$ is a solution to (1), and $\phi(3, 0, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi(2, t, j))$. With further repetition and by using the Cantor's diagonalization argument, one obtains a subsequence such that, for every $k = 2, 3, \dots$, $\phi_i(k, \cdot, \cdot)$ converge graphically, the limit $\phi(k, \cdot, \cdot)$ is a complete solution to (1), and $\phi(k, 0, 0) \in \Psi(\lim_{t+j \rightarrow \infty} \phi(k-1, t, j))$. This results in a desired solution ϕ . \square

For a solution ϕ as defined above, define

$$\omega(\phi) = \left\{ x \in \mathbb{R}^n \mid \exists k_l \nearrow \infty, (t_l, j_l) \in \text{dom } \phi(k_l, \cdot, \cdot) \text{ so that } x = \lim_{l \rightarrow \infty} \phi(k_l, t_l, j_l) \right\}.$$

PROPOSITION 5.12.

- (a) For every solution ϕ , the set $\omega(\phi)$ is closed.
- (b) Either the set $\omega(\phi) \cap \mathcal{B}(A)$ is nonempty or ϕ diverges out of $\mathcal{B}(A)$, in the sense that for every compact $K \subset \mathcal{B}(A)$ there exists $k_0 \in \mathbb{N}$ so that for every $k > k_0$, $K \cap \text{rge}\phi(k, \cdot, \cdot) = \emptyset$.
- (c) If ϕ is bounded relative to $\mathcal{B}(A)$, in the sense that there exists a compact set $K \subset \mathcal{B}(A)$ such that $\text{rge}\phi \subset K$, then $\omega(\phi)$ is nonempty, compact, and weakly forward and backward invariant:
 - (i) for every $x \in \omega(\phi)$ there exists a complete $\psi \in \mathcal{S}(x)$ with $\text{rge}\psi \subset \omega(\phi)$;
 - (ii) for every $x \in \omega(\phi)$ and $k_0 > 0$ there exists a complete $\psi \in \mathcal{S}(\omega(\phi))$ with $\text{rge}\psi \subset \omega(\phi)$ and $x \in \text{rge}\psi(k, \cdot, \cdot)$ for some $k > k_0$.

Proof. The omega limit of ϕ can be expressed as the outer limit of ranges of $\phi(k, \cdot, \cdot)$,

$$\omega(\phi) = \limsup_{k \rightarrow \infty} \text{rge}\phi(k, \cdot, \cdot),$$

and the outer limit of a sequence of sets is closed [42, Proposition 4.4]. Similarly, (b) follow from arguments similar to [42, Corollary 4.11]. In (c), nonemptiness and compactness follow from (a) and (b). The proof of weak invariance relies on a standard idea and resembles [16, Proposition 6.21]. Pick $x \in \omega(\phi)$ and $k_0 > 0$. By definition of $\omega(\phi)$, for all $l > k_0$ there exist $k_l > l$, $(t_l, j_l) \in \text{dom } \phi(k_l, \cdot, \cdot)$ such that $x = \lim_{l \rightarrow \infty} \phi(k_l, t_l, j_l)$. Pass to a subsequence so that $\phi(1 + k_l - k_0, 0, 0)$ converge. Using Theorem 5.11 pass to a further subsequence so that the sequence of solutions $\psi_i \in \mathcal{S}$ defined by $\psi_i(k, t, j) := \phi(k + k_i - k_0, t, j)$ is graphically convergent and its limit ψ is a solution, with $\psi(1, 0, 0) = \lim_{l \rightarrow \infty} \phi(1 + k_l - k_0, 0, 0)$. \square

COROLLARY 5.13. Suppose that the open set $U \subset \mathcal{B}(A)$ is forward invariant, in the sense that every $\phi \in \mathcal{S}(U)$ satisfies $\text{rge}\phi \subset U$. Suppose that there exist a continuous function $V : U \rightarrow \mathbb{R}$ which is nonincreasing along every solution $\phi \in \mathcal{S}(U)$. Then, every bounded relative to U solution ϕ converges to the largest weakly invariant subset Z of a level set $V^{-1}(r)$ for some $r \in \mathbb{R}$: for every $\varepsilon > 0$ and every large enough k , $\text{rge}\phi(k, \cdot, \cdot) \subset Z + \varepsilon\mathbb{B}$.

The simplest sufficient conditions for $V : U \rightarrow \mathbb{R}$ to be nonincreasing along every solution to $\phi \in \mathcal{S}(U)$ are that V be continuously differentiable and $\nabla V(x) \cdot f \leq 0$ for

all $x \in C$, $f \in F(x)$; $V(g) \leq V(x)$ for all $x \in D$, $g \in G(x)$; and $V(p) \leq V(x)$ for all $x \in A$, $p \in \Psi(x)$.

