Applications of convex analysis to consensus algorithms, pointwise asymptotic stability, and its robustness

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***Abstract*— Convex analysis and the theory of differential inclusions with maximal monotone right-hand sides suggests casting consensus algorithms as systems involving switching between such differential inclusions. Convergence of solutions to such switching systems is shown and applications to consensus are presented. Robustness of pointwise asymptotic stability for a single differential inclusion which has some monotonicity-related properties, but needs not be monotone, is shown.**

***Index Terms*— consensus, pointwise asymptotic stability, semistability, convex analysis, monotone mapping**

I. Introduction

This note is motivated by consensus algorithms for multi-agent systems [1], [2] in continuous time and by the question of robustness of the kind of stability they result in. Basic consensus algorithms lead to autonomous linear dynamics which are the steepest descent, or gradient flow, for a convex quadratic function. Constraints and consideration of projected gradient flow can be handled using nonsmooth convex functions and their subdifferentials. This leads to dynamics given by maximal monotone set-valued mappings, solutions to which are quite well-behaved and have a nonexpansive property; see [3] for a classical exposition.

One approach to changing communication topology in consensus problems is to consider switching between maximal monotone dynamics. The first contribution of this note is a result which shows when such switching leads to convergence of solutions to the set of common equilibria. If the common equilibria represent consensus, the result shows when a multi-agent system reaches consensus. This contribution unifies a variety of consensus results, similar to those in [1], [4], [5], [6], and [7], though continuously changing weights of the communication graph are not considered here. Convex-analytic methods have been used, to an extent, in [5], [6], and for discrete-time dynamics in [8], this note considers more general nonsmooth convex functions. Some related works also include applications of consensus to distributed optimization — see the discussion in [6]. The general idea behind the result has some parallels to results on common fixed points for families of nonexpansive mappings, for example [9], and for families of mappings with common convex Lyapunov functions, say [10]. [1]

For dynamics given by a maximal monotone mapping, including steepest descent for a convex function and the so-called saddle-point dynamics, every equilibrium is Lyapunov stable. If all solutions converge to the set of equilibria, the set is pointwise asymptotically stable. This property, also referred to as semistability, has been studied for differential equations [12], differential inclusions [13], difference inclusions [14], and hybrid systems [15], and is also present, as a special case, in numerous convergent convex optimization algorithms, under the name Fejér monotonicity. Robustness of pointwise asymptotic stability, for a difference inclusion, was addressed in [16], through the use of set-valued Lyapunov functions, proposed by [17]. For a differential inclusion given by a maximal monotone mapping, robustness was shown in [18]. The second contribution of this note is a robustness result generalizing that of [18], to a general differential inclusion, with constraints, and with a local Fejér monotonicity property, but not necessarily monotone. Robustness is understood similarly to [19], for differential inclusions, and [20], for differential inclusions with constraints.

The two contributions, discussed above, are in Section III and Section IV, respectively. In Section II, a brief introduction to maximal monotone mappings, to differential inclusions involving them, and to issues of convergence of solutions to such inclusions is provided.

II. Background

General references for convex and set-valued analysis are [21] and [22]. Early treatment of maximal monotone differential inclusions is in [3], essential elements of it can be found in [23], and a recent survey addressing both continuous and discrete-time systems is [24]. For further discussion and references on saddle-point dynamics, see [18].

A. Monotone mappings

A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *monotone* if for every $x_1, x_2 \in \mathbb{R}^n$ and every $y_1 \in M(x_1)$, $y_2 \in M(x_2)$,

$$(x_1 - x_2) \cdot (y_1 - y_2) \geq 0, \tag{1}$$

where $\cdot$ is the dot product. A monotone M is *maximal monotone* if the graph of M cannot be enlarged without violating monotonicity. Monotonicity is a generalization of positive semidefiniteness to the nonlinear setting. Indeed, for

[1] A quite different intersection of fixed point theory and Lyapunov analysis, for time varying systems, is in [11].

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an $n \times n$ matrix R and $M(x) = Rx$, positive semidefiniteness of R is equivalent to monotonicity of M. Other examples include subdifferential mappings of convex and of convex-concave functions, in the sense of convex analysis.

