Observers for Hybrid Dynamical Systems with Linear Maps and Known Jump Times

Pauline Bernard and Ricardo G. Sanfelice

Abstract— **This paper proposes a general framework for the state estimation of plants given by hybrid systems with linear flow and jump maps, in the favorable case where their jump events can be detected instantaneously. A candidate observer consists of a copy of the plant's hybrid dynamics with continuous-time and/or discrete-time correction terms adjusted by two constant gains, and with jumps triggered by those of the plant. Assuming that the time between successive jumps is known to belong to a given closed set allows us to formulate an augmented system with a timer which keeps track of the time elapsed between successive jumps and facilitates the analysis. Then, since the jumps of the plant and of the observer are synchronized, the error system has time-invariant linear flow and jump maps, and a Lyapunov analysis leads to sufficient conditions on the design of the gains for uniform asymptotic stability in three different settings: continuous and discrete updates, only discrete updates, or only continuous updates. Those conditions take the form of matrix inequalities, which we solve in examples including cases where the time between successive jumps is unbounded or tends to zero (Zeno behavior).**

I. Introduction

In many applications, estimating the state of a system is crucial, whether it be for control, supervision, or fault diagnosis purposes. Unfortunately, the problem of designing observers for hybrid systems with linear flow/jump maps in a general setting is unsolved. This issue arises mainly from the fact that hybrid systems combine both continuous-time and discrete-time dynamics, which in general leads to solutions from nearby initial conditions that have different jump times. Such a mismatch of time domains makes the formulation of observability/detectability and, in turn, observer design very challenging.

When the plant's jump times are unknown, the error system approach does not apply since the jumps of the observer and of the plant are not necessarily synchronized. Therefore, very few observer results exist. This problem is overcome in a particular case in [1], thanks to the fact that the jump map g is such that $g \circ g$ is the identity map, and in a more general setting in [2], thanks to a change of coordinates transforming the jump map into the identity map. Another path explored in the particular setting of switched systems is to estimate the plant's switching signal: its observability has been studied in [3], [4] and some designs exist based on *mode location observers* (see, e.g., [5], [6]).

On the other hand, in the context of (possibly switched) impulsive systems, jumps are assumed to occur at specific known times and are all separated by nonzero periods of flow. In that setting, observability and determinability have been extensively studied [7], [8], [9]. As for observer design, results are available when each mode is observable [10], or when the system is observable/determinable for any impulse time sequence containing more than a known finite number of jumps [11], [12]. In those references, however, the time elapsed between successive jumps must be lower bounded away from zero and upper bounded.

Another important hybrid setting for which observer results exist is when the system itself has continuous-time dynamics, but the measurement is sampled, i.e., only available at specific discrete times. For those sample-data systems and other systems with sporadic events, observers have been designed under specific assumptions on the time elapsed between successive events — or, in the case of periodic events, the sampling period. In [13], a design is proposed when the sampling period is sufficiently small. Then, it is extended to any sampling period in [14] (provided appropriate matrix inequalities are satisfied), and finally to the case of sporadic measurements in [15], i.e., when the time elapsed between sampling events varies in a known interval. Here again, the "inter-jump" duration must be lower bounded away from zero and upper bounded by known constants.

In this paper, we consider general hybrid systems as in [16] with linear flow and jump maps, and possibly an input whose value is considered known at all times. Under the assumption that the plant's jumps are detected instantaneously, a candidate observer is an impulsive system that jumps at the same time as the plant does and is fed with the known input and linear correction terms in either the flow or the jump maps, or both. Assuming that the time between successive jumps belongs to a known (possibly unbounded) closed set allows us to formulate (Section II) an augmented hybrid system with a timer that keeps track of the time elapsed between successive jumps. Then, we derive sufficient conditions for the design of the gains defining the observer's correction terms to ensure uniform global asymptotic stability in three different settings: both continuous-time and discrete-time updates (Section III), only discrete-time updates (Section IV), and finally only continuous-time updates (Section V).

Notation. $\mathbb{R}$ (resp. $\mathbb{N}$) denotes the set of real numbers (resp.

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P. Bernard (pmbernar@ucsc.edu) and R.G. Sanfelice (ricardo@ucsc.edu) are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA.

integers), and $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{R}_{>0} = (0, +\infty)$, $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$. The components of a square matrix P are denoted p_{ij}, and $\lambda_m(P)$ (resp. $\lambda_M(P)$) stands for its smallest (resp. largest) eigenvalue. The symbol $\star$ in a matrix denotes the symmetric blocks. $\mathbb{B}$ stands for a Euclidian ball of appropriate dimension, of radius 1 and center 0.

II. Hybrid observer

A. Problem statement

Consider a hybrid plant

$$\mathcal{H}_u \left\{ \begin{array}{rcll} \dot{x} & = & A_c\, x + B_c\, u_c & x \in C \\ x^+ & = & A_d\, x + B_d\, u_d & x \in D \\ & & & \\ y_c & = & H_c\, x & x \in C \\ y_d & = & H_d\, x & x \in D \end{array} \right. \tag{1}$$

with state x in $\mathbb{R}^n$, input u being the collection of a continuous-time input $u_c : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m_c}$ and a discrete-time input $u_d : \mathbb{N} \to \mathbb{R}^{m_d}$, and output $y = (y_c, y_d)$ with value in $\mathbb{R}^{p_c} \times \mathbb{R}^{p_d}$. We are interested in estimating the trajectories of the plant (1) when they are initialized in a given subset $\mathcal{X}_0$ of $\mathbb{R}^n$.

