Barrier Function Certificates for Forward Invariance in Hybrid Inclusions

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Abstract— **This paper proposes barrier functions for the study of forward invariance in hybrid systems modeled by hybrid inclusions. After introducing an appropriate notion of a barrier function, we propose sufficient conditions to guarantee forward invariance properties of a set for hybrid systems with nonuniqueness of solutions, solutions terminating prematurely, and Zeno solutions. Our conditions involve infinitesimal conditions on the barrier certificate and Minkowski functionals. Examples illustrate the results.**

I. Introduction

Forward invariance for dynamical systems is a useful property in many applications. Its importance is mainly due to its close relationship to safety. The main goal of safety analysis is to guarantee that the trajectories of the system remain outside an unsafe region, when they start from a particular set of initial conditions [1]. The interest in the study and the characterization of forward invariant sets dates back to the seminal work of Nagumo in [2], where what appears to be the first general characterization of forward invariance of a set proposed in terms of a cone condition. The so-called Nagumo Theorem guarantees that for each point in a given set, there exists at least one solution to the ordinary differential equation that remains in it. Extensions of this result, using similar type of cone conditions, are presented in [3] for differential inclusions, in [4] for impulse differential inclusions, and in [5] for hybrid inclusions. As the Nagumo-type conditions involve the computation of the contingent cone at the boundary of the considered set, which is a nontrivial task. However, when the considered set is defined using a scalar inequality, a barrier function candidate can be associated to this set. Hence, as we show in this paper under appropriate assumptions, it is possible to reformulate the invariance conditions using only the barrier function candidate defining the set and the data defining the system dynamics.

Different barrier notions are proposed in the literature for both continuous-time and discrete-time systems; see, e.g. [6], [7]. Some of these formulations involve conditions that have been shown to be necessary as well as sufficient in rather specific situations [8], [9]. To the best of our knowledge, barrier functions as a certificate of safety (or, equivalently, invariance) have been considered only for continuous-time systems and hybrid automata [10]. Building a barrier certificate of forward invariance for general hybrid systems such as *hybrid inclusions* [11] presents some challenges that make the extension not straightforward. A hybrid inclusion is defined as a differential inclusion with a constraint, which models the *flow* or continuous evolution of the system, and a difference inclusion with a constraint, modeling the *jumps* or discrete events. In particular, handling nonuniqueness of solutions in hybrid inclusions leads to weak forms of forward invariance properties that have been studied in [3], [12], [5], which have not been covered by the aformentioned works using barrier functions. Furthermore, having qualitative conditions of forward invariance interms of barrier functions is useful especially when control inputs can be used to force such a conditions [1], or when the forward invariance task is to be combined with a control task to be achieved inside the safety set [13].

In this paper, we introduce barrier functions to certify forward invariance properties in hybrid systems modeled as hybrid inclusions. We consider the forward invariance notions formulated in [5], which include weak invariance (or viability) and pre-invariance, where the prefix "pre" indicates that some solutions may have a bounded (hybrid) time domain. We define barrier functions as a scalar function of the state of the hybrid inclusion. Sufficient conditions in terms of infinitesimal inequalities – namely, without using information about solutions to the hybrid system – are proposed to guarantee that the set of points, denoted K, on which the barrier function is nonpositive is forward invariant (according to the different notions considered). More precisely, under mild conditions on the data defining the hybrid inclusion, we present conditions for which a barrier function guarantees forward pre-invariance of K–this result is in Theorem 1 . Under a condition on the gradient of the barrier function, which is typical in the literature (see, e.g., [14], [3]), we present conditions for weak pre-invariance as well as several special cases. It should be noted that the conditions in Theorem 1 require the barrier function to have, at points where flows are possible, a nonpositive derivative on a neighborhood of the said set and, after a jump from points where jumps are allowed, a nonpositive value. Exploiting properties of contractive sets, we relax the flow condition in Theorem 1 to one that holds only on the boundary of K.

Though stated for general hybrid inclusions, our contributions provide alternative methods, in terms of barrier

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functions, to most of the existing and well-established tools for the study of forward invariance usually stated in terms of cone conditions [4], [5]. To the best of our knowledge, this is the first time in the literature where the concept of barrier functions is used for general hybrid inclusions.

The remainder of the paper is organized as follows. Preliminaries and basic conditions are presented in Section II. Sufficient characterizations of forward invariance notions using barrier functions are in Section III. The contractivity notion hybrid systems is introduced and studied in Section IV. Further discussions on forward pre-invariance are presented in Section V. Due to space constraints, some of the proofs are omitted and will be published elsewhere.