Example 2.1: Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper (i.e., finite somewhere), lower semicontinuous (lsc), and convex function. It's *convex subdifferential* mapping is the set-valued mapping $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with the value at $x \in \mathbb{R}^n$ denoted by $\partial f(x)$ and given by

$$\{y \in \mathbb{R}^n \mid f(x') \geq f(x) + y \cdot (x' - x) \ \forall x' \in \mathbb{R}^n\}. \quad (2)$$

The subdifferential mapping is maximal monotone; [21, Theorem 24.9]. Monotonicity is easy to check directly, by adding the inequalities $f(x_2) \geq f(x_1) + y_1 \cdot (x_2 - x_1)$, $f(x_1) \geq f(x_2) + y_2 \cdot (x_1 - x_2)$. △

Example 2.2: Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set, and $f : \mathbb{R}^n \to \mathbb{R}$ be a convex (thus continuous) and differentiable on a neighborhood of C (thus continuously differentiable there) function. Define $f_C : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$f_C(x) = \begin{cases} f(x) & \text{if} \quad x \in C, \\ \infty & \text{if} \quad x \notin C. \end{cases} \quad (3)$$

Then f_C is proper, lsc, and convex, and its subdifferential, which is maximal monotone by Example 2.1, is given by

$$\partial f_C(x) = \begin{cases} \nabla f(x) & \text{if} \quad x \in \operatorname{int} C, \\ \nabla f(x) + N_C(x) & \text{if} \quad x \in \operatorname{bdry} C, \\ \emptyset & \text{if} \quad x \notin C. \end{cases} \quad (4)$$

Above, $\operatorname{int} C$ and $\operatorname{bdry} C$ stand for the interior and the boundary of C, and $N_C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the normal cone mapping, given by

$$N_C(x) = \{v \in \mathbb{R}^n \mid v \cdot (x' - x) \leq 0 \ \forall x' \in C\}. \quad △$$

Example 2.3: Let $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a convex-concave function: $x \mapsto h(x,y)$ is convex for every $y \in \mathbb{R}^m$ and $y \mapsto h(x,y)$ is concave for every $x \in \mathbb{R}^n$. It is then continuous [21, Theorem 35.1]. Suppose it is also differentiable (and then, automatically continuously differentiable). Then

$$(x,y) \mapsto \nabla_x h(x,y) \times (-\nabla_y h(x,y))$$

is maximal monotone. More generally, let h be as above, and let $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^m$ be nonempty, closed, and convex sets. There are different ways to extend h outside of $C \times D$ by infinite values. (Cf. (3).) The largest such extension is

$$h_{C,D}(x,y) = \begin{cases} h(x,y) & \text{if} \quad x \in C,\ y \in D, \\ \infty & \text{if} \quad x \notin C, \\ -\infty & \text{if} \quad x \in C,\ y \notin D. \end{cases} \quad (5)$$

The *convex-concave subdifferential* of $h_{C,D}$ is the mapping

$$(x,y) \mapsto \partial_x h_{C,D}(x,y) \times \tilde{\partial}_y h_{C,D}(x,y), \quad (6)$$

where $\partial_x h_{C,D}(x,y)$ is the convex subdifferential of the convex function $x \mapsto h(x,y)$ and $\tilde{\partial}_y h_{C,D}(x,y)$ is the negative of the convex subdifferential of the convex function $y \mapsto -h(x,y)$. The set-valued mapping

$$(x,y) \mapsto \partial_x h_{C,D}(x,y) \times \left(-\tilde{\partial}_y h_{C,D}(x,y)\right)$$

is maximal monotone. △

B. Monotone inclusions

Differential inclusions given by a negative of a maximal monotone operator, for example the continuous-time steepest descent, have several favorable properties. The results date back to [3] and before, and hold in Hilbert spaces; see [23, Theorem 1, Section 2, Chapter 2] for details.

Theorem 2.4: *Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone mapping and consider the differential inclusion*

$$\dot{x} \in -M(x) \quad (7)$$

Then:

(a) *For every $x_0 \in \operatorname{dom} M$ there exists a unique maximal solution to (7) with $x(0) = x_0$ and this solution is complete, i.e., defined on $[0, \infty)$.*

(b) *For any two complete solutions $x(\cdot), x'(\cdot)$ to (7),*

$$t \mapsto \|x(t) - x'(t)\|$$

is nonincreasing, where $\|\cdot\|$ stands for the Euclidean norm. In particular, the solutions to (7) depend continuously on initial conditions, in the uniform norm over $[0, \infty)$.

(c) *For every solution $x(\cdot)$ to (7), $\|\dot{x}(t)\|$ is nonincreasing, and, for almost all $t \geq 0$, $\dot{x}(t) = m(-M(x(t)))$, where $m(S)$ is the minimum norm element of the set S.*

In (a), $\operatorname{dom} M$ is the *effective domain* of M, namely $\{x \in \mathbb{R}^n \mid M(x) \neq \emptyset\}$. The nonexpansive property in (b) comes directly from the definition of monotonicity,

$$\frac{d}{dt}\frac{1}{2}\|x(t) - x'(t)\|^2 = (x(t) - x'(t)) \cdot (\dot{x}(t) - \dot{x}'(t)) \leq 0,$$

and implies that every equilibrium of (7), equivalently, every x such that $0 \in M(x)$, i.e., every element of $M^{-1}(0)$, is Lyapunov stable for (7). A different setting where this property holds, for systems of the form $\dot{x} = f(x)$, is when the vector field f is *geodesically monotone*, i.e., when there exists a Riemannian metric with a nonpositive Lie derivative in the directions of the vector field; see [25]. The property that the difference between every pair of solutions to $\dot{x} = f(x)$ has a decreasing distance is also known as *incremental stability* [26] and weak versions of the Lyapunov conditions in [27] can be employed to guarantee the nonexpansivity property.