REFERENCES

- [1] A. D. AMES, A. ABATE, AND S. SASTRY, *Sufficient conditions for the existence of Zeno behavior*, in Proceedings of the 44th IEEE Conference on Decision Control and European Control, 2005, pp. 696–701.
- [2] J.-P. AUBIN, *Impulse Differential Inclusions and Hybrid Systems*, Lecture notes, University of California, Berkeley, 1999.
- [3] S. P. BHAT AND D. S. BERNSTEIN, *Lyapunov analysis of semistability*, in Proceedings of the American Control Conference, 1999, pp.
- [4] S. P. BHAT AND D. S. BERNSTEIN, *Nontangency-based Lyapunov tests for convergence and stability in systems having a continuum of equilibria*, SIAM J. Control Optim., 42 (2003), pp. 1745–1775.
- [5] S. P. BHAT AND D. S. BERNSTEIN, *Arc-length-based Lyapunov tests for convergence and stability with applications to systems having a continuum of equilibria*, Math. Control Signals Systems, 22 (2010), pp. 155–184.
- [6] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales*, Birkhauser, Basel, 2001.
- [7] B. BROGLIATO, *Nonsmooth Mechanics: Models, Dynamics, and Control*, 3rd ed., Communications and Control Engineering Series, Springer, Cham, 2016.
- [8] P. J. L. CUIJPERS, M. A. REINIERS, AND A. G. ENGELS, *Beyond Zeno-Behaviour*, Technical report, Eindhoven University of Technology, 2001.
- [9] S. DASHKOVSKIY AND P. FEKETA, *Prolongation and Stability of Zeno Solutions to Hybrid Dynamical Systems*, arXiv:1507.01382, 2015.
- [10] A. DESHPANDE, *Control of Hybrid Systems*, Ph.D. thesis, University of California, Berkeley, 1994.
- [11] R. GOEBEL, *Set-valued Lyapunov functions for difference inclusions*, Automatica, 47 (2011), pp. 127–132.
- [12] R. GOEBEL, *Robustness of stability through necessary and sufficient Lyapunov-like conditions for systems with a continuum of equilibria*, Systems Control Lett., 65 (2014), pp. 81–88.
- [13] R. GOEBEL AND R. G. SANFELICE, *How well-posedness of hybrid systems can extend beyond Zeno times*, in Proceedings of the 55th IEEE Conference on Decision and Control, 2016, pp.
- [14] R. GOEBEL AND R. G. SANFELICE, *Notions and sufficient conditions for pointwise asymptotic stability in hybrid systems*, in Proceedings of the 10th IFAC Symposium on Nonlinear Control Systems, 2016, pp. 140–145.
- [15] R. GOEBEL, R. G. SANFELICE, AND A. R. TEEL, *Hybrid dynamical systems. robust stability and control for systems that combine continuous-time and discrete-time dynamics*, IEEE Control Syst. Mag., 29 (2009), pp. 28–93.
- [16] R. GOEBEL, R.G. SANFELICE, AND A.R. TEEL, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*, Princeton University Press, Princeton, NJ, 2012.
- [17] R. GOEBEL AND A. R. TEEL, *Lyapunov characterization of Zeno behavior in hybrid systems*, in Proceedings of the 47th IEEE Conference on Decision Control, 2008, pp. 2752–2757.
- [18] R. GOEBEL AND A. R. TEEL, *Pre-asymptotic stability and homogeneous approximations of hybrid dynamical systems*, SIAM Rev., 52 (2010), pp. 87–109.
- [19] J. W. GRIZZLE, G. ABBA, AND F. PLESTAN, *Asymptotically stable walking for biped robots: Analysis via systems with impulse effects*, IEEE Trans. Automat. Control, 46 (2001), pp. 51–64.
- [20] W. M. HADDAD, V. CHELLABOINA, AND Q. HUI, *Nonnegative and Compartmental Dynamical Systems*, Princeton University Press, Princeton, NJ, 2010.
- [21] S. H. J. HELJMANS, D. P. BORGERS, AND W. P. M. H. HEEMELS, *Solution concepts and analysis of spatially invariant hybrid systems: Exploring Zeno and beyond*, IFAC PapersOnLine, 49-18 (2016), pp. 164–169.
- [22] J. P. HESPANHA AND A. S. MORSE, *Stabilization of nonholonomic integrators via logic-based switching*, Automatica, 35 (1999), pp. 385–393.
- [23] Q. HUI, *Hybrid consensus protocols: An impulsive dynamical system approach*, Internat. J. Control, 83 (2010), pp. 1107–1116.
- [24] Q. HUI, *Semistability and robustness analysis for switched systems*, Eur. J. Control, 17 (2011), pp. 73–88.
- [25] Q. HUI AND W. M. HADDAD, *Semistability of switched dynamical systems. I. Linear system theory*, Nonlinear Anal. Hybrid Syst., 3 (2009), pp. 343–353.