Notation. For x, $y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ the 2−norm of x, $|x|_K := \inf_{y \in K} |x - y|$ the distance between x and the nonempty set K, and $\langle x, y\rangle$ the inner product between x and y. For a closed set $K \subset \mathbb{R}^n$, we use $\text{int}(K)$ to denote its interior, $\text{cl}(K)$ its closure, ∂K its boundary, and $U(K)$ to denote an open neighborhood around K, namely, $\text{cl}(K) \subset U(K)$. For $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O. $\mathbb{B}$ denotes the open unit ball in $\mathbb{R}^n$ centered at the origin. For a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$, $\nabla B(x)$ denotes the gradient of the function B evaluated at x. Finally $\mathcal{C}^1$ denotes the set of continuously differentiable functions.

II. Preliminaries and Basic Conditions

We consider general hybrid inclusions of the form

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \tag{1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$, the jump set $D \subset \mathbb{R}^n$, the flow and the jump set-valued maps, respectively, $F : C \rightrightarrows \mathbb{R}^n$ and $G : D \rightrightarrows \mathbb{R}^n$. A solution x to $\mathcal{H}$ is defined on a hybrid time domain denoted $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ where $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. The solution x is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$. Its domain of definition $\operatorname{dom} x$ is such that for each $(T, J) \in \operatorname{dom} x$, $\operatorname{dom} x \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0} ([t_j, t_{j+1}], j)$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$ and $t_0 = 0$; see [11].

Definition 1: (solution to $\mathcal{H}$) A function $x : \operatorname{dom} x \to \mathbb{R}^n$ defined on a hybrid time domain $\operatorname{dom} x$ and such that, for each $j \in \mathbb{N}$, $t \mapsto x(t, j)$ is absolutely continuous is a *solution* to $\mathcal{H}$ if

(S0) $x(0, 0) \in \text{cl}(C) \cup D$;

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t \ : \ (t, j) \in \operatorname{dom} x \}$ has nonempty interior

$$\begin{array}{lll} x(t, j) \in C & \text{for all} & t \in \text{int}(I^j), \\ \dot{x}(t, j) \in F(x(t, j)) & \text{for almost all} & t \in I^j; \end{array} \tag{2}$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \qquad x(t, j + 1) \in G(x(t, j)). \tag{3}$$

•

A solution x to $\mathcal{H}$ starting from x_o is said to be complete if it is defined on an unbounded hybrid time domain; that is, the set $\operatorname{dom} x$ is unbounded. It is said to be maximal if there is no solution y to $\mathcal{H}$ such that $x(t, j) = y(t, j)$ for all $(t, j) \in \operatorname{dom} x$ with $\operatorname{dom} x$ a proper subset of $\operatorname{dom} y$.

A. Forward invariance notions for hybrid inclusions

For a set $K \subset C \cup D$, following [5], we introduce the forward invariance notions considered in this paper.

Definition 2 (Weak forward pre-invariance): The set K is said to be weakly forward pre-invariant if for each $x_o \in K$, at least one maximal solution x starting from x_o satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. •

Definition 3 (Forward pre-invariance): The set K is said to be forward pre-invariant if for each $x_o \in K$, each maximal solution x starting from x_o satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. •

Definition 4 (Weak forward invariance): The set K is said to be weakly forward invariant if for each $x_o \in K$, at least one maximal solution x starting from x_o is complete and satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. •

Definition 5 (Forward invariance): The set K is said to be forward invariant if for each $x_o \in K$, each maximal solution x starting from x_o is complete and satisfies $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. •

In [3], the forward pre-invariance property is named invariance and the weak forward pre-invariance is named viability.

B. Anatomy of sets

Different types of cones have been used in the study of differential inclusions. In the following, we recall from [3] the definition of some of them, for a closed set $K \in \mathbb{R}^n$, that are used in this paper.

Definition 6: The *contingent cone* of K at x is given by

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}. \tag{4}$$

The *Dubovtsky-Miliutin* cone of K at x is given by

$$D_K(x) := \{ v \in \mathbb{R}^n : \exists \epsilon, \alpha > 0 : x + (0, \alpha](v + \epsilon\mathbb{B}) \subset K \}. \tag{5}$$

•

C. Basic assumptions

Our results are obtained under the following standing assumptions.