A solution x to a hybrid system is called a *hybrid arc* and is defined on a *hybrid time domain* denoted $\operatorname{dom} x$. A hybrid time domain $\mathcal{D}$ is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for any (T', J') in $\mathcal{D}$, there exists a sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$ such that

$$\mathcal{D} \cap ([0, T'] \times \{0, 1, \ldots, J'\}) = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j) \ .$$

For a hybrid arc x, we denote $\operatorname{dom}_t x$ (resp. $\operatorname{dom}_j x$) the projection of $\operatorname{dom} x$ on the first (resp. second) dimension, $T(x) = \sup \operatorname{dom}_t x$, $J(x) = \sup \operatorname{dom}_j x$, $t_j(x)$ the time stamp associated to jump j uniquely characterized by

$$(t_j(x), j-1) \in \operatorname{dom} x \quad , \quad (t_j(x), j) \in \operatorname{dom} x$$

and $\mathcal{T}(x) = \{t_j(x) \ : \ j \in \operatorname{dom}_j x \cap \mathbb{N}_{>0}\}$. For $u_c : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m_c}$ and $u_d : \mathbb{N} \to \mathbb{R}^{m_d}$, we say that a hybrid arc x is solution to $\mathcal{H}_u$ with output $y = (y_c, y_d)$ if $\operatorname{dom} x = \operatorname{dom} y$, for all $j \in \mathbb{N}$ and

- for all t in $(t_j(x), t_{j+1}(x))$, $x(t, j)$ is in C
- for almost all t in $(t_j(x), t_{j+1}(x))$, we have $\dot{x}(t, j) = A_c\, x(t, j) + B_c\, u_c(t)$
- for all t in $[t_j(x), t_{j+1}(x)]$, $y_c(t, j) = H_c\, x(t, j)$

and for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$, $x(t, j)$ is in D, $y_d(t, j) = H_d\, x(t, j)$, and

$$x(t, j+1) = A_d\, x(t, j) + B_d\, u_d(j) \ .$$

This solution is maximal if it cannot be continued into a solution with larger domain. We denote $\mathcal{S}_{\mathcal{H}_u}(\mathcal{X}_0)$ the set of maximal solutions of $\mathcal{H}_u$ with initial condition in $\mathcal{X}_0$ and input u. We will also need the following definition.

Definition 2.1: For a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$, an input u, and a subset $\mathcal{X}_0$ of $\mathbb{R}^n$, we will say that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds if for any hybrid arc x in $\mathcal{S}_{\mathcal{H}_u}(\mathcal{X}_0)$,

- $0 \leq t - t_j(x) \leq \sup \mathcal{I} \quad \forall (t, j) \in \operatorname{dom} x$
- $t_{j+1}(x) - t_j(x) \in \mathcal{I}$
  - $\forall j \in \mathbb{N}_{>0}$ if $J(x) = +\infty$
  - for $\forall j \in \{1, \ldots, J(x) - 1\}$ if $J(x) < +\infty$

In other words, the set $\mathcal{I}$ describes the possible lengths of the flow intervals between successive jumps. The role of the first item in Definition 2.1 is to bound the length of the intervals of flow which are not covered by the second item, namely possibly the first $[0, t_1(x)]$ and the last $\operatorname{dom}_t x \cap [t_{J(x)}(x), +\infty)$ (when they are defined). Our goal is the following.

Problem 1: Design an observer assuming we know

- the value of the input u at all times,
- when the plant's jumps occur,
- the outputs y_c during flows and/or y_d at the jumps,
- some information about the flow time between successive jumps, namely a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds.

The existence of a set $\mathcal{I}$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds is not a problem because it always holds for $\mathcal{I} = \mathbb{R}_{\geq 0}$. But as we will see later, it is advantageous to select $\mathcal{I}$ as tight as possible, namely to have as much information about the duration of flow between successive jumps as possible. The following example shows how $\mathcal{I}$ can be chosen depending on $\mathcal{X}_0$.

Example 2.2: Consider a bouncing ball with gravity coefficient $\mathfrak{g} > 0$ and restitution coefficient $\lambda > 0$, modelled as system (1) with[1,2]

$$A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad A_d = \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix} \tag{2}$$
$$C = \mathbb{R}_{\geq 0} \times \mathbb{R} \ , \quad D = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 = 0 \, , \, x_2 \leq 0\}$$
$$B_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad B_d = 0 \quad , \quad u_c \equiv -\mathfrak{g}$$

If $\lambda < 1$, any maximal solution x is such that[3] $T < +\infty$ and $J = +\infty$. The time between two successive jumps $t_{j+1} - t_j$ tends to 0 when j tends to $+\infty$, and its upper bound increases with $|x(0,0)|$. So we can take $\mathcal{I} = [0, \tau_M]$ with $\tau_M \geq 0$, if $\mathcal{X}_0$ is bounded. Otherwise, we must take $\mathcal{I} = \mathbb{R}_{\geq 0}$.