The minimum norm property in (c) leads to solutions to (7) being referred to as "slow" or "lazy". In general, the minimum norm element does not depend continuously on x:

Example 2.5: Let f_C be as in Example 2.2, with $C = [0, \infty)^n$ being the nonnegative cone. For $x \in C$, for $i = 1, 2, \ldots, n$, the i-th coordinates of points in $\partial f_C(x)$ are

$$[\partial f_C(x)]_i = \begin{cases} [\nabla f(x)]_i & \text{if} \quad x_i > 0, \\ [\nabla f(x)]_i + (-\infty, 0] & \text{if} \quad x_i = 0. \end{cases}$$

Then the minimum norm element $m(\partial f_C(x))$ depends discontinuously on $x \in C$: its i-th coordinate $[m(\partial f_C(x))]_i$ is

$$\begin{cases} [\nabla f(x)]_i & \text{if} \quad x_i > 0 \text{ or } x_i = 0,\ [\nabla f(x)]_i \leq 0, \\ 0 & \text{if} \quad x_i = 0,\ [\nabla f(x)]_i > 0. \end{cases} \quad △$$

Above, the minimum norm element of $\partial f_C(x)$ is the projection of $\nabla f(x)$ onto the set C. This is true in general. The result below is from the recent [28, Corollary 2], though some of the equivalences go back to [29], [3] and the relation to variational inequalities in (c) can be found in [23].

Proposition 2.6: *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set and $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, differentiable on a neighborhood of C. For any $x, v \in \mathbb{R}^n$, the following are equivalent:*

(a) $v = P_{T_C(x)}\left(-\nabla f(x)\right)$
(b) $v = \lim_{\lambda \searrow 0} \left[P_C(x + \lambda(-\nabla f(x))) - x\right]/\lambda$
(c) $v \in -\left(\nabla f(x) + N_{T_C(x)}(v)\right)$
(d) $v = -m\left(\nabla f(x) + N_C(x)\right)$

Above, the *projection* $P_C(x)$ of $x \in \mathbb{R}^n$ onto C is the unique $c \in C$ which minimizes $\|x - c\|$. The *tangent cone to* C at $x \in C$ is $T_C(x) = \{z \in \mathbb{R}^n \mid z \cdot w \leq 0 \ \forall w \in N_C(x)\}$.

As a consequence of Theorem 2.4 and Proposition 2.6, for f and C as in Example 3, the solutions to

$$\dot{x} = P_{T_C(x)}\left(-\nabla f(x)\right) \tag{8}$$

exist from every initial condition in C, are unique, the maximal solutions are complete, and they depend continuously on initial conditions. This extends to saddle dynamics, dating back to [30]. For h, C, and D as in Example 2.3, the solutions to

$$\dot{x} = P_{T_C(x)}\left(-\nabla_x h(x,y)\right), \ \dot{y} = P_{T_D(y)}\left(\nabla_y h(x,y)\right) \tag{9}$$

exist from every initial condition in $C \times D$, are unique, the maximal solutions are complete, and they depend continuously on initial conditions.

C. Convergence and stability

If $M = \partial f$ where $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is proper, lsc, and convex, then (7) is the steepest descent

$$\dot{x} \in -\partial f(x), \tag{10}$$

and the set of equilibria of (7) is the set of minimizers of f — indeed, by (2), $0 \in \partial f(x)$ if and only if x minimizes f. Suppose that the closed and convex, but not necessarily bounded, set $A := \arg\min f$ is nonempty. Let $x^* \in A$. For a complete solution x to (10),

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|x(t) - x^*\|^2 &= (x(t) - x^*) \cdot \dot{x}(t) \\ &\leq f(x^*) - f(x(t)) \\ &= \min f - f(x(t)) \leq 0, \end{aligned}$$

where the inequality follows from (2). Then there exists a sequence $t_i \nearrow \infty$ such that $f(x(t_i)) \searrow \min f$. Since x is bounded, the sequence $x(t_i)$ has a convergent subsequence, and since f is lsc, x has a cluster point in A. Since each $a \in A$ is Lyapunov stable, x converges to that cluster point. These arguments motivate the next definition.