Standing assumptions. The data of the hybrid inclusion $\mathcal{H} = (C, F, G, D)$ is such that the flow map F is outer semicontinuous and locally bounded [1] on C, $F(x)$ is nonempty and convex for all $x \in C$, and $G(x)$ is nonempty for all $x \in D$. •

[1]Outer semicontinuous mappings have closed values. If additionally, are locally bounded, the values are compact [15].

III. Characterizations of Forward Invariance Notions Using Barrier Functions

Given a hybrid system $\mathcal{H} = (C, F, D, G)$, we consider closed sets K subset of $C \cup D$ collecting points where a barrier function candidate is nonpositive.

Definition 7: A function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a *barrier function candidate* defining the set $K \subset C \cup D$ if

$$K = \{x \in C \cup D : B(x) \leq 0\}. \quad (6)$$

•

By construction, if B is continuous, the set K is closed relative to $C \cup D$. When $C \cup D$ is closed, K is automatically closed. Furthermore, when $C \cup D = \mathbb{R}^n$, the barrier candidate B defines the set K as in [10].

Remark 1: In the literature (see, e.g., [6], [16]) barrier function candidates[2] are defined as scalar functions that are positive, locally bounded on int(K), and approach infinity as their argument converges to ∂K. The key difference between the notions therein and the one in Definition 7 is that, in the former case, solutions that start in int(K) cannot reach the boundary ∂K, which in turn renders int(K) invariant (in the appropriate sense). •

A. Sufficient conditions for forward pre-invariance

The following result provides infinitesimal conditions guaranteeing that the set K defined by a barrier function is pre-invariant. It generalizes the results in [7] for continuous-time systems and hybrid automata to the case of hybrid inclusions.

Theorem 1: Consider a barrier function candidate B defining the closed set K as in (6) that is $\mathcal{C}^1$ on a neighborhood of $\partial K \cap C$. The set K is forward pre-invariant if

$$\langle \nabla B(x), \eta \rangle \leq 0 \quad \forall \eta \in F(x),\ \forall x \in (U(\partial K) \backslash K) \cap C, \quad (7)$$
$$B(\eta) \leq 0 \quad \forall \eta \in G(x) \cap (C \cup D),\ \forall x \in D \cap K, \quad (8)$$
$$B(\eta) > 0 \quad \forall \eta \in G(x) \backslash (C \cup D),\ \forall x \in D \cap K. \quad (9)$$

□

Sketch of Proof. We prove the statement by contradiction. We have consider two cases:

- Suppose there exist a solution x jumping from K to $\mathbb{R}^n \backslash K$. This implies, using (9) that $B(x(t, j+1)) > 0$ with $x(t, j+1) \in G(x(t, j))$. However, $x(t, j) \in K \cap D$, hence using (8), it follows that $B(x(t, j+1)) \leq 0$ for all $x(t, j+1) \in G(x(t, j))$ and a contradiction follows.
- Now, suppose there exists a solution x that leaves the set K by flowing. That is, $B(x(t, 0)) > 0$ for all $t \in (t_1', t_2']$ and $x((t_1', t_2'], 0) \subset (U(\partial K) \backslash K) \cap C$ for some $0 \leq t_1' < t_2'$. Furthermore, having $B(x(\cdot, 0))$ absolutely continuous on the interval $[t_1', t_2']$, implies that
$$B(x(t_2', 0)) - B(x(t_1', 0)) = \int_{t_1'}^{t_2'} \langle \nabla B(x(t, 0)), \dot{x}(t, 0) \rangle dt > 0,$$
where $\dot{x}(t, 0) \in F(x(t, 0))$ for almost all $t \in (t_1', t_2']$. However, using (7), we conclude that $\langle \nabla B(x(t, 0)), \eta \rangle \leq 0$ for all $t \in (t_1', t_2']$ and for all $\eta \in F(x(t, 0))$, which yields to a contradiction. ■

[2] Barrier functions are also called *potential functions in the literature [17].*

Remark 2: Condition (7) in Theorem 1 requires to check the "time derivative" of B on a neighborhood $U(\partial K)$ of the set ∂K (relative to C), and any neighborhood of any size would suffice. In the particular cases when $C \cup D = \mathbb{R}^n$, or $K = \{x \in \mathbb{R}^n : B(x) \leq 0\}$, or also when $G(K \cap D) \subset C \cup D$, condition (9) is not required. •

Remark 3: If the flow condition (7) and the jump condition (8) in Theorem 1 are satisfied with different barrier function candidates, the statement therein still holds as long as each barrier function defines the same set K as in (6). The same comment applies to all the results that follow. •

The following example illustrates Theorem 1.