If now $\lambda > 1$, any maximal solution x initialized in $\mathbb{R}^2 \setminus \{(0,0)\}$ is such that $T = +\infty$, $J = +\infty$. The time between two successive jumps $t_{j+1} - t_j$ tends to $+\infty$ when j tends to $+\infty$, and its lower bound decreases with $|x(0,0)|$. Therefore, if there exists $\delta > 0$ such that $\mathcal{X}_0$ is a subset of $\mathbb{R}^n \setminus \delta\mathbb{B}$, one can take $\mathcal{I} = [\tau_m, +\infty)$ with $\tau_m > 0$. Otherwise, we need $\mathcal{I} = \mathbb{R}_{\geq 0}$.

Finally if $\lambda = 1$, any maximal solution x initialized in $\mathbb{R}^2 \setminus \{(0,0)\}$, is such that $T = +\infty$, $J = +\infty$, and the time between two successive jumps $t_{j+1} - t_j$ is constant for all $j \geq 1$, and increases with $|x(0,0)|$. The maximal solution

[1] The coefficient -1 in A_d is arbitrary because $x_1 = 0$ in the jump set. We take -1 because, numerically, if x_1 is negative when the jump condition is detected, it is useful to change its sign after the jump in order for the flow condition to be verified at the next iteration.

[2] Several definitions of H_c and H_d will be considered later.

[3] To simplify the notation, we write T, J and t_j for $T(x)$, $J(x)$, $t_j(x)$.

initialized at $(0,0)$ is discrete, i.e., $T=0$ and $J=+\infty$. We can take $\mathcal{I}$ of the form:

- $\mathcal{I}=[0,\tau_M]$ with $\tau_M \geq 0$, if $\mathcal{X}_0$ is bounded.
- $\mathcal{I}=[\tau_m,+\infty)$ with $\tau_m>0$, if there exists $\delta>0$ such that $\mathcal{X}_0$ is a subset of $\mathbb{R}^n \setminus \delta\mathbb{B}$.
- $\mathcal{I}=[\tau_m,\tau_M]$ with $\tau_m>0$ and $\tau_M>0$, if there exists $\delta>0$ such that $\mathcal{X}_0$ is a bounded subset of $\mathbb{R}^n \setminus \delta\mathbb{B}$.
- otherwise, $\mathcal{I}=\mathbb{R}_{\geq 0}$. △

B. Proposed hybrid observer

Since the plant's jumps and the value of the input are assumed to be known, we propose to use an impulsive observer of the form[4]

$$\hat{\mathcal{H}}_{u,y}(\mathcal{T}) \begin{cases} \dot{\hat{x}}(t) = A_c\hat{x}(t)+B_cu_c(t) \\ \qquad\qquad + L_c(y_c(t)-H_c\hat{x}(t)) \quad \text{if } t\notin\mathcal{T} \\ \hat{x}(t_j^+) = A_d\hat{x}(t_j)+B_du_d(j) \\ \qquad\qquad + L_d(y_d(t_j)-H_d\hat{x}(t_j)) \quad \text{if } t_j\in\mathcal{T} \end{cases} \tag{3}$$

where $\mathcal{T}$ can be taken equal to the plant's jump times, namely $\mathcal{T}(x)$ with x solution to $\mathcal{H}_u$.

To use the hybrid framework from [16] and express the fact that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0,\mathcal{I})$ is satisfied, we will consider the augmentation of $\mathcal{H}_u$ in (1) given by the hybrid system

$$\mathcal{H}_u^\tau \begin{cases} \left.\begin{array}{rcl} \dot{x} &=& A_cx+B_cu_c \\ \dot{\tau} &=& 1 \end{array}\right\} (x,\tau)\in C^\tau \\ \\ \left.\begin{array}{rcl} x^+ &=& A_dx+B_du_d \\ \tau^+ &=& 0 \end{array}\right\} (x,\tau)\in D^\tau \\ \\ \begin{array}{rclr} y_c &=& H_cx & (x,\tau)\in C^\tau \\ y_d &=& H_dx & (x,\tau)\in D^\tau \end{array} \end{cases} \tag{4}$$

with, denoting $\tau_M=\sup\mathcal{I}$,

$$C^\tau=\mathbb{R}^n\times([0,\tau_M]\cap\mathbb{R}_{\geq 0}) \quad , \quad D^\tau=\mathbb{R}^n\times\mathcal{I} \tag{5}$$

and the interconnection of $\mathcal{H}_u^\tau$ with $\hat{\mathcal{H}}_{u,y}(\mathcal{T})$ (after rewriting (3) as a hybrid dynamical system as in [16]) resulting in the hybrid system

$$\hat{\mathcal{H}}_u^\tau \begin{cases} \left.\begin{array}{l} \dot{x}=A_cx+B_cu_c \\ \dot{\hat{x}}=A_c\hat{x}+B_cu_c+L_c(H_cx-H_c\hat{x}) \\ \dot{\tau}=1 \end{array}\right\} (x,\hat{x},\tau)\in \hat{C}^\tau \\ \\ \left.\begin{array}{l} x^+=A_dx+B_du_d \\ \hat{x}^+=A_d\hat{x}+B_du_d+L_d(H_dx-H_d\hat{x}) \\ \tau^+=0 \end{array}\right\} (x,\hat{x},\tau)\in \hat{D}^\tau \end{cases} \tag{6}$$

with

$$\hat{C}^\tau=\mathbb{R}^n\times\mathbb{R}^n\times([0,\tau_M]\cap\mathbb{R}_{\geq 0})\ ,\ \hat{D}^\tau=\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{I} \tag{7}$$