A maximal monotone $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called *demipositive*, following [31], if there exists $a \in M^{-1}(0)$ such that, for every convergent sequence x_i and every bounded sequence $v_i \in M(x_i)$, if $(x_i - a, v_i) \to 0$ then $\lim_{i\to\infty} x_i \in M^{-1}(0)$. The original definition, given in a broader setting, considered weak convergence of x_i. Here, since the graph of a maximal monotone M is closed, the definition reduces to: there exists $a \in M^{-1}(0)$ such that, if $v \cdot (x - a) = 0$ for some $v \in M(x)$ then $x \in M^{-1}(0)$. For a proper, lsc, and convex f, ∂f is demipositive, and arguments very similar to those above show that if a maximal monotone M is demipositive, with a nonempty set of equilibria, then every complete solution to (7) converges to an equilibrium.

For a convex-concave function, like h or $h_{C,D}$ in Example 2.3, the gradient or subdifferential (6) need not be demipositive. It is if h is strictly convex in x, strictly concave in y, but not in general: for example, consider $h(x,y) = x^2 + xy$. Consequently, solutions to *saddle-point dynamics*

$$\dot{x} \in -\partial_x h_{C,D}(x,y), \ \dot{y} \in \tilde{\partial}_y h_{C,D}(x,y), \tag{11}$$

need not, in general, converge to the set of equilibria of (11), which is exactly the set of saddle points of $h_{C,D}$. Recall that (x^*, y^*) is a *saddle point* of $h_{C,D}$ if

$$h_{C,D}(x^*, y) \leq h_{C,D}(x^*, y^*) \leq h_{C,D}(x, y^*)$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, equivalently, due to the construction of $h_{C,D}$, for all $(x,y) \in C \times D$. Recall too that solutions to saddle-point dynamics (11) are the same as solutions to the projected gradient dynamics (9).

The following result is [32, Theorem 4.1]. It extends to nondifferentiable h, see [18], but is stated here in the differentiable case for simplicity.

Theorem 2.7: *In the setting of Example 2.3, suppose that the set $X^* \times Y^*$ of saddle points of $h_{C,D}$ is nonempty, and either*

(a) *for every $(x^*, y^*) \in X^* \times Y^*$ and every $y \notin Y^*$, $h_{C,D}(x^*, y) < h_{C,D}(x^*, y^*)$,*

or

(b) *for every $(x^*, y^*) \in X^* \times Y^*$ and every $x \notin X^*$, $h_{C,D}(x^*, y^*) < h_{C,D}(x, y^*)$,*

then every complete solution to (11) *converges to a saddle point of $h_{C,D}$.*

The result is proven using an invariance argument. Thanks to Theorem 2.4, a standard invariance principle applies, for example [33, Lemma 4.3, Theorem 4.4] in Khalil's textbook. In contrast, [34] used a hybrid system invariance principle for saddle-point dynamics; incorrectly, as pointed out in [35]. In turn, [35] used a projected dynamics result from [36].

III. Switching between monotone inclusions and application to consensus

Consider the switching system

$$\dot{x} \in -M_q(x), \quad q(\cdot) \in \mathcal{S} \tag{12}$$

where $\mathcal{S}$ is a set of switching signals.

Assumption 3.1:

(a) *$Q = \{1, 2, \ldots, p\}$ for some $p \in \mathbb{N}$. For every $q \in Q$, $M_q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a demipositive maximal monotone mapping with closed* $\operatorname{dom} M_q$.

(b) *For every $q(\cdot) \in \mathcal{S}$ there exists a dwell-time $\tau_D > 0$.*

(c) *For every complete $q(\cdot) \in \mathcal{S}$ there exists $T > 0$ such that, for every $t \in [0, \infty)$,*

$$q([t, t+T]) = Q.$$

(d) $A := \bigcap_{q \in Q} A_q \neq \emptyset$, *where* $A_q := M_q^{-1}(0)$.

Switching signals are considered to be piecewise continuous and right-continuous. The dwell time assumption in (b) means that, for each $q(\cdot) \in \mathcal{S}$, there exists $\tau_D > 0$ so that discontinuities of $q(\cdot)$ occur at times $t_1, t_2, \ldots$ (dependent on $q(\cdot)$), where $0 < t_1 < t_2 < \ldots$ and $t_{i+1} - t_i \geq \tau_D$ for $i = 1, 2, \ldots$. The assumption in (c) means that each switching signal $q(\cdot)$ "visits" every $q \in Q$ during every interval of length T, and T can depend on $q(\cdot)$.

Directly from Theorem 2.4, one obtains that under Assumption 3.1 (a), (b), for every $q(\cdot) \in \mathcal{S}$ and every $x_0 \in \operatorname{dom} M_{q(0)}$ there exists a unique maximal solution to (12). This solution is complete under additional assumptions on domains of M_q, for example if $\operatorname{dom} M_q$ are equal to each another, over all $q \in Q$.