Example 1: Consider the bouncing ball example modeled as $\mathcal{H}$ in (1) with $F(x) := [x_2 \quad -\gamma]^\top$ for all $x \in C$,

$$C := \left\{x \in \mathbb{R}^2 : x_1 > 0, \text{ or } x_1 = 0 \text{ and } x_2 \geq 0\right\},$$

$G(x) := [0 \quad -\lambda x_2]^\top$ for all $x \in D$, and $D := \left\{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\right\}$. The constants $\gamma > 0$ and $\lambda \in [0, 1]$ are the gravity acceleration and the restitution coefficient, respectively. Furthermore, the barrier function $B(x) := 2\gamma x_1 + (x_2 - 1)^2 + 2(x_2 - 1)$, according to (6), defines the set

$$K := \left\{x \in C \cup D : 2\gamma x_1 + (x_2 - 1)^2 + 2(x_2 - 1) \leq 0\right\}.$$

The set K can be seen as the sublevel set where the total energy of the ball is less or equal than $1/2$. To conclude forward pre-invariance of the set K using Theorem 1, first, notice that $\langle \nabla B(x), F(x) \rangle = 0$ for all $x \in C$. Moreover, for each $x \in K \cap D$, we have $B(G(x)) = 2\gamma x_1 + \lambda^2 x_2^2 - 1 \leq 2\gamma x_1 + x_2^2 - 1 \leq 0$ since $\lambda \in [0, 1]$. □

B. Sufficient conditions for weak forward pre-invariance

Barrier functions defining closed sets have been used for differential inclusions in [3] and [14] to characterize weak forward invariance notions for systems with nonunique solutions. That is, in those references, a characterization of the contingent cone in terms a barrier candidate defining the set K is proposed, where, it is assumed that the gradient (or the proximal subgradient in the nonsmooth case) of the barrier candidate does not vanish on the boundary of K.

Our results in this subsection characterize weak forward invariance notions in terms of the barrier function candidate even when the gradient vanishes on the boundary of K, provided that the following assumption holds.

Assumption 1: Consider $\mathcal{C}^1$ barrier function candidate B defining the closed set $K \subset \mathbb{R}^n$ as $K = \{x \in \mathbb{R}^n : B(x) \leq 0\}$. Then, for each $x_o \in \partial K$, there exists $\epsilon^* > 0$ such that

$$\nabla B(x) \neq 0 \quad \forall x \in (x_o + \epsilon^* \mathbb{B}) \backslash K. \quad (10)$$

•

Assumption 1 will play an important role in establishing sufficient conditions for weak forward pre-invariance when the gradient ∇B vanishes on the boundary ∂K. Note that Assumption 1 is not necessarily satisfied for general barrier candidates as illustrated in the following example:

Example 2: Consider the case where $C \cup D = \mathbb{R}$ and

$$B(x) := \begin{cases} \sin^2(1/x)\exp(-x^{-2}) + \exp(-x^{-4}) & x > 0 \\ 0 & x \leq 0, \end{cases} \tag{11}$$

with K defined as in (6), $\partial K = \{x_o\} = \{0\}$ and $D_{\mathbb{R}\setminus K}(x_o) = \mathbb{R}_{>0}$. However, for all $\epsilon^* > 0$, the function B, which is infinitely smooth [3], is not monotone on any interval of the form $(0, \epsilon^*) = (x_o + \epsilon^* \mathbb{B}) \setminus K$. Indeed, for the two sequences $a_n := 1/(2\pi n)$ and $b_n := 1/(2\pi n + (\pi/2))$, $n = 1, 2, ..$, it is easy to verify that $B(b_n) < B(b_{n-1})$ and $B(a_n) < B(b_n) > B(a_{n-1})$. Hence, using the continuity of ∇B, for each $\epsilon^* > 0$ one can find $x \in (0, \epsilon^*)$ such that $\nabla B(x) = 0$. □