The models $\mathcal{H}_u^\tau$ and $\hat{\mathcal{H}}_u^\tau$ are such that the timer τ has to reach $\mathcal{I}$ before a jump can occur and is forced to jump when reaching τ_M (if finite). This enables to relate the behavior of $\mathcal{H}_u$, $\hat{\mathcal{H}}_{u,y}(\mathcal{T})$, $\mathcal{H}_u^\tau$, and $\hat{\mathcal{H}}_u^\tau$ as follows.

Lemma 2.3: Consider a subset $\mathcal{X}_0$ of $\mathbb{R}^n$, a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$ and denote $\tau_M=\sup\mathcal{I}\leq+\infty$. For any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0,\mathcal{I})$ holds, for any maximal solution x of $\mathcal{H}_u$ initialized in $\mathcal{X}_0$, and for any maximal solution $\hat{x}$ of $\hat{\mathcal{H}}_{u,y}(\mathcal{T}(x))$, we have $\operatorname{dom} x=\operatorname{dom}\hat{x}=\mathcal{D}$, and there exists a function τ defined on $\mathcal{D}$ such that (x,τ) is solution to $\mathcal{H}_u^\tau$ and $(x,\hat{x},\tau)$ is solution to $\hat{\mathcal{H}}_u^\tau$.

Proof: By definition of $\mathcal{T}(x)$, x and $\hat{x}$ have the same time domain $\mathcal{D}=\operatorname{dom} x=\operatorname{dom}\hat{x}$. Besides, since $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0,\mathcal{I})$ holds, the function τ defined by $\tau(t,j):=t-t_j(x)$ for all (t,j) in $\mathcal{D}$ gives the result. ■

We conclude that any property obtained for $\mathcal{H}_u^\tau$ or $\hat{\mathcal{H}}_u^\tau$ will be extendable to $\mathcal{H}_u$ and the cascade $\mathcal{H}_u$-$\hat{\mathcal{H}}_{u,y}(\mathcal{T})$, respectively, as long as $\mathcal{H}_u$ is initialized in $\mathcal{X}_0$ and $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0,\mathcal{I})$ holds.

Example 2.4: As mentioned in the introduction, the proposed framework also applies to the case where the plant itself has continuous-time dynamics

$$\dot{x}=A\,x+B\,u \quad , \quad y=Hx$$

but the output y is only available at discrete times t_j, which occur either periodically or sporadically. In that case, one can use an observer of the type (3) with $L_c=0$, $\mathcal{T}=\{t_1,t_2,\ldots,t_j,\ldots\}$, $A_c=A$, $B_c=B$, $A_d=I$, $B_d=0$, $u_c\equiv u$, $u_d\equiv 0$, $H_d=H$, and L_d to be designed. If we know that the time elapsed between two successive sampling events is in a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$, then the interconnection between the system and the observer can be modelled exactly by $\hat{\mathcal{H}}_u^\tau$. For instance, $\mathcal{I}$ is a singleton in the case of a periodic sampling, and $\mathcal{I}$ is a compact interval of $\mathbb{R}_{>0}$ in the case of sporadic sampling as done in [15]. △

III. Continuous and discrete updates

The following theorem gives a first sufficient condition to ensure global exponential stability.

Theorem 3.1: Consider a subset $\mathcal{X}_0$ of $\mathbb{R}^n$, a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$. Assume there exist scalars a_c and a_d, matrices $L_c\in\mathbb{R}^{n\times p_c}$ and $L_d\in\mathbb{R}^{n\times p_d}$, and a positive definite symmetric matrix $P\in\mathbb{R}^{n\times n}$ such that:

$$(A_c-L_cH_c)^\top P+P(A_c-L_cH_c)\leq a_cP \tag{8a}$$
$$(A_d-L_dH_d)^\top P(A_d-L_dH_d)\leq e^{a_d}P \tag{8b}$$
$$a_c\tau+a_d<0 \qquad \forall\tau\in\mathcal{I} \tag{8c}$$

Then, there exist $\gamma>0$ and $\theta>0$ such that for any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0,\mathcal{I})$ holds, every maximal solution x of $\mathcal{H}_u$ initialized in $\mathcal{X}_0$ and every maximal solution $\hat{x}$ of $\hat{\mathcal{H}}_{u,y}(\mathcal{T}(x))$ are complete and verify for all $(t,j)\in\operatorname{dom} x=\operatorname{dom}\hat{x}$

$$\left|x(t,j)-\hat{x}(t,j)\right|\leq\gamma\left|x(0,0)-\hat{x}(0,0)\right|e^{-\theta(t+j)}\ . \tag{9}$$

Sketch of the Proof: First observe that there always exists[5] a positive scalar a such that $a_c\tau+a_d\leq-a(\tau+1)$ for all

[4]In the following, the solutions $\hat{x}$ to this impulsive observer will be considered hybrid, with a domain inherited from the impulse sequence.