Theorem 3.2: *Under Assumption 3.1, every solution to* (12) *has $x(\cdot)$ bounded, and every complete solution is such that $\lim_{t\to\infty} x(t)$ exists and belongs to A.*

The proof relies on $V(x) := \|x - a\|^2$ which is nonincreasing along every solution, for any $a \in A$. Considering $M_q = \partial f_q$ in Theorem 3.2, where f_q are convex functions, yields convergence to common minimizers of f_q.

Corollary 3.3: *For $q \in Q$, let $f_q : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper, lsc, and convex function. Let*

$$A := \bigcap_{q \in Q} A_q \quad \text{where } A_q := \arg\min f_q.$$

If $A \neq \emptyset$ and Assumption 3.1 (b), (c) holds, then every complete solution to

$$\dot{x} \in -\partial f_q(x), \quad q(\cdot) \in \mathcal{S}, \tag{13}$$

converges to a point in A.

The nonexpansive property ensures that in Theorem 3.2 and in Corollary 3.3, every $a \in A$ is Lyapunov stable and thus A is pointwise asymptotically stable. Similarly, in the setting of Theorem 2.7, the set of saddle point is pointwise asymptotically stable. The formal definition of this property is postponed until Section IV.

Remark 3.4: *A nice and related result appears in [37]. There, the system*

$$\dot{x} = -m\left(N_C(x) + \operatorname{con}\{\partial g_1(x), \partial g_2(x), \ldots, g_p(x)\}\right) \tag{14}$$

is studied, where m is the minimum norm element, C is a closed convex set and g_q are finite-valued convex functions on a Hilbert space, and con *stands for the convex hull. Convergence to Pareto points of g_q in C is concluded, and such points reduce to common minimizers, if the latter exist. For comparison, recall that solutions to* (13) *satisfy $\dot{x} = -m(\partial f_q(x))$, and one can consider f_q constructed from g_q and C through* (3)*. Then* (14) *is $\dot{x} = -m\left(\operatorname{con}\{\partial f_1(x), \ldots, \partial f_p(x)\}\right)$. The set of velocities $\operatorname{con}\{\partial f_1(x), \ldots, \partial f_p(x)\}$ can be related to arbitrary switching between velocities in $\partial f_q(x)$, but the minimum norm selection destroys this relationship.*

The breadth of the setting of Corollary 3.3 is illustrated through the following examples, often related to the questions of consensus. Let $n = km$, where k represents the number of m-dimensional agents. For convenience, $x \in \mathbb{R}^n$ is $(x_1, x_2, \ldots, x_k)$, with $x_i \in \mathbb{R}^m$. Let $M \subset \mathbb{R}^n$ be the *consensus subspace*:

$$CS = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_k\}.$$

It is said that a complete solution to (13) *reaches consensus* if $\lim_{t\to\infty} x(t)$ exists and belongs to CS; in other words if the limits $\lim_{t\to\infty} x_i(t)$ are the same, for $i = 1, 2, \ldots, k$. Then, under the assumptions of Corollary 3.3,

- If A is nonempty and $A \subset CS$, then complete solutions reach consensus for every initial condition.
- If A is nonempty and there exists a point $a \in A$ such that $a \notin CS$, then, for some initial conditions, complete solutions do not reach consensus.

For the conclusions of the examples below, let Assumption 3.1 (b), (c) hold.

Example 3.5: For $q \in Q$, let $a_{ij}(q) = a_{ji}(q) \geq 0$ for $i, j = 1, \ldots, k$. Let

$$l_q(x) = \frac{1}{4} \sum_{i,j=1}^{k} a_{ij}(q)(x_i - x_j)^2, \tag{15}$$

which is a convex quadratic function. Then $\dot{x} \in -\partial l_q(x)$ reduces to $\dot{x} = -\nabla l_q(x)$, which, for a given $q \in Q$, becomes

$$\dot{x}_i = \sum_{j=1}^{k} a_{ij}(q)\left(x_j - x_i\right), \quad i = 1, \ldots, k. \tag{16}$$

By (15), $CS \subset \arg\min l_q$ for every $q \in Q$. Complete solutions reach consensus if $\cap_{q\in Q} \arg\min l_q \subset CS$. △

In the setting of Example 3.5, let the symmetric matrices $\{a_{ij}(q)\}_{i,j=1,2,\ldots k}$ represent undirected communication graphs G_q between agents, where an edge between agents i and j in the q-th mode is represented by $a_{ij} = a_{ji} > 0$. If the union of the communication graphs, over all $q \in Q$, is connected, equivalently, if it has a spanning tree, then $\cap_{q\in Q} \arg\min l_q = CS$, and consequently, $A = CS$.