The following result provides sufficient conditions for weak forward pre-invariance. One of the conditions involves the extension of the set K to $\mathbb{R}^n$, namely,

$$K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}. \tag{12}$$

Theorem 2: Consider a barrier function candidate B defining the closed set K as in (6) that is $\mathcal{C}^1$ on a neighborhood of $\partial K \cap C$. Under Assumption 1, the set K is weakly forward pre-invariant if

$$\forall x \in (U(\partial K) \setminus K) \cap \text{int}(C),\ \exists \eta \in F(x) : \langle \nabla B(x), \eta \rangle \leq 0, \tag{13}$$

$$\forall x \in K \cap D,\ \exists \eta \in G(x) \cap (C \cup D) : B(\eta) \leq 0, \tag{14}$$

and for all $x \in (\partial K_e \cap \partial C) \setminus D$ with $F(x) \cap T_C(x) \neq \emptyset$,

$$F(x_1) \cap T_{K \cap C}(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial(K \cap C) \cap \partial C. \tag{15}$$

□

Sketch of proof. As a first step, we show that for each initial condition in $K \setminus \partial C$, there exists a maximal solution remaining in K provided that it does not reach the set $\partial C \cap K$. After that, we show that for each $x_o \in K \cap \partial C$, either:

1) There exists a nontrivial solution starting from x_o that remains in the set K, or
2) All of the maximal solutions starting from $x_o \in K \cap \partial C$ are trivial.

When, in addition, $x_o \in D$, we show that item 1) holds. Otherwise, we have the following two cases for x_o:

1) $x_o \in (\partial K_e \cap \partial C) \setminus D$. In this case, if $F(x_o) \cap T_C(x_o) = \emptyset$, item 2) holds. Otherwise, we establish item 1) by showing that

$$F(x) \cap T_{K \cap C}(x) \neq \emptyset \quad \forall x \in U(x_o) \cap \partial(K \cap C). \tag{16}$$

2) $x_o \in (\text{int}(K_e) \cap \partial C) \setminus D$. In this case, if item 2) doesn't hold and since $x_o \in \text{int}(K_e)$, we conclude that if there exists a nontrivial solution flowing from x_o then this solution must satisfy $x((0, T), 0) \subset K$ for some $T > 0$.

■

Remark 4: We notice that when characterizing weak forward invariance notions, in the particular situations when $C \cup D = \mathbb{R}^n$, $K = \{x \in \mathbb{R}^n : B(x) \leq 0\}$, or $G(x) \subset C \cup D$ for all $x \in K \cap D$, condition (14) can be replaced by

$$\forall x \in K \cap D,\ \exists \eta \in G(x) : B(\eta) \leq 0. \tag{17}$$

•

Remark 5: In Theorem 2, condition (15) cannot be expressed in terms of a barrier function candidate. The reason is that when $x_o \in (\partial C \cap \partial K) \setminus D$, if the inequality in condition (13) holds, it would allow us to only conclude that $F(x_o) \cap T_{K_e}(x_o) \neq \emptyset$. However, this property is not enough to conclude weak forward pre-invariance of K. Indeed, when $F(x_o) \cap T_{K_e}(x_o) \not\subset T_C(x)$, no solution x starting from x_o can satisfy $x([0, T], x_o) \subset K$ for any $T > 0$. Note that if there exists at least one solution starting from x_o and flowing in $C \setminus K$, the set K is actually not weakly forward pre-invariant since the solution does not remain in K. •

Example 3: We revisit the hybrid system introduced in Example 1. We assume, further, that the coefficient λ in G is given by the set $[1, 2]$, we can show weak forward pre-invariance of the same set for $\mathcal{H}$ with $G(x) = [0, -\lambda[1, 2]x_2]^\top$. Indeed, Assumption 1 is satisfied since B is quadratic and ∂K does not include the origin which is the only element where ∇B vanishes. Also, condition (13) is satisfied since $\langle \nabla B(x), F(x) \rangle = 0$ for all $x \in \mathbb{R}^2$. Moreover, for each $x \in K \cap D = K_e \cap D$ we can find $\eta = [0 \ \ -x_2] \in G(x) \cap K$. Hence, (14) is also satisfied. The last step consists in checking (15) holds for all $x \in (\partial K_e \cap \partial C) \setminus D$ only if $F(x) \cap T_C(x) \neq \emptyset$. Indeed, in this example, we have $(\partial K_e \cap \partial C) \setminus D = \{x_o\} := \{[0 \ \ 1]^\top\}$. Furthermore, there exists a neighborhood $U(x_o)$ around x_o such that $U(x_o) \cap \partial(K \cap C) \cap \partial C = \{x \in \mathbb{R}^2 : x_2 \in [1, 1 - \epsilon],\ x_1 = 0\} := K_1$ for some $\epsilon \geq 0$. For all $x \in K_1 \setminus \{(0, 1)\} \subset \text{int}(K_e)$, it is easy to see that $F(x) \in T_C(x) = T_{K \cap C}(x)$. Also, when $x = (0, 1)$, we can show that $F(x) \in T_{K \cap C}(x)$ using the definition of the contingent cone, the fact that $K_e \cap C = K \cap C$, and the particular geometry of intersection between K_e and C. Hence, weak forward pre-invariance follows. □