[5]If $\mathcal{I}$ is unbounded, necessarily a_c is negative according to (8c).

τ in $\mathcal{I}$. Also, by definition of $\hat{C}^\tau$ and $\hat{D}^\tau$ in $\hat{\mathcal{H}}_u^\tau$, for any solution $\phi = (x, \hat{x}, \tau)$ to $\hat{\mathcal{H}}_u^\tau$, we have for all $(t,j) \in \mathrm{dom}\,\phi$,

$$t_{i+1} - t_i \in \mathcal{I} \text{ for } i \in \{1, \dots, j-1\}\ , \quad t - t_j \in [0, \tau_M] \cap \mathbb{R}_{\geq 0}\ .$$

From that, it is possible to show that there exists M such that for any solution $\phi = (x, \hat{x}, \tau)$ to $\hat{\mathcal{H}}_u^\tau$,

$$a_c t + a_d j \leq M - a(t+j) \qquad \forall (t,j) \in \mathrm{dom}\,\phi\ . \tag{10}$$

Applying [16, Proposition 3.29] with $V(x, \hat{x}, \tau) = (\hat{x} - x)^\top P (\hat{x} - x)$, and Lemma 2.3 gives the result. ■

Note that from conditions (8a)-(8c), we recover the fact that if $0 \in \mathcal{I}$, namely there are Zeno or eventually discrete solutions, then a_d must be negative, i.e., the innovation term in the discrete dynamics of the observer must make the error contractive at jumps; similarly if $\sup \mathcal{I} = +\infty$, then a_c must be negative, i.e., the innovation term in the continuous dynamics must make the error contractive during flow.

The interesting property of conditions (8a)-(8c) is that they are affine (and thus convex) in τ, which means that it is sufficient to check them at the boundaries of the set $\mathcal{I}$ only.

Corollary 3.2: Consider a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$ with $\tau_m = \min \mathcal{I}$, $\tau_M = \sup \mathcal{I}$. Assume there exist scalars a_c and a_d, matrices $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that (8a)-(8b) *are satisfied. If any of the following conditions is verified*

1) $a_c \leq 0$ *and* $a_d < 0$,
2) $a_c < 0$ *and* $a_c \tau_m + a_d < 0$,
3) $a_c > 0$, $\tau_M < +\infty$, *and* $a_c \tau_M + a_d < 0$,

then (8a)-(8c) *hold.*

Example 3.3: Consider a bouncing ball modelled by (2) with a restitution coefficient $\lambda \in (0,1)$, and suppose that x_1 is measured at all (hybrid) times, i.e.

$$H_c = H_d = \begin{pmatrix} 1 & 0 \end{pmatrix}\ . \tag{11}$$

The continuous pair (A_c, H_c) is observable, and since $\lambda < 1$, the discrete pair (A_d, H_d) is detectable. We will show that it is possible to find P, L_c, and L_d such that (8a)-(8b) are satisfied with $a_c < 0$ and $a_d < 0$. Applying Corollary 3.2, we will then be able to deduce that (8a)-(8c) hold for $\mathcal{I} = \mathbb{R}_{\geq 0}$ and any set of initial conditions, and thus obtain a global hybrid observer via Theorem 3.1.

Since (A_d, H_d) is detectable, we start by looking for P and L_d such that (8b) holds with $a_d < 0$. To that end, we follow Lemma 1.1 given in Appendix and solve

$$F A_d - \Lambda F = B H_d$$

with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$, $|\lambda_i| < 1$ and $B = (1,1)^\top$. Straightforward computations show that if $\lambda_2 = -\lambda$, there exist[6] solutions given by

$$F = \begin{pmatrix} \frac{-1}{1+\lambda_1} & 0 \\ \frac{-1}{1-\lambda} & f \end{pmatrix} \tag{12}$$

[6]The nonuniqueness of solutions and the constraint on λ_2 are due to the fact that (A_d, H_d) is detectable but not observable.

where f is a degree of freedom which should be nonzero to ensure F is invertible. Applying Lemma 1.1 with the identity matrix for P_0, (8b) is verified with

$$P = F^\top F = \begin{pmatrix} \frac{1}{(1+\lambda_1)^2} + \frac{1}{(1-\lambda)^2} & \frac{-f}{1-\lambda} \\ \frac{-f}{1-\lambda} & f^2 \end{pmatrix} \tag{13}$$

$$L_d = F^{-1} B = \begin{pmatrix} -(1+\lambda_1) \\ \frac{1}{f}\left(1 - \frac{1+\lambda_1}{1-\lambda}\right) \end{pmatrix} \tag{14}$$

$$a_d = \ln(\max(\lambda_1^2, \lambda^2)) < 0\ . \tag{15}$$

Now we look for L_c such that

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) < 0\ . \tag{16}$$

Denoting $L_c = (\ell_1, \ell_2)^\top$ and $P L_c = (\alpha_1, \alpha_2)^\top$, we get