Corollary 3.6: *With the notation above, if the union of communication graphs G_q over all $q \in Q$ is connected and Assumption 3.1 (b), (c) holds, then every complete solution to* (16) *reaches consensus.*

The connectedness assumption combined with Assumption 3.1 (c) is similar to "periodic connectedness" assumptions often made in the consensus literature. Thus, Corollary 3.6 essentially recovers results reaching consensus for switching and periodically connected communication graphs in the literature; e.g., [4, Theorem 3.12]. For the purpose of finite-time consensus, one can take, in (15), powers of $x_i - x_j$ in $[1, 2)$, which also leads to a convex function; c.f. [38].

Example 3.7: For $i = 1, 2, \ldots, k$, let $C_i \subset \mathbb{R}^n$ be a nonempty, closed, convex set, and let $d_i : \mathbb{R}^n \to \mathbb{R}$ be

$$d_i(x_i) := \frac{1}{2}\left(\mathrm{dist}_{C_i}(x_i)\right)^2 = \frac{1}{2}\left(x_i - P_{C_i}(x_i)\right)^2.$$

Above, $P_{C_i}(x_i)$ is the projection of x_i onto C_i. Then d_i is a differentiable convex function, with $\arg\min d_i = C_i$ and $\nabla d_i(x_i) = x_i - P_{C_i}(x_i)$. With l_q as in Example 3.5, consider

$$f_q(x) = l_q(x) + \sum_{i=1}^{k} d_i(x_i).$$

Then $\dot{x} = -\nabla f_q(x)$ becomes, for $i = 1, 2, \ldots, k$,

$$\dot{x}_i = \sum_{j=1}^{k} a_{ij}(q)\,(x_j - x_i) + P_{C_i}(x_i) - x_i. \tag{17}$$

Such systems were analyzed, for example, in [5]; similar ones are in [7]. In both [5], [7] a different time dependence of a_{ij} was considered. Consensus here corresponds to finding the intersection of the sets C_i. A generalization of (17), to

$$\dot{x}_i = \sum_{j=1}^{k} a_{ij}(q)\,(x_j - x_i) - \nabla g_i(x_i), \tag{18}$$

where $g_i : \mathbb{R}^m \to \mathbb{R}$ is a differentiable convex function with nonempty $\arg\min g_i$, and $\dot{x}_i = -\nabla g_i(x_i)$ is the local dynamics of the i-th agent, was considered in [6]. This case still fits in the framework of Corollary 3.3. △

Example 3.8: Let $a_{ij}(q)$ be as in Example 3.5, let C_i be as in Example 3.7. Consider, for $i = 1, 2, \ldots, k$,

$$\dot{x}_i = P_{T_{C_i}(x_i)}\left(\sum_{j=1}^{k} a_{ij}(q)\,(x_j - x_i)\right), \tag{19}$$

which is the dynamics (16) projected onto the convex set C_i. These dynamics are discontinuous. As explained in Section II-B, solutions to (19) are the same as solutions to $\dot{x} \in -\partial f_q$, where $C = C_1 \times C_2 \times \cdots \times C_k$ and f_q is obtained from l_q in Example 3.5 by the construction (3), i.e.,

$$f_q(x) = \begin{cases} l_q(x) & \text{if} \quad x \in C, \\ \infty & \text{if} \quad x \notin C. \end{cases}$$

Consequently, for every initial condition in C, and every switching signal $q(\cdot) \in Q$, there exists a unique complete solution $x(\cdot)$. If

$$\emptyset \neq \left(\bigcap_{q \in Q} \arg\min l_q\right) \cap \left(\bigcap_{i=1}^{k} C_i\right) \subset CS$$

complete solutions to (19) reach consensus. △

IV. Robustness of PAS through Fejér monotonicity

In the setting of Theorem 3.2, the set of common equilibria of M_q is pointwise asymptotically stable (PAS). This section shows that the PAS property is robust, if, locally around the attractor, a nonexpansive property similar to what is guaranteed by monotonicity, holds. The setting is that of a constrained differential inclusion

$$\dot{x} \in F(x), \qquad x \in C, \tag{20}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping and $C \subset \mathbb{R}^n$. A generalization to the switching case should be possible. A *solution* to (20) is a locally absolutely continuous function $x : I \to \mathbb{R}^n$, where I is an interval containing and beginning at 0, such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in I$ and $x(t) \in C$ for all $t \in I$.

A set $A \subset \mathbb{R}^n$ is *PAS* for (20) if

- every $a \in A$ is Lyapunov stable: for every $\varepsilon > 0$ there exists $\delta > 0$ so that, for every solution $x(\cdot)$ to (20), if $\|x(0)-a\| < \delta$ then $\|x(t)-a\| < \varepsilon$ for all $t \in \mathrm{dom}\, x(\cdot)$; and
- every solution $x(\cdot)$ to (20) is bounded, and if it is complete, then it is convergent and $\lim_{t\to\infty} x(t) \in A$.

If A is a singleton, then PAS the same as asymptotic stability. If A is compact, PAS implies asymptotic stability. If A is unbounded, the two properties are not comparable.