From the conditions in Theorem 2, existence of a solution from the set $\partial K \cap \text{int}(C) \cap D$ that remains in K is guaranteed when both the flow and the jump conditions (13) and (14) are satisfied. However, the weak forward pre-invariance property is already satisfied even when only one of the two conditions (13) and (14) is fulfilled in $\partial K \cap \text{int}(C) \cap D$. This is summarized in the following result.

Corollary 1: Consider a barrier function candidate B defining the closed set K as in (6) that is $\mathcal{C}^1$ on a neighborhood of $\partial K \cap C$. Under Assumption 1, the set K is weakly

[3] This can be shown by differentiating B for $x > 0$ and changing to $y = \frac{1}{x^2}$.

forward pre-invariant if the following conditions hold:

- For each $x \in (U(\partial K \backslash D)\backslash K) \cap \text{int}(C)$,
$$\exists \eta \in F(x) : \langle \nabla B(x), \eta \rangle \leq 0. \tag{18}$$
- For each $x \in K \backslash C$,
$$\exists \eta \in G(x) \cap (C \cup D) : B(\eta) \leq 0. \tag{19}$$
- For each $x \in \partial K \cap \text{int}(C) \cap D$, either (19) holds or the following condition hold
$$\forall x_1 \in (U(x)\backslash K) \cap C, \ \exists \eta \in F(x_1) : \langle \nabla B(x_1), \eta \rangle \leq 0. \tag{20}$$
- For each $x \in \partial K_e \cap \partial C$,
  - if $F(x) \cap T_C(x) = \emptyset$ and $x \in D$, then (19) holds;
  - if $F(x) \cap T_C(x) \neq \emptyset$, then either (19) or the following conditions hold:
$$F(x_1) \cap T_{K \cap C}(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial(K \cap C) \cap \partial C, \tag{21}$$
$$\forall x_1 \in U(x) \backslash (K \cap C) \cap \text{int}(C), \quad \exists \eta \in F(x_1) : \langle \nabla B(x_1), \eta \rangle \leq 0. \tag{22}$$
- For each $x \in \text{int}(K_e) \cap \partial C \cap D$, either (19) or the following condition holds:
$$F(x_1) \cap T_C(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial C. \tag{23}$$

□

In Corollary 1, the flow conditions hold on an external neighborhood of the boundary ∂K. However, when the gradient ∇B does not vanish at some elements of $\partial K_e \cap C$, the flow conditions can be relaxed to hold only for some elements of ∂K_e. This is summarized in the following result.

Corollary 2: Consider a barrier function candidate B defining the closed set K as in (6) that is $\mathcal{C}^1$ on a neighborhood of $\partial K \cap C$. The set K is weakly forward pre-invariant if the following conditions hold:

- We have
$$\nabla B(x) \neq 0 \qquad \forall x \in \partial K \backslash D \cap \text{int}(C), \tag{24}$$
$$\forall x \in U(\partial K \backslash D) \cap \partial K \cap \text{int}(C), \quad \exists \eta \in F(x) \ : \ \langle \nabla B(x), \eta \rangle \leq 0. \tag{25}$$
- For each $x \in K \backslash C$,
$$\exists \eta \in G(x) \cap (C \cup D) \ : \ B(\eta) \leq 0. \tag{26}$$
- For each $x \in \partial K \cap \text{int}(C) \cap D$, either (26) holds or, $\nabla B(x) \neq 0$ and
$$\forall x_1 \in U(x) \cap \partial K \cap C, \ \exists \eta \in F(x_1) \ : \langle \nabla B(x_1), \eta \rangle \leq 0. \tag{27}$$
- For each $x \in \partial K_e \cap \partial C$,
  - if $F(x) \cap T_C(x) = \emptyset$ and $x \in D$, then (26) holds;
  - if $F(x) \cap T_C(x) \neq \emptyset$, then either (26) or $\nabla B(x) \neq 0$ hold, and the following conditions hold:
$$F(x_1) \cap T_{K \cap C}(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial(K \cap C) \cap \partial C, \tag{28}$$
$$\forall x_1 \in U(x) \cap \partial K \cap \text{int}(C), \quad \exists \eta \in F(x_1) : \langle \nabla B(x_1), \eta \rangle \leq 0. \tag{29}$$
- For each $x \in \text{int}(K_e) \cap \partial C \cap D$, either (26) or the following condition holds.
$$F(x_1) \cap T_C(x_1) \neq \emptyset \quad \forall x_1 \in U(x) \cap \partial C. \tag{30}$$