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) = \begin{pmatrix} -2\alpha_1 & -\alpha_2 + p_{11} \\ -\alpha_2 + p_{11} & 2p_{12} \end{pmatrix}\ ,$$

so that

$$(16) \iff \alpha_1 > 0 \quad , \quad -4p_{12}\alpha_1 > (-\alpha_2 + p_{11})^2\ .$$

We conclude that by choosing f, λ_1, α_1, and α_2 such that

$$f > 0\,, \quad |\lambda_1| < 1\,, \quad \alpha_1 > \frac{(-\alpha_2 + p_{11})^2}{4f}(1 - \lambda) \tag{17}$$

with p_{11} defined in (13), (8a)-(8b) are satisfied with $a_c < 0$, $a_d < 0$, P given in (13) and the gains $L_c = P^{-1}(\alpha_1, \alpha_2)^\top$ and L_d in (14). This gives a global observer for the bouncing ball with $\lambda < 1$ with item 1) of Corollary 3.2. △

Remark 3.4: In Example 3.3, the restrictions on f, α_1, and α_2 show that, in the favorable case where both the continuous and the discrete dynamics are detectable, it is not sufficient to choose independently $A_c - L_c H_c$ Hurwitz and $A_d - L_d H_d$ Schur. Indeed, their descent directions could be incompatible: jumps could destroy what has been achieved during flow, or vice versa. To avoid this phenomenon, (8a) and (8b) should be solved with the same P, and $a_c \leq 0$ and $a_d < 0$. By the Schur complement, this is equivalent to solving the LMIs

$$\begin{aligned} A_c^\top P + P A_c - (\tilde{L}_c H_c + H_c^\top \tilde{L}_c^\top) < 0 \\ \begin{pmatrix} P & (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \end{aligned} \tag{18}$$

in $(P, \tilde{L}_c, \tilde{L}_d)$ and take $L_c = P^{-1}\tilde{L}_c$ and $L_d = P^{-1}\tilde{L}_d$. Note that the problem of finding common quadratic Lyapunov functions for several continuous-time or several discrete-time systems has been studied in the context of switched systems and quadratic stabilization. But we are not aware of any result concerning the existence of a common quadratic function for a continuous-time system and a discrete-time system.

IV. Particular case: updates at jumps only

We now consider the case where only y_d is known, namely the measurement is known only at jump times. Therefore, we build an observer with $L_c = 0$. Of course, without the assumption that A_c is already Hurwitz, we cannot allow eventually continuous solutions to exist and we need

$\mathcal{I}$ bounded. The following result follows from combining Theorem 3.1 and Corollary 3.2.

Corollary 4.1: [Update at jumps] Consider a subset $\mathcal{X}_0$ of $\mathbb{R}^n$ and a compact subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$. Assume there exist scalars $a_c \in \mathbb{R}$ and $a_d < 0$, a matrix $L_d \in \mathbb{R}^{n\times p_d}$, and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A_c^\top P + PA_c \leq a_c P \tag{19a}$$

$$(A_d - L_d H_d)^\top P(A_d - L_d H_d) \leq e^{a_d} P \tag{19b}$$

$$a_c \tau_M + a_d < 0 \tag{19c}$$

with $\tau_M = \max \mathcal{I}$. Then, there exist $\gamma > 0$ and $\theta > 0$ such that for any input u making $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ hold, every maximal solution x of $\mathcal{H}_u$ initialized in $\mathcal{X}_0$, and every maximal solution $\hat{x}$ of $\hat{\mathcal{H}}_{u,y}(\mathcal{T}(x))$, with $L_c = 0$ and L_d as above, are complete and verify

$$\Big|x(t,j) - \hat{x}(t,j)\Big| \leq \gamma \Big|x(0,0) - \hat{x}(0,0)\Big| e^{-\theta(t+j)} \qquad \forall (t,j) \in \operatorname{dom} x \ (= \operatorname{dom} \hat{x}) \ . \tag{20}$$

Example 4.2: Consider a bouncing ball modelled by (2) with $\lambda \in (0,1)$, but as a difference to Example 3.3, assume that the measurement is only available at jumps, namely

$$H_c = 0 \quad , \quad H_d = (1,0) \ . \tag{21}$$

As seen in Example 2.2, for any compact subset K of $\mathbb{R}_{\geq 0} \times \mathbb{R}$, there exists $0 \leq \tau_K < +\infty$ such that $\mathcal{C}_{\mathcal{H}_u}(K, \mathcal{I})$ holds with $\mathcal{I} = [0, \tau_K]$. We will now determine conditions for P and L_d to verify (19a)-(19c). We have already found in Example 3.3 matrices P and L_d verifying (19b) with $a_d < 0$. They are given by (13)-(15) with f nonzero and $|\lambda_1| < 1$. It now remains to choose λ_1 and f such that the rest of the constraints are satisfied. Computing $A_c^\top P + PA_c$ we get that

$$\text{(19a)} : A_c^\top P + PA_c \leq a_c P \quad \iff \quad a_c \geq \frac{p_{11}}{|\det F|} \ .$$