The inclusion (20) is *Fejér monotone* with respect to a set $A \subset \mathbb{R}^n$ if, for every solution x to (20), for every $a \in A$,

$$\|x(t) - a\| \le \|x(0) - a\| \quad \forall t \in \mathrm{dom}\, x(\cdot). \tag{21}$$

The inclusion (20) is *locally Fejér monotone* with respect to A if there exists a neighborhood U of A such that (21) holds for every solution x with $x(0) \in U$ and every $a \in A$. The terminology is borrowed from optimization [39], where it is usually applied to discrete-time dynamics.

A set $A \subset \mathbb{R}^n$ is *robustly PAS* for (20) if there exists a continuous function $\rho : \mathbb{R}^n \to [0, \infty)$, with $\rho(x) = 0$ if and only if $x \in A$, such that A is PAS for

$$\dot{x} \in F_\rho(x), \qquad x \in C_\rho, \tag{22}$$

where the set-valued mapping $F_\rho : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

$$F_\rho(x) = \overline{\mathrm{con}} F\left(x + \rho(x)\mathbb{B}\right) + \rho(x)\mathbb{B} \quad \forall x \in \mathbb{R}^n$$

and $C_\rho = \{x \in \mathbb{R}^n \mid (x + \rho(x)\mathbb{B}) \cap C \neq \emptyset\}$. The perturbation of F, given by F_ρ, is what was considered by [19]. The perturbation of C, given by C_ρ, is what was considered, for example, in [20].

Theorem 4.1: *Suppose that*

(a) $A \subset \mathbb{R}^n$ *is nonempty, compact, and PAS for* (20)*;*
(b) (20) *is locally Fejér monotone with respect to* A*;*
(c) C *is nonempty and closed,* F *is locally bounded and outer semicontinuous relative to* C *and for every* $x \in C$, $F(x)$ *is nonempty and convex.*

Then A *is robustly PAS.*

In fact, in the setting of Theorem 4.1, the local Fejér monotonicity can be almost preserved — the perturbation ρ can be small enough to ensure that for every solution $x(\cdot)$ to (22) with $x(0) \in W$, for every $a \in A$,

$$\|x(t) - a\| \le (1 + \varepsilon)\,\|x(0) - a\| \quad \forall t \in \mathrm{dom}\, x(\cdot). \tag{23}$$

Example 4.2: Consider the setting of Example 3.8, but with no q-dependence of the dynamics. That is, consider

$$\dot{x}_i = P_{T_{C_i}(x_i)}\left(\sum_{j=1}^{k} a_{ij}\left(x_j - x_i\right)\right). \tag{24}$$

Suppose that $\arg\min l = CS$, where l is the quadratic function (15), and that $A := CS \cap \bigcap_{i=1}^{k} C_i$ is nonempty and compact. Then A is PAS for (24), as concluded in Example 3.8, and the property is robust, thanks to Theorem 4.1.

The theorem cannot be applied directly to (24), as the dynamics need not be continuous. However, for $C = C_1 \times C_2 \times \cdots \times C_k$, let l_C be the convex function obtained from l, where l is the right-hand side of (15), and C via the construction (3). Then (24) has the same solutions as $\dot{x} \in -\partial l_C(x)$ and, in fact, $\dot{x} = m(-\partial l_C(x))$, by Proposition 2.6. Theorem 4.1 cannot be applied to the inclusion, as $-\partial l_C$ is not locally bounded at the boundary of C. However, since projections are nonexpansive, $\|m(-\partial l_C(x))\| \leq \|\nabla l(x)\| \leq L\|x\|$ for all $x \in C$ and some $L \geq 0$. Consider

$$F(x) = -\partial l_C(x) \cap L\|x\|\mathbb{B},$$

which defines a set-valued mapping that satisfies the assumptions of Theorem 4.1, and $m\left(-\partial l_C(x)\right) \in F(x)$ for every $x \in C$. Theorem 4.1 can be applied to F. △

REFERENCES

[1] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[2] K.-K. Oh, M.-C. Park, and H.-S. Ahn. A survey of multi-agent formation control. *Automatica J. IFAC*, 53:424–440, 2015.

[3] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[4] W. Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Automat. Control*, 50(5):655–661, 2005.

[5] G. Shi, K.H. Johansson, and Y. Hong. Reaching an optimal consensus: dynamical systems that compute intersections of convex sets. *IEEE Trans. Automat. Control*, 58(3):610–622, 2013.

[6] G. Shi, A. Proutiere, and K.H. Johansson. Network synchronization with convexity. *SIAM J. Control Optim.*, 53(6):3562–3583, 2015.

[7] Z. Qiu, S. Liu, and L. Xie. Distributed constrained optimal consensus of multi-agent systems. *Automatica J. IFAC*, 68:209–215, 2016.