□

C. Sufficient conditions for non-pre invariance notions

To guarantee the completeness of maximal solutions required in forward and weak forward invariance, solutions cannot escape in finite time while in the set $K \cap C$ and solutions cannot terminate at points in $(K \cap \partial C)\backslash D$. We have the following result.

Proposition 1: Consider a closed set $K \subset C \cup D$. Suppose that no maximal solution starting from K has a finite time escape within $K \cap C$ and every maximal solution from $(K \cap \partial C)\backslash D$ is nontrivial[4]. Then,

- If K is weakly forward pre-invariant then K is weakly forward invariant.
- If K is forward pre-invariant then K is forward invariant.

□

Remark 6: Finite-time escape inside the K is avoided when the latter set is compact or when F is (globally) bounded on the set $K \cap C$. The existence of nontrivial solutions from each $x \in (K \cap \text{int}(C))\backslash D$ follows, for instance, when condition (23) holds. •

Example 4: In Example 3, the set K is compact and, for each initial condition in $(K \cap \text{int}(C))\backslash D$, there exists a nontrivial solution. Hence, when $\lambda \leq 1$, the forward invariance follows. Similarly, when λ is replaced by the set $[1, 2]$, weak forward invariance follows. □

IV. Stronger forms of forward invariance for $C-$sets

One possible way to guarantee forward pre-invariance while requiring the flow condition (7) to hold only on ∂K, using barrier functions, is by considering strict inequalities instead of the weak inequalities in Theorem 2, as proposed in [10]. However, we show that such strict conditions are much stronger than typically needed, as they induce a pre-contractivity property. Roughly speaking, a pre-contractive set is forward pre-invariant and whenever a solution reach its boundary, it moves back towards the interior. In this section, inspired by [18], we propose a general definition of contractivity notions for the so-called $C-$sets under hybrid

[4] A solution x is nontrivial if $\operatorname{dom} x$ has at least two points.

inclusions. Furthermore, necessary and sufficient characterizations in terms of barrier candidates defining the set are proposed.

We recall that a set $K \subset C \cup D$ is a $C-$set if it is compact, convex and includes the origin in its interior, moreover, the corresponding Minkowski functional at $x \in \mathbb{R}^n$ is given by

$$\Psi_K(x) := \inf \{\mu \geq 0 : \ x \in \mu K\}. \tag{31}$$

Following [18], we define pre-contractivity of C-sets for hybrid inclusions.

Definition 8 (Pre-contractivity for $C-$sets): A $C-$set $K \subset C \cup D$ is said to be pre-contractive if

$$\limsup_{h \to 0^+} \frac{\Psi_K(x + \eta h) - 1}{h} < 0 \quad \forall x \in \partial K \cap C \text{ and } \forall \eta \in F(x) \cap T_C(x), \tag{32}$$

$$\Psi_K(\eta) < 1 \quad \forall x \in (D \cap K), \quad \forall \eta \in G(x). \tag{33}$$

•

In the following, we relate pre-contractivity to properties of solutions.

Lemma 1: A $C-$set K is forward pre-invariant. Moreover, for each $x_o \in \partial K$ and each nontrivial solution x starting from x_o, there exists $T > 0$ and $J \in \mathbb{N}$ such that $x(t,j) \in \text{int}(K)$ for all $(t,j) \in \text{dom}\, x \cap ([0,T] \times \{0,1,...,J\}) \setminus \{(0,0)\}$. □

Next, we propose an equivalent characterization of pre-contractivity in terms of barrier function candidates.