Since

$$\text{(19c)} : a_c \tau_K + a_d < 0 \quad \iff \quad a_c < \frac{-a_d}{\tau_K} \ ,$$

we finally conclude that it suffices to have[7]

$$\frac{\frac{1}{(1+\lambda_1)^2} + \frac{1}{(1-\lambda)^2}}{|f|}(1+\lambda_1) < \frac{-\ln(\max(\lambda_1^2, \lambda^2))}{\tau_K} \tag{22}$$

to satisfy both (19a) and (19c). This is achieved by choosing any λ_1 such that $|\lambda_1| < 1$ and $|f|$ sufficiently large. We conclude that for any compact subset K in $\mathbb{R}_{\geq 0} \times \mathbb{R}$, there exists $\alpha_K > 0$ such that by choosing $L_c = 0$ and $L_d = (\ell_1, \ell_2)^\top$ verifying

$$-2 < \ell_1 < 0 \quad , \quad \ell_2 = \alpha \left(1 + \frac{\ell_1}{1-\lambda}\right) \ , \tag{23}$$

with $0 < |\alpha| < \alpha_K$, we get a uniformly globally exponentially stable (UGES) observer for $\mathcal{H}_u$ initialized in K. Note that since $\lambda < 1$, whatever the initial condition of $\mathcal{H}_u$, the duration between two successive jumps tends to 0 and becomes eventually smaller than τ_K. Therefore, any choice of ℓ_1 and ℓ_2 satisfying (23) for some nonzero α gives a globally convergent observer for $\mathcal{H}_u$ (but maybe without uniformity and stability with respect to the initial error). △

[7] If $\tau_K = 0$, i.e, there are only discrete solutions, the inequality holds trivially, because it is sufficient to take $A_d - L_d H_d$ Schur.

V. Particular case: continuous updates only

When $\mathcal{I}$ is unbounded, it is not possible to implement an observer with discrete updates only: continuous updates are necessary. And when the continuous dynamics are detectable, it may be sufficient to use only continuous updates (with $L_d = 0$). The following corollary follows from Theorem 3.1 and Corollary 3.2.

Corollary 5.1: [Continuous update] Consider a subset $\mathcal{X}_0$ of $\mathbb{R}^n$ and a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$. Assume there exist scalars $a_d \in \mathbb{R}$ and $a_c < 0$, a matrix L_c in $\mathbb{R}^{n\times p_c}$, and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \tag{24a}$$

$$A_d^\top P A_d \leq e^{a_d} P \tag{24b}$$

$$a_c \tau_m + a_d < 0 \tag{24c}$$

where $\tau_m = \min \mathcal{I}$. Then, there exist $\gamma > 0$ and $\theta > 0$ such that for any input u making $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ hold, every maximal solution x of $\mathcal{H}_u$ initialized in $\mathcal{X}_0$, and every maximal solution $\hat{x}$ of $\hat{\mathcal{H}}_{u,y}(\mathcal{T}(x))$, with L_c as above and $L_d = 0$, are complete and verify

$$|x(t,j) - \hat{x}(t,j)| \leq \gamma \Big|x(0,0) - \hat{x}(0,0)\Big| e^{-\theta(t+j)} \qquad \forall (t,j) \in \operatorname{dom} x \ (= \operatorname{dom} \hat{x}) \ . \tag{25}$$

Example 5.2: Consider again the bouncing ball (2) but this time with a restitution coefficient $\lambda \geq 1$. As seen in Example 2.2, for any $\delta > 0$, there exists $\tau_m > 0$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathbb{R}^2 \setminus \delta\mathbb{B}, \mathcal{I})$ holds with $\mathcal{I} = [\tau_m, +\infty)$. Suppose the height of the ball is measured continuously. The discrete dynamics being no longer detectable, the design from Example 4.2 is no longer possible. So we want to find a gain L_c such that (24a)-(24c) are satisfied. Since $A_c - L_c H_c$ is in companion form, it can be diagonalized with a Vandermonde matrix if its eigenvalues are real and distinct. Indeed, suppose we choose its eigenvalues λ_1 and λ_2 distinct and negative (such that $\lambda_1 + \lambda_2 = -\ell_1$ and $\lambda_1 \lambda_2 = \ell_2$). Then, the Vandermonde matrix

$$V_\lambda = \begin{pmatrix} -\frac{1}{\lambda_2} & -\frac{1}{\lambda_1} \\ 1 & 1 \end{pmatrix}$$

is invertible and we have

$$V_\lambda^{-1}(A_c - L_c H_c)V_\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \Lambda$$

namely, $(A_c - L_c H_c) = V_\lambda \Lambda V_\lambda^{-1}$. Since $\Lambda^\top + \Lambda \leq -2 \min |\lambda_i| I$, by taking $P_\lambda = (V_\lambda^{-1})^\top V_\lambda^{-1}$ straightforward computations give

$$(A_c - L_c H_c)^\top P_\lambda + P_\lambda(A_c - L_c H_c) \leq -2 \min |\lambda_i| P_\lambda$$

namely (24a) is satisfied with $a_c = -2\min|\lambda_i|$. Now, replacing P by P_λ in (24b), we get

$$\text{(24b)} \iff M_\lambda^\top M_\lambda \leq e^{a_d} I \quad \text{with} \quad M_\lambda = V_\lambda^{-1} A_d V_\lambda \ .$$