[8] M. Zhu and S. Martínez. On distributed convex optimization under inequality and equality constraints. *IEEE Trans. Automat. Control*, 57(1):151–164, 2012.

[9] R. DeMarr. Common fixed points for commuting contraction mappings. *Pacific J. Math.*, 13:1139–1141, 1963.

[10] S. Reich and A.J. Zaslavski. Asymptotic behavior of dynamical systems with a convex Lyapunov function. *J. Nonlinear Convex Anal.*, 1(1):107–113, 2000.

[11] T.A. Burton and T. Furumochi. Fixed points and problems in stability theory for ordinary and functional differential equations. *Dynam. Systems Appl.*, 10(1):89–116, 2001.

[12] S.P. Bhat and D.S. Bernstein. Nontangency-based Lyapunov tests for convergence and stability in systems having a continuum of equilibria. *SIAM J. Control Optim.*, 42(5):1745–1775, 2003.

[13] Q. Hui, W.M. Haddad, and S.P. Bhat. Finite-time semistability and consensus for nonlinear dynamical networks. *IEEE Trans. Automat. Control*, 53(8):1887–1900, 2008.

[14] R. Goebel. Set-valued Lyapunov functions for difference inclusions. *Automatica*, 47(1):127–132, 2011.

[15] R. Goebel and R.G. Sanfelice. Pointwise asymptotic stability in a hybrid system and well-posed behavior beyond Zeno. *SIAM J. Control Optim.*, 56(2):1358–1385, 2018.

[16] R. Goebel. Robustness of stability through necessary and sufficient Lyapunov-like conditions for systems with a continuum of equilibria. *Systems Control Lett.*, 65:81–88, 2014.

[17] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Automat. Control*, 50(2):169–182, 2005.

[18] R. Goebel. Stability and robustness for saddle-point dynamics through monotone mappings. *Systems Control Lett.*, 108:16–22, 2017.

[19] F.H. Clarke, Y.S. Ledyaev, and R.J. Stern. Asymptotic stability and smooth Lyapunov functions. *J. Diff. Eq.*, 149(1):69–114, 1998.

[20] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[21] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[22] R.T. Rockafellar and R. J-B Wets. *Variational Analysis*. Springer, 1998.

[23] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, 1984.

[24] J. Peypouquet and S. Sorin. Evolution equations for maximal monotone operators: asymptotic analysis in continuous and discrete time. *J. Convex Anal.*, 17(3-4):1113–1163, 2010.

[25] G. Isac and S.Z. Németh. *Scalar and asymptotic scalar derivatives: theory and applications*. Springer, 2008.

[26] V. Fromion, S. Monaco, and D. Normand-Cyrot. Asymptotic properties of incrementally stable systems. *IEEE Transactions on Automatic Control*, 41(5):721–723, 1996.

[27] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.

[28] B. Brogliato, A. Daniilidis, C. Lemaréchal, and V. Acary. On the equivalence between complementarity systems, projected systems and differential inclusions. *Systems Control Lett.*, 55(1):45–51, 2006.

[29] C. Henry. An existence theorem for a class of differential equations with multivalued right-hand side. *J. Math. Anal. Appl.*, 41:179–186, 1973.

[30] K. Arrow and L. Hurwicz. A gradient method for approximating saddle points and constrained maxima [reprint of RAND Corporation, June 13, 1951, paper P-223]. In *Traces and emergence of nonlinear programming*, pages 45–60. Birkhäuser/Springer Basel AG, Basel, 2014.

[31] R.E. Bruck Jr. Asymptotic convergence of nonlinear contraction semigroups in Hilbert space. *J. Funct. Anal.*, 18:15–26, 1975.

[32] V.I. Venets. Continuous algorithms for solution of convex optimization problems and finding saddle points of convex-concave functions with the use of projection operations. *Optimization*, 16(4):519–533, 1985.

[33] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, 3rd edition, 2002.

[34] D. Feijer and F. Paganini. Stability of primal-dual gradient dynamics and applications to network optimization. *Automatica J. IFAC*, 46(12):1974–1981, 2010.

[35] A. Cherukuri, E. Mallada, and J. Cortés. Asymptotic convergence of constrained primal-dual dynamics. *Systems Control Lett.*, 87:10–15, 2016.

[36] A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. Springer, 2012.

[37] H. Attouch, G. Garrigos, and X. Goudou. A dynamic gradient approach to Pareto optimization with nonsmooth convex objective functions. *J. Math. Anal. Appl.*, 422(1):741–771, 2015.

[38] L. Wang and F. Xiao. Finite-time consensus problems for networks of dynamic agents. *IEEE Trans. Automat. Control*, 55(4):950–955, 2010.

[39] P.L. Combettes. Fejér monotonicity in convex optimization. In *Encyclopedia of Optimization*, pages 1016–1024. Springer, New York, second edition, 2009.