Proposition 2: A $C-$set $K \subset \text{int}(C \cup D)$ is pre-contractive if and only if there exists a Lipschitz continuous barrier function candidate B defining the set K as in (6) such that

$$\limsup_{h \to 0^+} \frac{B(x + \eta h)}{h} < 0 \quad \forall x \in \partial K \cap C \text{ and } \forall \eta \in F(x) \cap T_C(x), \tag{34}$$

$$B(\eta) < 0 \quad \forall x \in K \cap D, \ \forall \eta \in G(x), \tag{35}$$

$$B(\eta) \geq 0 \quad \forall x \in K \cap D, \ \forall \eta \in G(x) \backslash (C \cup D). \tag{36}$$

□

Remark 7: A pre-contractive C-set $K \subset C \cup D$ admitting a nontrivial flowing solution from each point in $(\partial(K \cap C) \cap \partial C) \backslash D$ is said to be contractive. Furthermore, the condition on the existence on nontrivial flows starting from $(\partial(K \cap C) \cap \partial C) \backslash D$ is satisfied when, for example,

$$F(x_1) \cap T_C(x_1) \neq \emptyset \quad \forall x_1 \in U(x_o) \cap \partial(K \cap C) \cap \partial C, \quad \forall x_o \in (\partial(K \cap C) \cap \partial C) \backslash D. \tag{37}$$

•

V. Final Remarks

In general, if we allow nonstrict inequalities instead of (34)-(36), we may fail to guarantee forward pre-invariance due to the following reasons:

1) When $\nabla B(x_o) = 0$ for some $x_o \in \partial K \cap C$, even if $F(x_o) \subset D_{\mathbb{R}^n \backslash K}(x_o)$, condition (34) with a nonstrict inequality is satisfied. However, according to [3, Theorem 4.3.4], there exists of a solution starting from x_o and flowing outside the set K. Hence, K is not forward pre-invariant.
2) When solutions starting from x_o are nonunique, even if $\nabla B(x_o) \neq 0$, if $\langle \nabla B(x), \eta \rangle = 0$ for each $\eta \in F(x)$, a solution may leave the set K.

When the flow map satisfies certain regularity conditions outside the set K, such as Lipschitz continuity, it is possible to show that relaxed barrier conditions on the boundary allow to conclude forward pre-invariance, see [19] for the case of continuous-time differential equations. Due to space constraints, this study is not included in this paper.

References

[1] P. Wieland and F. Allgöwer. Constructive safety using control barrier functions. *IFAC Proceedings Volumes*, 40(12):462–467, 2007.
[2] M. Nagumo. Über die lage der integralkurven gewöhnlicher differentialgleichungen. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 24:551–559, 1942.
[3] J. P. Aubin. *Viability Theory*. Birkhauser Boston Inc., Cambridge, MA, USA, 1991.
[4] J. P. Aubin, L. Lygeros, M. Quincampoix, S. Sastry, and N. Seube. Impulse differential inclusions: A viability approach to hybrid systems. *IEEE Transactions on Automatic Control*, 47(1):2–20, 2002.
[5] J. Chai and R. G. Sanfelice. Forward invariance of sets for hybrid dynamical systems (Part I). *IEEE Transactions on Automatic Control*, pages 1–1, 2019.
[6] A. G. Wills and W. P. Heath. Barrier function based model predictive control. *Automatica*, 40(8):1415 – 1422, 2004.
[7] S. Prajna. *Optimization-based methods for nonlinear and hybrid systems verification*. PhD thesis, California Institute of Technology, 2005.
[8] S. Prajna and A. Rantzer. On the necessity of barrier certificates. *IFAC Proceedings Volumes*, 38(1):526–531, 2005.
[9] R. Wisniewski and C. Sloth. Converse barrier certificate theorems. *IEEE Transactions on Automatic Control*, 61(5):1356–1361, 2016.
[10] S. Prajna, A. Jadbabaie, and G. J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE Transactions on Automatic Control*, 52(8):1415–1428, 2007.
[11] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems*, 29(2):28–93, 2009.
[12] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*, volume 178. Springer Science & Business Media, 2008.
[13] M. Z. Romdlony and B. Jayawardhana. Stabilization with guaranteed safety using control lyapunov–barrier function. *Automatica*, 66:39–47, 2016.
[14] F. H. Clarke. *Optimization and Nonsmooth Analysis*, volume 5. 1990.
[15] R. Goebel and A. R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573–587, 2006.
[16] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada. Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, 62(8):3861–3876, 2017.
[17] H. G. Tanner, A. Jadbabaie, and G. J. Pappas. Stable flocking of mobile agents, Part I: Fixed topology. In *42nd Conference on Decision and Control*, volume 2, pages 2010–2015. IEEE, 2003.
[18] F. Blanchini. Survey paper: Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.
[19] R. M. Redheffer. The theorems of bony and brezis on flow-invariant sets. *The American Mathematical Monthly*, 79(7):740–747, 1972.