- [26] Q. HUI, W. M. HADDAD, AND S. P. BHAT, *Finite-time semistability and consensus for nonlinear dynamical networks*, IEEE Trans. Automat. Control, 53 (2008), pp. 1887–1900.
- [27] Q. HUI, W. M. HADDAD, AND S. P. BHAT, *Semistability, finite-time stability, differential inclusions, and discontinuous dynamical systems having a continuum of equilibria*, IEEE Trans. Automat. Control, 54 (2009), pp. 2465–2470.
- [28] Q. HUI, W. M. HADDAD, AND S. P. BHAT, *Finite-time semistability, Filippov systems, and consensus protocols for nonlinear dynamical networks with switching topologies*, Nonlinear Anal. Hybrid Syst., 4 (2010), pp. 557–573.
- [29] Q. HUI, W. M. HADDAD, AND S. P. BHAT, *On robust control algorithms for nonlinear network consensus protocols*, Internat. J. Robust Nonlinear Control, 20 (2010), pp. 269–284.
- [30] K. H. JOHANSSON, M. EGERSTEDT, J. LYGEROS, AND S. SASTRY, *On the regularization of Zeno hybrid automata*, Systems Control Lett., 38 (1999), pp. 141–150.
- [31] K. H. JOHANSSON, J. LYGEROS, J. ZHANG, AND S. SASTRY, *Simulation of Zeno hybrid automata*, in Proceedings of the 38th IEEE Conference on Decision and Control, 1999, pp. 3538–3543.
- [32] M. KONEČNÝ, W. TAHA, F. A. A. BARTHA, J. DURACZ, A. DURACZ, AND A. D. AMES, *Enclosing the behavior of a hybrid automaton up to and beyond a Zeno point*, Nonlinear Anal. Hybrid Syst., 20, 2016, pp. 1–20.
- [33] A. LAMPERSKI AND A. D. AMES, *Lyapunov-like conditions for the existence of Zeno behavior in hybrid and Lagrangian hybrid systems*, in Proceedings of the 46th IEEE Conference Decision and Control, 2007, pp. 115–120.
- [34] A. LAMPERSKI AND A. D. AMES, *On the existence of Zeno behavior in hybrid systems with non-isolated Zeno equilibria*, in Proceedings of the 47th IEEE Conference on Decision and Control, 2008, pp. 2776–2781.
- [35] A. LAMPERSKI AND A. D. AMES, *Lyapunov theory for Zeno stability*, IEEE Trans. Automat. Control, 58 (2013), pp. 100–112.
- [36] C. G. MAYHEW, R. G. SANFELICE, AND A. R. TEEL, *Quaternion-based hybrid controller for robust global attitude tracking*, IEEE Trans. Automat. Control, 56 (2011), pp. 2555–2566.
- [37] L. MOREAU, *Stability of multiagent systems with time-dependent communication links*, IEEE Trans. Automat. Control, 50 (2005), pp. 169–182.
- [38] J. OH, B. DRINCIC, AND D. S. BERNSTEIN, *Nonlinear feedback models of hysteresis*, IEEE Control Syst. Mag., 29 (2009), pp. 100–119.
- [39] Y. OR AND A. D. AMES, *Stability and completion of Zeno equilibria in Lagrangian hybrid systems*, IEEE Trans. Automat. Control, 56 (2011), pp. 1322–1336.
- [40] A. Y. POGROMSKY, W. P. M. H. HEEMELS, AND H. NIJMEIJER, *On solution concepts and well-posedness of linear relay systems*, Automatica, 39 (2003), pp. 2139–2147.
- [41] A. PURI AND P. VARAIYA, *Decidability of hybrid systems with rectangular differential inclusions*, in Computer Aided Verification, Lecture Notes in Comput. Sci. 818, Springer, New York, 1994.
- [42] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational Analysis*, Springer, New York, 1998.
- [43] R. G. SANFELICE, *Analysis and Design of Cyber-Physical Systems: A Hybrid Control Systems Approach*, CRC Press, Boca Raton, FL, 2015.
- [44] J. SHEN, *Robust non-Zenoness of piecewise affine systems with applications to linear complementarity systems*, SIAM J. Optim., 24 (2014), pp. 2023–2056.
- [45] D. E. STEWART, *Dynamics with Inequalities: Impacts and Hard Constraints*, SIAM, Philadelphia, 2011.
- [46] J. ZHANG, K. H. JOHANSSON, J. LYGEROS, AND S. SASTRY, *Zeno hybrid systems*, Internat. J. Robust Nonlinear Control, 11 (2001), pp. 435–451.
- [47] H. ZHENG, E. A. LEE, AND A. D. AMES, *Beyond Zeno: Get on with it!*, Hybrid Systems: Computation and Control, Lecture Notes in Comput. Sci. 3927, Springer, New York, 2006, pp. 568–582.