This means that the smallest value e^{a_d} can take is the maximal eigenvalue of the positive definite matrix $M_\lambda^\top M_\lambda$. In our case,

$$M_\lambda = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_1 - \lambda\lambda_2 & \lambda_2(1-\lambda) \\ -\lambda_1(1-\lambda) & -\lambda_2 + \lambda\lambda_1 \end{pmatrix}$$

What is interesting in M_λ is that it is homogeneous of degree 0 in λ_i : taking (λ_1, λ_2) or $(\mu\lambda_1, \mu\lambda_2)$ for any nonzero value of μ gives the same M_λ, and thus the same $M_\lambda^\top M_\lambda$, and thus the same a_d, while a_c is transformed into μa_c ! We conclude from this reasoning that for any $\tau_m > 0$, for any choice of negative distinct $(\lambda_{1,0}, \lambda_{2,0})$, the conditions (24a)-(24c) are satisfied with P_λ and $L_c = (-(\lambda_1+\lambda_2), \lambda_1\lambda_2)^\top$ if we choose $(\lambda_1, \lambda_2) = (\mu\lambda_{1,0}, \mu\lambda_{2,0})$ for $\mu > 0$ sufficiently large. In other words, for any $\delta > 0$, we can choose any $\rho = \frac{\lambda_2}{\lambda_1}$ in $\mathbb{R}_{>0} \setminus \{1\}$, and then take $(\lambda_1, \lambda_2) = (-\mu, -\mu\rho)$ for a sufficiently large $\mu > 0$. This corresponds in fact to a high gain design with $L_c = (\mu(1+\rho), \rho\mu^2)$ and μ sufficiently large. Taking $L_d = 0$ finally gives a UGES observer for $\mathcal{H}_u$ initialized in $\mathbb{R}^2 \setminus \delta\mathbb{B}$.

Observe also that in fact, with any positive ℓ_1 and ℓ_2, $A_c - L_c H_c$ is Hurwitz, so there exist P and $a_c < 0$ such that (24a) holds. Then, there exists a_d such that (24b) is verified, and for any $\tau_m > \frac{a_d}{-a_c}$, we have (24c). In the case where $\lambda > 1$, for any initial condition different from the origin, the duration between two successive jumps tends to $+\infty$ and becomes larger than τ_m at some point. Therefore, we actually have a globally convergent observer for $\mathcal{H}_u$ initialized in $\mathbb{R}^2 \setminus \{0\}$ by choosing any ℓ_1 and ℓ_2 positive. △

The reasoning of Example 5.2 is based on the homogeneity of M_λ, which comes from the diagonality of A_d in this particular example. It is not always the case. On the other hand, the way of expressing P with Vandermonde matrices is possible as soon as (A_c, H_c) is observable, because one can always find a change of coordinates that transforms $(A_c - L_c H_c)$ into a block-companion form.

VI. Conclusion

Under the assumption that the jumps of the system can be detected, we have given sufficient conditions for asymptotic convergence of an impulse observer for general hybrid systems with linear flow/jump maps. Those conditions take the form of matrix inequalities which can often be solved thanks to LMI solvers. An improvement of our results could be to find sufficient conditions linked to detectability/observability to guarantee their solvability. Also, we have assumed that the jumps of the plant and of the observer are synchronized, but the instantaneous detection of the plant's jumps may be unrealistic in practice. A further study of the robustness with respect to delays in the observer jumps is thus necessary. Preliminary results based on [17] show that semiglobal practical stability may be obtained under certain conditions.

Appendix

Lemma 1.1: Consider a matrix A in $\mathbb{R}^{n\times n}$, a matrix H in $\mathbb{R}^{p\times n}$, a matrix B in $\mathbb{R}^{n\times p}$ and a diagonal matrix $\Lambda = diag(\lambda_1, \dots, \lambda_n)$. If there exists an invertible matrix F in $\mathbb{R}^{n\times n}$ such that

$$FA - \Lambda F = BH \tag{26}$$

then, for any positive definite diagonal matrix P_0, taking $P = F^\top P_0 F$ and $L = F^{-1}B$ gives

$$(A - LH)^\top P + P(A - LH) \leq 2 \max_i \lambda_i \, P \tag{27}$$

$$(A - LH)^\top P (A - LH) \leq \max_i \lambda_i^2 \, P \tag{28}$$

and the eigenvalues of $(A - LH)$ are $(\lambda_1, \dots, \lambda_n)$.

Proof: $A - LH = F^{-1}\Lambda F$ and thanks to the diagonality of P_0, $\Lambda^\top P_0 + P_0 \Lambda \leq 2\max_i \lambda_i P_0$ and $\Lambda^\top P_0 \Lambda \leq \max_i \lambda_i^2 P_0$. ■

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